Genus 3 curves

The goal of these notes is to ~~give//prov~~ provide the details of the genus 3 example given in the lectures. Namely we will prove that:

1. Given two biholomorphic ~~quadr~~ quartics $C_1 \cong C_2$ in $\mathbb{P}^2$ there exists $A \in \mathrm{Aut}(\mathbb{P}^2) = \mathbb{P}GL_3$ s.t. $A(C_1) = C_2$;

2. $$\mathcal{M}_3 \setminus \{\text{quartics in } \mathbb{P}^2\} = \left\{\begin{matrix}\text{hyperelliptic genus 3} \\ \text{curves}\end{matrix}\right\}.$$

proof of 1

Obs Let $C \subseteq \mathbb{P}^2$ be a quartic. Then by adjunction formula

$$K_C = \underset{\substack{\shortparallel \\ \mathcal{O}(-3)}}{K_{\mathbb{P}^2}}\Big|_C \otimes \mathcal{O}(4)\Big|_C \cong \mathcal{O}(1)\Big|_C$$

has $\dim = \cancel{g} = 3$

So $x_0, x_1, x_2 \in H^0(C, \mathcal{O}(1)|_C = K_C)$ is a basis.

Now consider the commutative diagram

$$\begin{tikzcd}
& K_{C_1} \arrow[d, equal] & & K_{C_2} \arrow[d, equal] & \\
& \mathcal{O}(1)|_{C_1} \arrow[dl] \arrow[d] & & \mathcal{O}(1)|_{C_2} \arrow[ll, "\varphi^*"'] \arrow[d] \arrow[dr] & \\
\mathcal{O}(1) \arrow[dr] & C_1 \arrow[d] & & C_2 \arrow[ll, "\cong", "\varphi"'] \arrow[d] & \mathcal{O}(1) \arrow[dl] \\
& \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^2 &
\end{tikzcd}$$

coordinates Y_0, Y_1, Y_2

we want to extend φ to $\tilde{\varphi} \in \mathrm{Aut}(\mathbb{P}^2) = \mathbb{P}GL_3$

coordinates X_0, X_1, X_2

Since Y_0, Y_1, Y_2 is a basis of $H^0(C_1, \mathcal{O}(1)|_{C_1})$ we can write

$$\varphi^* X_i = \sum_{j=0}^{2} a_{ij} Y_j \qquad i = 0,1,2 \qquad \text{for } a_{ij} \in \mathbb{C}$$

Now

1-dim vector spaces

$$\begin{array}{ccccc}
\varphi^* X_i : [Y] & \longrightarrow & \varphi[Y] & \xrightarrow{X_i} & \mathbb{C} \\
\Psi & & \Psi & & \\
(y_0, y_1, y_2) & \mapsto & (x_0, x_1, x_2) & \longmapsto & x_i
\end{array}$$

$0 \neq y \in [Y]$

So $\varphi([Y]) \overset{!}{=} [\varphi^* x_0(y), \varphi^* x_1(y), \varphi^* x_2(y)] = [\sum a_{0j} y_j, \sum a_{1j} y_j, \sum a_{2j} y_j] = [A][Y]$

where $[A = (a_{ij})] \in \mathbb{P}GL_3$. ∎

Remark ~~The same proof gives // This approach is one of the possible way to approaches // pro // This // ar~~ The same proof gives $\mathrm{Aut}(\mathbb{P}^n) = \mathbb{P}GL_{n+1}$

~~proof of 2~~

We need to introduce the so called canonical map of C.

Canonical curves

Let $g \geq 2$.

Lemma $\boxed{K_C \text{ is base point free}}$

proof

Suppose $p \in C$ is a base point for K_C (i.e. $w(p)=0$ $\forall w \in K_C(C)$) Then

$$h^0(K_C \otimes \mathcal{O}_{-p}) = h^0(K_C) - g \geq 2$$

By RR

$$h^0(\mathcal{O}_p) = h^1(\mathcal{O}_p) + 1 - g + \deg(\mathcal{O}_p) =$$

$$= h^0(K_C \otimes \mathcal{O}_p) + 1 - g + 1 = 2$$

$\Rightarrow \exists\ f : C \xrightarrow{1:1} \mathbb{P}^1$ non constant with a simple pole at p

$\Rightarrow C \cong \mathbb{P}^1$. Contradiction. ∎

Def The map

$$i_C : C \to \mathbb{P}^{g-1}$$
$$p \mapsto [w_1(p) : \ldots : w_g(p)]$$

is called canonical mapping of C

Question: When is i_C an embedding?

Obs: i_C is injective $\Leftrightarrow \forall p \neq q \in C\ \exists w \in K_C(C) : w(p)=0, w(q)\neq 0$

proof

($\Leftarrow$) Clear.

($\Rightarrow$) $i_C(p) \neq i_C(q) \Leftrightarrow \mathrm{rk}\begin{pmatrix} w_1(p) & \cdots & w_g(p) \\ w_1(q) & & w_g(q)\end{pmatrix} = 2$

$\Leftrightarrow$ doing elementary operations on the columns of this matrix we get $\begin{pmatrix} 1 & \boxed{0} & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \cdots & 0\end{pmatrix}$ ∎

▲ i_K is an immersion $\iff$ $\forall p \in C$ $\exists \omega \in K_C(C)$ vanishing exactly to order 1 at p

proof

$p \in C$ and let z be coordinates around p. Write $w_i = f_i\, dz$ $i = 1, \dots g$

Then
$$i_C(z) = [f_1(z) : \dots : f_g(z)]$$

If $i_C(z) \in U_i$, then $i_C(z) = \left(\frac{f_1(z)}{f_i(z)}, \dots, \hat{1}, \dots, \frac{f_g(z)}{f_i(z)} \right)$

and $\frac{\partial}{\partial z}(f_j/f_i) = \frac{f_j' f_i - f_i' f_j}{f_i^2} \neq 0 \iff \det \begin{pmatrix} f_i & f_j \\ f_i' & f_j' \end{pmatrix} \neq 0$

Thus $(d\, i_C)_z$ is inj $\iff$ $rk \begin{pmatrix} f_1 & \dots & f_g \\ f_1' & & f_g' \end{pmatrix} = 2$

$\iff$ doing elementary operations on the columns of this matrix we get $\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix}$

Proposition

$$\boxed{i_C : C \to \mathbb{P}^{g-1} \text{ is an embedding} \iff C \text{ is NOT hyperelliptic.}}$$

proof

From ▲ and ▲▲ follow that

i_C is an embedding $\iff$ $\forall p, q \in C$ (not necessarily distinct)
$$h^0(K_C \otimes \mathcal{O}_{-p-q}) < h^0(K_C \otimes \mathcal{O}_{-p}) = g-1$$

holomorphic differentials that are 0 at p

$\cap$ ✗ ← lemma

holomorphic differentials

By R-R
$$h^0(K \otimes \mathcal{O}_{-p-q}) = h^1(K \otimes \mathcal{O}_{-p-q}) + 1 - g + (2g-2-2) =$$
$$= h^0(\mathcal{O}_{p+q}) + g - 3$$

So

i_C is an embedding $\iff$ $\forall p,q \in C$ $h^0(\mathcal{O}_{p+q}) + g - 3 < g - 1 \iff \forall p,q \in C$ $h^0(\mathcal{O}_{p+q}) < 1$

$\iff \mathcal{O}_{p+q}(C) = 0 \quad \forall p, q \in C$

$\iff \nexists\ f : C \xrightarrow{2:1} \mathbb{P}^1$.

∎

Corollary

Suppose $g=3$. Then

C is NOT hyperelliptic $\Leftrightarrow$ C is isomorphic to some quartic in $\mathbb{P}^2$

proof.

($\Leftarrow$) $K_C = \mathcal{O}(1)|_C$, $x_0, x_1, x_2 \in H^0(C, \mathcal{O}(1)|_C = K_C)$ basis

$\Rightarrow$ $i_C : C \hookrightarrow \mathbb{P}^2$ is just the inclusion map

$\Rightarrow$ i_C is embedding $\Rightarrow$ C is NOT hyperelliptic.

($\Rightarrow$) Suppose C is not hyperelliptic. Then

$$i_C : C \hookrightarrow \mathbb{P}^2$$

is an embedding

$\Rightarrow$ $C \cong V(F) \subseteq \mathbb{P}^2$

$\uparrow$ smooth of degree d

Claim: it must be $d=4$

proof of the claim

From the genus formula

$$3 = g = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$$

$$\Leftrightarrow \quad 6 = d^2-3d+2 \Leftrightarrow (d-4)(d+1)=0 \underset{d>1}{\Leftrightarrow} d=4$$

∎

This proves 2