A tour of the geometry of points in affine space

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§I. Counting partitions

How can we write n as a sum of positive numbers?

The list of partitions of $n = 3$ is

$$3\,,\quad 2+1\,,\quad 1+1+1\,,$$

and the list of partitions of $n = 4$ is

$$4\,,\quad 3+1\,,\quad 2+2\,,\quad 2+1+1\,,\quad 1+1+1+1\,.$$

$$\boxed{\text{Let } \; p(n) \; = \; \text{Number of partitions of } n}$$

So $p(3) = 3$ and $p(4) = 5$.

A formula for $p(n)$?

There is no direct formula for $p(n)$, but there is a formula for the **generating series**:

$$\boxed{\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right)}$$

Expand the right side

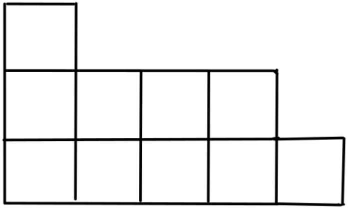
$$\begin{aligned} \sum_{n=0}^{\infty} p(n) q^n &= \left(\frac{1}{1-q^1} \right) \left(\frac{1}{1-q^2} \right) \left(\frac{1}{1-q^3} \right) \cdots \\ &= 1 + q^1 + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots \end{aligned}$$

The **product formula** for the counting of partitions was found by Leonhard Euler (1707-1783):

Express partitions as diagrams:

$$10 = 5 + 4 + 1$$

can be pictured as



The diagram may be viewed as stacking squares in the corner of a 2-dimensional room (stable for both coordinate directions of gravity).

What about 3-dimensions ?

We would like to stack 3-dimensional boxes in the corner of a 3-dimensional room.

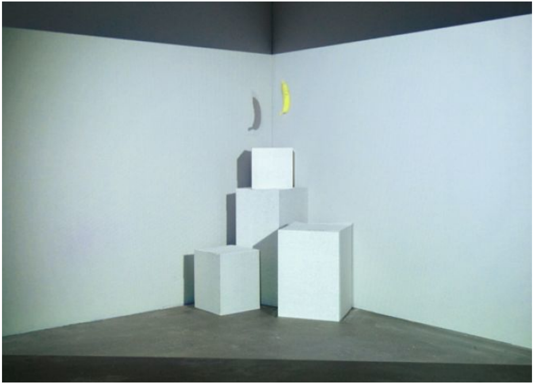
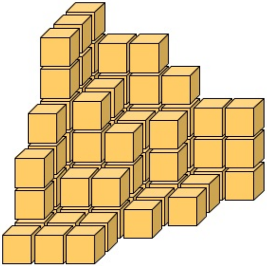
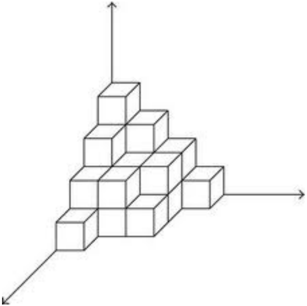


Photo of the installation **Five Boxes** by the Icelandic artist **Egill Sæbjörnsson**.

Photo courtesy of the **Reykjavik Art Museum**.

A 3-dimensional partition is a stacking of boxes in the corner of a room (which is stable for any of the three coordinate directions of gravity):





Let $P(n)$ = Number of 3-dimensional partitions of n

We see $P(1) = 1$, $P(2) = 3$, $P(3) = 6$, ...

A formula for $P(n)$

Again, there is no direct formula for $P(n)$, but there is a formula for the **generating series**:

$$\sum_{n=0}^{\infty} P(n) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right)^k$$

The formula is due to Percy MacMahon (1854-1929). Before his mathematical career, he was a Lieutenant in the British army. He was said to be at least partially inspired by stacking cannon balls.

A formula for counting partitions in 4-dimensions ?

2-dim $$\boxed{\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right)}$$

3-dim $$\boxed{\sum_{n=0}^{\infty} P(n)q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right)^k}$$

MacMahon proposed $\prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right)^{\binom{k+1}{2}}$ for the generating series of 4-dimensional partitions.

He was wrong! Formulas for dimensions 4 and higher are unknown.

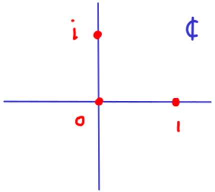
His 4-dim proposal is correct for $n \leq 5$. For $n = 6$ boxes, he proposes 141, while the correct number is 140.

§II. Points in affine space: dimensions 1 and 2

We will study the r-dimensional complex affine space $\mathbb{C}^r$ and consider configurations of n distinct unordered points of $\mathbb{C}^r$.

A configuration of 3 points in $\mathbb{C}^1$:

$$\{0, 1, i\} \subset \mathbb{C}^1$$



The configuration space $\mathbb{C}^r[n]$ parameterizes all such configurations of n distinct unordered points of $\mathbb{C}^r$.

- The $r = 1$ case is simple:

$$\mathbb{C}^1[n] = \{\text{monic degree n polynomials in } x \text{ with no double roots}\}$$

by multiplication of linear factors

$$\{0, 1, i\} \mapsto (x - 0)(x - 1)(x - i) = x^3 - (1 + i)x^2 + ix\,.$$

To capture the **collisions of points**, we take the space of all monic polynomials

$$\mathbb{C}^1[n] \subset \{\text{all monic degree n polynomials in } x\} = \mathbb{C}^n \, .$$

• The $r = 2$ case is much more interesting: how are we to capture the collisions of points in $\mathbb{C}^2$?

Algebraic geometry provides a deep solution to the question of collisions via the Hilbert scheme.

Let x, y be the two coordinates of $\mathbb{C}^2$. To each configuration

$$\{p_1, p_2, \ldots, p_n\} \in \mathbb{C}^2$$

of distinct points, we associate the ideal of polynomials $\mathcal{I} \subset \mathbb{C}[x, y]$ which vanish on these points

$$\{p_1, p_2, \ldots, p_n\} \ \mapsto \ \mathcal{I} = \{\, f \in \mathbb{C}[x, y] \mid \forall i \, , \ f(p_i) = 0 \,\} \ .$$

The quotient ring has dimension n as a $\mathbb{C}$-vector space:

$$\dim_{\mathbb{C}} \left(\mathbb{C}[x,y]/\mathcal{I} \right) = n\,.$$

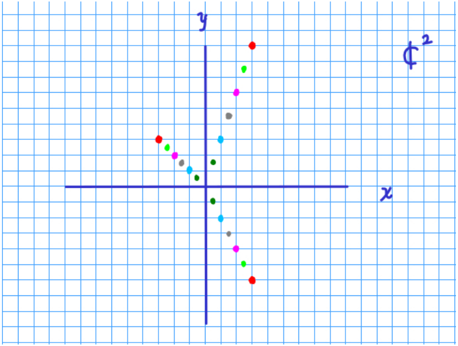
An idea due to Alexander Grothendieck is to parameterize all ideals $\mathcal{I} \subset \mathbb{C}[x,y]$ of codim n by a space he called the Hilbert scheme.

The Hilbert scheme is an example of a moduli space in algebraic geometry:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \left\{ \mathcal{I} \subset \mathbb{C}[x,y] \;\middle|\; \dim_{\mathbb{C}} \left(\mathbb{C}[x,y]/\mathcal{I} \right) = n \right\},$$

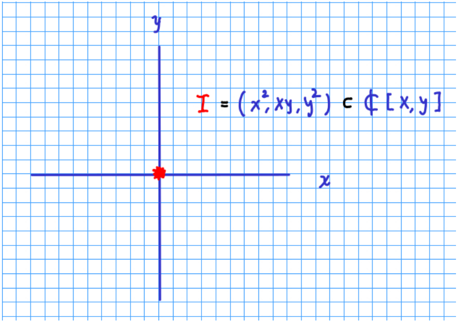
and we have $\mathbb{C}^2[n] \subset \mathrm{Hilb}^n(\mathbb{C}^2)$.

Collision of point configurations in $\mathbb{C}^2[3]$



Limit configuration in $\mathrm{Hilb}^3(\mathbb{C}^2)$ satisfying

$\dim_{\mathbb{C}} \left(\mathbb{C}[x,y]/(x^2, xy, y^2)\right) = 3$



§III. Geometry of $\mathrm{Hilb}^n(\mathbb{C}^2)$

$\mathrm{Hilb}^n(\mathbb{C}^2)$ is a **nonsingular** complex manifold (or algebraic variety) of dimension 2n by Fogarty (1968).

- **Euler characteristic**

The first question about the topology of a space:
what is the Euler characteristic?

Theorem [Ellingsrud-Strømme 1987, Göttsche 1994].
The **generating series** of Euler characteristics is:

$$\sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n(\mathbb{C}^2)) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right)$$

We recognize the right side as **counting partitions**.

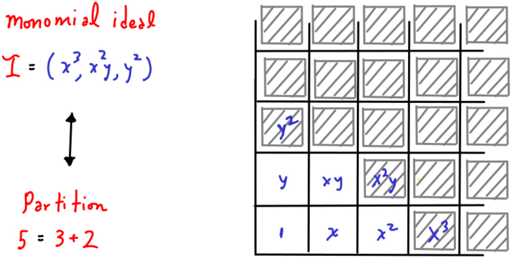
A coincidence?

An ideal $\mathcal{I} \subset \mathbb{C}[x, y]$ is monomial if $\mathcal{I}$ is generated by monomials in x and y. For example:

$$\mathcal{I} = (x^2, xy, y^2) \text{ is mononial,} \quad \mathcal{I} = (x + y, y^3) \text{ is not.}$$

Monomial ideals of codimension n are in bijective correspondence with partitions of n.

The diagram of the corresponding partition is defined by the n monominals which are not in $\mathcal{I}$.



Calculation of $\chi(\mathrm{Hilb}^n(\mathbb{C}^2))$ by Ellingsrud-Strømme (1987) and Cheah (1996) in four steps:

• The group $\mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{C}^2$ by scaling the coordinates

$$(\lambda_1, \lambda_2) \cdot (x, y) = (\lambda_1 x, \lambda_2 y)$$

and therefore $\mathbb{C}^* \times \mathbb{C}^*$ also acts on $\mathrm{Hilb}^n(\mathbb{C}^2)$.

• Since $\chi(\mathbb{C}^*) = 0$, we have:

$$\boxed{\chi(\mathrm{Hilb}^n(\mathbb{C}^2)) \;=\; \text{Number of fixed points}}$$

• The fixed points of the action are monomial ideals.

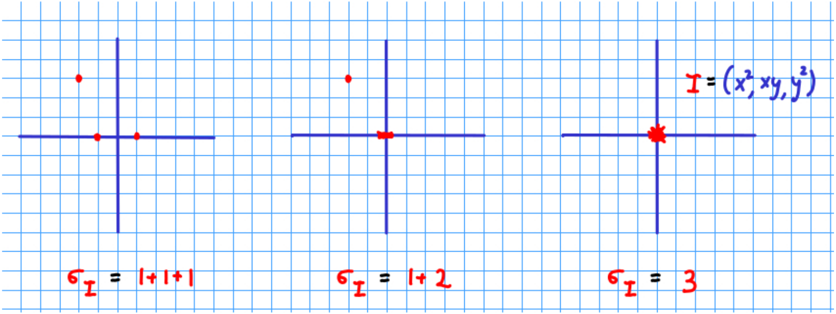
• Monomial ideals in $\mathbb{C}[x, y]$ of codimension n are in bijective correspondence with partitions of n.

- **Full cohomology** $H^\star(\mathrm{Hilb}^n(\mathbb{C}^2))$

We can ask next: what does the cohomology look like?

To every $\mathcal{I} \in \mathrm{Hilb}^n(\mathbb{C}^2)$, we can associate a partition $\sigma_{\mathcal{I}}$ of n by the pattern of collisions.

Examples for $n = 3$ are:



Given any partition σ of n, we define $\mathrm{N}(\sigma) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ by:

$$\mathrm{N}(\sigma) = \overline{\left\{ \mathcal{I} \in \mathrm{Hilb}^n(\mathbb{C}^2) \;\middle|\; \sigma_{\mathcal{I}} = \sigma \right\}}.$$

Theorem [Nakajima 1997, Grojnowski 1996]. A $\mathbb{Q}$-basis of the cohomology of $\mathrm{Hilb}^n(\mathbb{C}^2)$ is determined by the subvarieties $\mathrm{N}(\sigma)$ as σ varies over all partitions of n.

The result allows for a geometric understanding of the full cohomology. The sum

$$\bigoplus_{n=0}^{\infty} H^\star(\mathrm{Hilb}^n(\mathbb{C}^2))$$

is naturally the Fock space representation of the Heisenberg algebra, and there is a natural (additive) isomorphism:

$$\bigoplus_{n=0}^{\infty} H^\star(\mathrm{Hilb}^n(\mathbb{C}^2)) \;\stackrel{\sim}{=}\; \Lambda\,,$$

where Λ is the ring of symmetric polynomials in variables $\{x_i\}_{i=1}^{\infty}$.

Under the isomorphism,

$$\bigoplus_{n=0}^{\infty} H^\star(\mathrm{Hilb}^n(\mathbb{C}^2)) \ni [\mathrm{N}(\sigma)] \quad \longleftrightarrow \quad \frac{1}{|\mathrm{Aut}(\sigma)|}\, p^\sigma \in \Lambda\,,$$

where p^σ is the power sum symmetric function:

$$\sigma = 1+1+3\,, \quad p^\sigma = p_1^2 \cdot p_3\,, \quad p_i = {x_1}^i + {x_2}^i + {x_3}^i + \cdots\,.$$

The connection to representation theory was first conjectured by C. Vafa and E. Witten (1994) based on a study of the orbifold cohomology of the quotient $(\mathbb{C}^2)^n/\Sigma_n$.

The geometry of $\mathrm{Hilb}^n(\mathbb{C}^2)$ was used by M. Haiman (2001) to prove properties of Macdonald polynomials and the n! conjecture.

- **Quantum cohomology** $QH^\star(\mathrm{Hilb}^n(\mathbb{C}^2))$

The symmetric product $(\mathbb{C}^2)^n/\Sigma_n$ is **singular**, but otherwise a much more naive geometry. The Hilbert scheme admits a map

$$\mathrm{Hilb}^n(\mathbb{C}^2) \longrightarrow (\mathbb{C}^2)^n/\Sigma_n$$

which is a **resolution of singularities**.

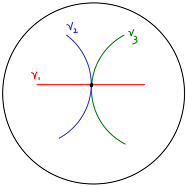
As suggested by Vafa and Witten (1994), there is a deep connection between the geometry of

$$\mathrm{Hilb}^n(\mathbb{C}^2) \quad \text{and} \quad \left[(\mathbb{C}^2)^n/\Sigma_n\right]^{\mathrm{orb}},$$

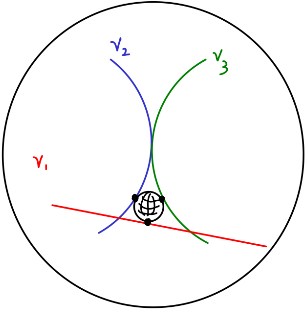
where the **orbifold structure** is taken on the symmetric product.

20 year project to compute and prove an equivalence in quantum cohomology: Chen-Ruan (2002), Bryan-Graber (2009), Coates-Corti-Iritani-Tseng (2009), Maulik-Oblomkov (2009), Okounkov-P (2010), P-Tseng (2019).

The **classical cup product** in cohomology (for manifolds) carries the data of the intersection product of triples of cycles.



The **quantum product** carries a richer set of data: the enumeration of rational curves meeting triples of cycles.



Theorem [Okounkov-P 2010]. The quantum cohomology of $\text{Hilb}^n(\mathbb{C}^2)$ is generated as an algebra by the class

$$\mathsf{N}(2 + \underbrace{1 + \cdots + 1}_{n-2}) .$$

While quantum cohomology concerns the enumeration of Riemann spheres, the full Gromov-Witten theory carries the enumerative geometery of curves of all genera.

Theorem [P-Tseng 2019]. The full Gromov-Witten theories of $\text{Hilb}^n(\mathbb{C}^2)$ and $\left[(\mathbb{C}^2)^n/\Sigma_n\right]^{\text{orb}}$ are isomorphic.

Philosophy: $\text{Hilb}^n(\mathbb{C}^2)$ is a perfect resolution of singularities of the symmetric product which carries exactly the same quantum geometry.

Of course there are many beautiful directions related to $\mathrm{Hilb}^n(\mathbb{C}^2)$ which I have not covered:

▲ Euler characteristics of Hilbert schemes of points of plane curve singularities $C \subset \mathbb{C}^2$ and the HOMFLY-PT polynomials of their links [Oblomkov-Shende 2012, Maulik 2016].

▲ Exact formulas for tautological integrals and K-theoretic invariants [Lehn 1999, Carlsson 2008, Carlsson-Okounkov 2012, Voisin 2019, Marian-Oprea-P 2022, Moreira 2022, Göttsche-Mellit 2022].

▲ Stable cohomology of $\mathrm{Hilb}^n(\mathbb{C}^\infty)$ [Hoyois, Jelisiejew, Nardin, Totaro, Yakerson 2021].

▲ Holomorphic symplectic geometry of $\mathrm{Hilb}^n(\mathbb{C}^2)$, $\mathrm{Hilb}^n(\mathrm{A})$, $\mathrm{Hilb}^n(K3)$. There is far too much activity to summarize, see the webpage www.erc-hyperk.org of the ERC Synergy Grant HyperK led by Debarre, Huybrechts, Macri, Voisin.

§IV. **Geometry of** $\mathrm{Hilb}^n(\mathbb{C}^3)$

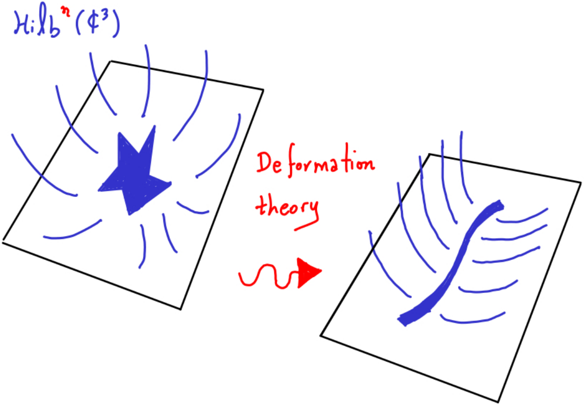
Unlike the case of $\mathbb{C}^2$, the Hilbert scheme

$$\mathrm{Hilb}^n(\mathbb{C}^3) = \left\{ \mathcal{I} \subset \mathbb{C}[x, y, z] \;\middle|\; \dim_{\mathbb{C}} \left(\mathbb{C}[x, y, z]/\mathcal{I} \right) = n \right\}$$

parameterizing ideals in 3 variables is a terrible space (**singular**, **many irreducible components**, **unknown nilpotent structure**). Not a central topic of study until recently.

Starting in the 1990s, there was an effort made in algebraic geometry to define integration on algebraic moduli spaces predicted by path integral techniques [Li-Tian, Behrend-Fantechi].

The idea is to use **deformation theory** in algebraic geometry. Though moduli spaces, such as the Hilbert scheme, are ill-behaved, we have some understanding of their local structure.



If we view $\text{Hilb}^n(\mathbb{C}^3)$ as essentially the space of 3 commuting nxn matrices A, B, C in the space of all nxn matrices, then the defining equations are given by the critical locus $dF = 0$ where

$$F = \text{Trace}([A, B]C)\,.$$

The outcome is a **virtual fundamental class** and a well-defined theory of integration on $\mathrm{Hilb}^n(\mathbb{C}^3)$.

- **Integration**

Theorem [Maulik-Nekrasov-Okounkov-P 2006]:

$$\boxed{\sum_{n=0}^{\infty} q^n \int_{[\mathrm{Hilb}^n(\mathbb{C}^3)]^{\mathrm{vir}}} 1 \;=\; \prod_{k=1}^{\infty} \left(\frac{1}{1-(-q)^k} \right)^k}$$

which is MacMahon's series for counting 3-dimensional partitions (up to a sign).

- **Sign**

While $\mathrm{Hilb}^n(\mathbb{C}^3)$ is **singular**, there is a **Zariski tangent space**

$$\mathrm{Tan}_{\mathcal{I}}^{\mathrm{vir}} = \mathrm{Ext}^1(\mathcal{I}, \mathcal{I})\,.$$

Conjecture [Okounkov-P 2006]. For all $\mathcal{I} \in \mathrm{Hilb}^n(\mathbb{C}^3)$,

$$\dim_{\mathbb{C}} \mathrm{Tan}_{\mathcal{I}}^{\mathrm{vir}} = n \mod 2\,.$$

- **Virtual motive**

Theorem [Behrend-Bryan-Szendrői 2013]:

$$\boxed{\sum_{n=0}^{\infty} q^n \, [\mathrm{Hilb}^n(\mathbb{C}^3)]^{\mathrm{vir}}_{\mathrm{mot}} = \prod_{k=1}^{\infty} \prod_{\ell=0}^{k-1} \frac{1}{1 - \mathbb{L}^{\ell+2-\frac{k}{2}} q^k}}$$

where $\mathbb{L}$ is the Lefschetz motive corresponding to $\mathbb{C}^1$.

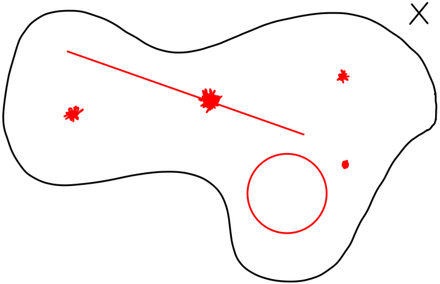
The result refines the integration calculation.

We end here at the beginning of several rich directions.

▲ Donaldson-Thomas theory: the virtual geometry of the moduli of sheaves on varieties of low dimension.

▲ Gromov-Witten/Donaldson-Thomas correspondence relating sheaf counting to curve counting.

Richest context so far is for 3-dim algebraic varieties X:



Recent study in 4-dim [Borisov-Joyce 2017, Oh-Thomas 2022].

An example of how **box counting** influences everything in 3-dimensions:

Conjecture [Oblomkov-Okounkov-P 2020]. The normalized generating series of DT invariants

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_m}(\gamma_m)\right\rangle^X_\beta \Big/ \left\langle 1\right\rangle^X_0$$

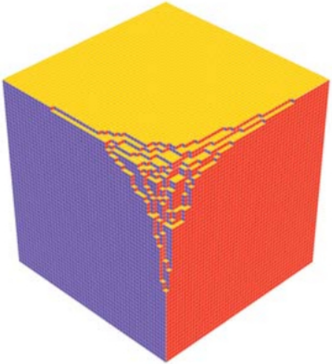
for a 3-fold X in class $\beta \in H_2(X,\mathbb{Z})$ is polynomial in the series

$$\left(q\frac{d}{dq}\right)^i F_3(-q)$$

with coefficients in the ring of rational functions in q.

$$F_3(q) = \sum_{k=1}^{\infty} k^2 \frac{q^k}{1-q^k} = \frac{q\frac{d}{dq}\mathrm{M}(q)}{\mathrm{M}(q)}\,, \quad \mathrm{M}(q) = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)^k .$$

▲ Mirror symmetry relating sheaves in one geometry to curves in a mirror geometry.



Limit shape as a mirror [Kenyon-Okounkov 2007].

The End