COHOMOLOGY OF MODULI OF STABLE POINTED CURVES OF LOW GENUS. I

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First I give some motivating examples for the theory of weights in the cohomology of an algebraic variety. Perhaps they are not so useful if you have never seen the theory before.

After this I introduce mixed Hodge theory. By this I really mean that I state a bunch of useful properties that will be used in the rest of the course as a black box. I also consider a concrete example, namely the mixed Hodge structure on H^1 of a punctured curve.

References for these first parts are: for less formal introductions, [Deligne 1975; Danilov 1996]; for a formal treatment, the book [Peters and Steenbrink 2008]; and Deligne's original papers Hodge I, Hodge II, Hodge III, Weil I, Weil II.

Finally we apply the mixed Hodge theory to moduli of curves. I show that $H^\bullet(\overline{\mathcal{M}}_{0,n})$ is spanned by strata and that the space of relations is spanned by pullbacks of the WDVV relation on $\overline{\mathcal{M}}_{0,4}$.

I also include in the end a brief explanation of how what we say about $H^\bullet(\overline{\mathcal{M}}_{0,n})$ can be interpreted in the language of operads.

This part follows the treatment of [Getzler 1995] with some differences.

1. Motivation for the theory of weights

The fundamental principle in the theory of weights is that if X is a smooth proper variety, then $H^i(X)$ is a 'pure object' of 'weight i'. These pure objects are the building blocks for the cohomology of an arbitrary variety. For any variety, $H^i(X)$ is an iterated extension of pure objects.

The main motivation for believing that such a theory should exist comes from étale cohomology and the Weil conjecture. Let me recall this. Let X be a smooth projective variety over $\mathbf{F}_q$ and $\overline{X}$ its base change to an algebraic closure. We are interested in counting the number of points of X over finite fields, i.e. finding

$$\#X(\mathbf{F}_{q^k}),$$

which is the fixed points of Frob_q acting on $X(\overline{\mathbf{F}}_q)$. The idea is that this may be computed using the Lefschetz fixed point theorem for a suitable cohomology theory. This theory is given by the ℓ-adic cohomology. It is defined algebraically and canonically enough that $H^i(\overline{X}, \mathbf{Q}_\ell)$ obtains an action of $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$ and we can talk about the trace of Frobenius on cohomology.

Theorem 1.1. *Let X be a smooth projective variety over $\mathbf{F}_q$ as above. For any ℓ not dividing q we have:*

(1) $\#X(\mathbf{F}_{q^k}) = \sum_i (-1)^i \mathrm{tr}(\mathrm{Frob}_q^k \mid H^i(\overline{X}, \mathbf{Q}_\ell))$

(2) All eigenvalues of Frob_q *on* $H^i(\overline{X}, \mathbf{Q}_\ell)$ *are algebraic and have absolute value* $q^{i/2}$ *under any embedding into the complex numbers. (We say that* H^i *is 'pure of weight* i*'.)*

Remark 1.2. The first part generalizes to an arbitrary variety if we put H_c^i instead of H^i. The second remains true if instead of $q^{i/2}$ we take $q^{k/2}$ for some $0 \leq k \leq i$. (Different eigenvalues can have different absolute value.) This is somewhat natural because we expect both non-compactness and singularities to decrease the number of $\mathbf{F}_q$-points.

MAIN IDEA: Any map between cohomology groups which is 'algebraically defined' or 'sufficiently natural' should be equivariant with respect to Frob_q-action.[1] In particular there should be *no nonzero maps* between cohomology groups of different weight.

Example 1.3. Let U be a smooth variety and X a smooth compactification, such that $X \setminus U = \bigcup D_i$ is a normal crossing divisor, with any intersection of the D_i again smooth. We consider the Leray spectral sequence for

$$f : U \to X.$$

One can compute that each sheaf $R^k f_* \mathbf{Q}_\ell$ is a sum of sheaves of the form $j_* \mathbf{Q}_\ell \otimes \mathbf{Q}_\ell(-k)$, where j is the inclusion of a k-fold intersection of distinct boundary divisors. Over the complex numbers this is not so strange, if you think of the stalk of $R^k f_* \mathbf{Q}_\ell$ at a point x as the cohomology of $U \cap V$ for a sufficiently small neighbourhood V of x in X. But let's just accept it.

(The factor $\mathbf{Q}_\ell(-k)$ makes the example not as simple as one would like. But it's not complicated. $\mathbf{Q}_\ell(-k)$ is a particular 1-dimensional Galois representation, the kth tensor power of the inverse of the cyclotomic character. We have $\mathrm{tr}(\mathrm{Frob}_q \mid \mathbf{Q}_\ell(-k)) = q^k$. So tensoring with $\mathbf{Q}_\ell(-k)$ just has the effect of multiplying all Frobenius eigenvalues by q^k, so that the weight is increased by $2k$. This is called a *Tate twist*.)

What one finds is that we have a spectral sequence

$$E_2^{pq} = \bigoplus_{i_1 < \cdots < i_q} H^p(D_{i_1} \cap \cdots \cap D_{i_q}) \otimes \mathbf{Q}_\ell(-q) \implies H^{p+q}(U).$$

Rahul pointed out that I'm using q for two distinct things now: the cardinality of a finite field and an index in a spectral sequence. But I don't think any other variable name is acceptable for either one of them, so I will stick to this.

Now the intersection of boundary divisors is smooth and projective, so $H^p(D_{i_1} \cap \cdots \cap D_{i_q})$ is a pure object of weight p. Taking into account the Tate twist we get that E_2^{pq} has weight $p + 2q$. So the weights look as in Figure 1.

It's clear that the E_2 differential is compatible with weights, but that all further differentials go between cohomology groups with *different* weight. So all further differentials should be zero, according to the principle of weights, and the Leray spectral sequence should degenerate after the first differential.

[1] At least up to a Tate twist.

2	4	5	6	7
1	2	3	4	5
0	0	1	2	3
	0	1	2	3

Figure 1. Weights at E_2^{pq} of Leray spectral sequence for f.

In characteristic zero, this degeneration is a famous result of Deligne, proven in [Deligne 1971]. His proof uses Hodge theory and represents the Leray spectral sequence as the hypercohomology spectral sequence for a complex of sheaves on X given by holomorphic differential forms possibly with logarithmic poles along $X \setminus U$.

Example 1.4. Let A be a differential graded algebra (dga), so that its cohomology $H^\bullet(A)$ is a graded algebra. It is not always true that A and $H^\bullet(A)$ are quasi-isomorphic as dga's. However, there is a notion of an A_∞-algebra S, which means that S is equipped with a sequence of multilinear maps $\mu_n : S^{\otimes n} \to S$ for $n \geq 1$ of degree $2 - n$ satisfying certain identities. These identities say that μ_1 is a differential, so $\mu_1 \circ \mu_1 = 0$ and S is in particular a chain complex. μ_2 is a product compatible with the differential, so that S becomes a dga. μ_3 is a chain homotopy (w.r.t. μ_1) between the two maps $S^{\otimes 3} \to S$ that can be constructed by composing μ_2 in two different ways. μ_4 is a homotopy between homotopies, and so on.

There is then Kadeishvili's theorem [Kadeišvili 1980], which says the following. Let A be a dga, and consider it as an A_∞-algebra by putting $\mu_n = 0$ for $n > 2$. Then one can define operations μ_n for $n > 2$ on $H^\bullet(A)$ making it an A_∞-algebra such that A and $H^\bullet(A)$ are quasi-isomorphic as A_∞-algebras. The μ_n are unique up to homotopy.

In the particular case when A is the singular cochain complex of a topological space X, we obtain certain multilinear operations[2] on $H^\bullet(X)$. These are exactly the much more classical *Massey products* [Massey 1958] in the cohomology of X.

Now let's say that X is a smooth projective variety over $\mathbf{F}_q$ as before, and let's say we have an étale analogue of the singular cochain complex.[3] Then we would get Massey products in the étale cohomology, multilinear operations μ_n of degree $2 - n$. But if we believe in the principle of weights we should expect all of them to vanish, which should tell us that $H^\bullet(X)$ is quasi-isomorphic as a dga to the singular cochains on X.

[2]Defined up to some ambiguity since μ_n were only unique up to homotopy.
[3]Such an analogue exists by [Deligne 1980].

Again let's consider the case of characteristic zero. A famous theorem of Deligne–Griffiths–Morgan–Sullivan [Deligne et al. 1975] says that if X is a compact Kähler manifold (in particular a smooth projective variety) then the dga's $\mathrm{Sing}^\bullet(X)$ and $H^\bullet(X)$ are quasi-isomorphic.[4] One says that the space X is *formal*.

Again the proof uses Hodge theory.

The upshot is that the principle of weights seems to make accurate and highly nontrivial predictions for cohomology over an *arbitrary* base field. (This is maybe not so surprising because you can spread over a finitely generated domain, count points, apply comparison isomorphism between different cohomology theories, etc.) Also, if we extrapolate from these two examples, then it seems that the weights in the cohomology of a complex algebraic variety should have something to do with *Hodge theory*. This turns out to be true.

2. Mixed Hodge theory

From now on all algebraic varieties are complex and all cohomology is rational. Deligne's mixed Hodge theory [Deligne 1971; Deligne 1974] tells us the following facts, that we can assume as a black box.

Fact. If X is a variety, then $H^k(X)$ has an increasing filtration

$$\cdots \subset W_i \subset W_{i+1} \subset$$

such that $W_{-1} = 0$ and $W_{2k} = H^k(X)$. We call $\mathfrak{gr}_r^W H^k(X)$ the *weight r component* of $H^k(X)$. If $H^k(X) = \mathfrak{gr}_r^W H^k(X)$ then we say that $H^k(X)$ is *pure of weight r*.

Remark 2.1. It is maybe surprising that we get a weight *filtration* and not a weight *grading* on the cohomology groups. If you only think in terms of Frob_q you might expect a direct sum decomposition into eigenspaces. From the point of view of Example 1.3 it is natural, though. Whenever you have a spectral sequence converging to something you get a canonically defined filtration on that object. The first step of the filtration is the image of the first row of the spectral sequence, the second step is the image of the first two rows, and so on. If you consider the Leray spectral sequence for $f\colon U \to X$ as in Example 1.3 then the first step of the filtration on $H^k(X)$ is pure of weight k, the second has weights k and $k+1$, the third has weights k, $k+1$ and $k+2$, and so on. So we get an increasing filtration with the rth component of the associated graded pure of weight $k+r$.

More facts. If $f\colon X \to Y$ is a morphism then the weight filtration is strictly compatible, i.e.

$$f^*(H^k(Y)) \cap W_r H^k(X) = f^*(W_r H^k(Y)).$$

So it is functorial in a strong sense. There are natural weight filtrations also on relative cohomology, compactly supported cohomology, Borel–Moore homology. The weight filtration is compatibly with Künneth theorem, that is, the natural tensor product filtration on

$$H^\bullet(X) \otimes H^\bullet(Y)$$

coincides with the weight filtration on $H^\bullet(X \times Y)$ under the Künneth isomorphism. In particular cup product respects weights. If $Z \subset Y \subset X$ then the maps in the long exact sequence

$$\cdots \to H^k(X,Y) \to H^k(X,Z) \to H^k(Y,Z) \to H^{k+1}(X,Y) \to \cdots$$

[4] With real coefficients. This was later improved to rational coefficients [Sullivan 1977].

are compatible with weights. If X is smooth then the Poincaré duality pairing

$$H^k(X) \otimes H_c^{2d-k}(X) \to H_c^{2d}(X)$$

is compatible with weights.

So far all we have said is that weights are compatible with most natural operations on cohomology groups. Here is a more substantial result that will be used over and over again in this minicourse.

Theorem 2.2. *Let X be an algebraic variety. We denote by H_k the Borel–Moore homology.*

(1) $H_k(X)$ has weights in the interval $[-k, 0]$ for any k.
(2) The cycle class map $A_k(X) \to H_{2k}(X)$ has its image in the lowest weight subspace $W_{-2k}H^{2k}(X)$.
(3) if $U \subset X$ is an open immersion, then $W_{-k}H_k(X) \to W_{-k}H_k(U)$ is surjective.
(4) if $X \to Y$ is a proper surjection, then $W_{-k}H_k(X) \to W_{-k}H_k(Y)$ is surjective.

Let us make some remarks and draw some consequences of this.

(1) For any variety X, $H_c^k(X)$ is the dual of $H_k(X)$ and will therefore have weights in the interval $[0, k]$. This is what we expect according to Remark 1.2.
(2) If X is compact, then $H^k(X) = H_c^k(X)$, so its weights are in the interval $[0, k]$ (a priori they are in $[0, 2k]$.)
(3) The fundamental class in $H_c^{2d}(X)$, where d is the dimension of X, is pure of weight $2d$, by the second claim of the theorem.
(4) If X is smooth then we have the Poincaré duality pairing $H^k(X) \otimes H_c^{2d-k}(X) \to H_c^{2d}(X)$. Since $H_c^{2d-k}(X)$ has weights *at most* $2d-k$ and $H_c^{2d}(X)$ has weights *exactly* $2d$, compatibility of Poincaré duality with weights implies that $H^k(X)$ has weights in the interval $[k, 2k]$.
(5) If X is smooth and compact then $H^k(X)$ is thus pure of weight k, which is as it should be according to our guiding principles.
(6) If X is smooth, then the cycle class map $A^k(X) \to H^{2k}(X)$ lands in $W_{2k}H^{2k}(X)$.
(7) The third claim of the theorem follows quite easily from the first since the cokernel of $H_k(X) \to H_k(U)$ is contained in $H_{k-1}(X \setminus U)$.
(8) In the second point of the theorem we do not only get something in the lowest weight subspace but we get a class of type $(-k, -k)$ in the (p, q)-decomposition. (I haven't defined what this means.) The Hodge conjecture asserts that the converse is true: $A_k(X)$ surjects onto the $(-k, -k)$ part of $H_{2k}(X)$.

The third and fourth claims may be thought of as analogues of the fact that $A_k(X) \to A_k(U)$ is surjective if $U \subset X$ is an open immersion, and $A_k(X) \to A_k(Y)$ is surjective for a proper surjection. The second claim shows a nice compatibility. Hence considering weights in the cohomology allows one to use certain convenient properties about Chow groups, while still working with a full-fledged cohomology theory.

The third and fourth claims have the following useful special cases.

(1) Let $U \subset X$ be a Zariski open in a smooth proper variety. Then

$$W_k H^k(U) = \mathrm{Im}(H^k(X) \to H^k(U)).$$

(Indeed the map $W_k H^k(X) \to W_k H^k(U)$ is surjective by the third claim in Theorem 2.2 and Poincaré duality. But X is proper so $W_k H^k(X) = H^k(X)$ by the weight bound for compact varieties.)

(2) Let $\widetilde{X} \to X$ be a resolution of singularities of a compact variety. Then

$$H^k(X)/W_{k-1}H^k(X) = \mathrm{Im}(H^k(X) \to H^k(\widetilde{X})).$$

(Since cohomology is dual to Borel–Moore homology on a compact variety, injectivity of $\mathfrak{gr}_k^W H^k(X) \to \mathfrak{gr}_k^W H^k(\widetilde{X})$ follows from the fourth claim in Theorem 2.2. But $\widetilde{X}$ is in addition smooth, so $W_{k-1}H^k(X)$ goes to zero by the weight bound for smooth varieties.)

Finally I state a result that will be very useful for us.

Lemma 2.3. *Let X be a smooth variety, Z a closed subvariety of (complex) codimension c, and $\widetilde{Z}$ a resulution of singularities. Then there is a short exact sequence*

$$W_k\left(H^{k-2c}(\widetilde{Z})(-c)\right) \to W_k H^k(X) \to W_k H^k(U) \to 0.$$

Proof. We have an exact sequence

$$H_c^k(Z) \leftarrow H_c^k(X) \leftarrow H_c^k(U).$$

Apply $\mathfrak{gr}_k^W$ to get also injectivity on the left,

$$\mathfrak{gr}_k^W H_c^k(Z) \leftarrow \mathfrak{gr}_k^W H_c^k(X) \leftarrow \mathfrak{gr}_k^W H_c^k(U) \leftarrow 0$$

(since $U \to X$ is an open immersion). Now $\mathfrak{gr}_k^W H_c^k(\widetilde{Z}) \leftarrow \mathfrak{gr}_k^W H_c^k(Z)$ is injective, so the composite

$$\mathfrak{gr}_k^W H_c^k(\widetilde{Z}) \leftarrow \mathfrak{gr}_k^W H_c^k(X) \leftarrow \mathfrak{gr}_k^W H_c^k(U) \leftarrow 0$$

is also exact. But now all spaces are smooth and we can apply Poincaré duality to get the lemma. □

3. The twice punctured elliptic curve

Let us consider a single concrete example of a mixed Hodge structure before moving on to moduli of curves. Let $\overline{C}$ be a smooth compact genus one curve, Z two distinct points, and $C = \overline{C} \setminus Z$. We have a long exact sequence in cohomology (recall $H_c^\bullet(C) = H^\bullet(\overline{C}, Z)$)

$$\cdots \to H_c^k(C) \to H^k(\overline{C}) \to H^k(Z) \to H^{k+1}(C) \to \cdots$$

We find that $H_c^2(C) \cong H_c^2(\overline{C})$ and that H_c^1 sits in a short exact sequence

$$0 \to H^0(Z)/H^0(\overline{C}) \to H_c^1(C) \to H^1(\overline{C}) \to 0.$$

We apply Poincaré duality and find the short exact sequence

$$0 \to H^1(\overline{C}) \to H^1(C) \to \mathrm{Ker}\left(H^0(Z)(-1) \to H^2(\overline{C})\right) \to 0.$$

Here the first term is pure of weight 1, the last term is pure of weight 2, and the short exact sequence exactly exhibits the 2-step weight filtration on $H^1(C)$.

We remark that the pure part of the cohomology is precisely the part that is in the restriction of the compactification, as was said in the previous section.

We can represent cohomology classes on C as Poincaré duals of $\mathcal{C}^\infty$ oriented submanifolds, so classes in $H^1(C)$ can be drawn as curves on the punctured torus. Then $H^1(\overline{C})$ is spanned by the two curves around the meridians; these curves define the pure subspace of $H^1(C)$.

The other type of generator of $H^1(C)$ is a curve connecting the two punctures. But this is not a subspace: there are many different paths between the two punctures, and the difference of two such paths will in general be given by a collection of curves around the meridians of the torus, so a class in the pure subspace. So elements of $\mathfrak{gr}_2^W H^1(C)$ can not be interpreted as curves, but they *can* be faithfully represented as the difference of the endpoints of the curves. Such a difference is precisely an element of $\operatorname{Ker}\big(H^0(Z)(-1) \to H^2(\overline{C})\big)$. We will see later again the idea that elements of $\mathfrak{gr}_{k+1}^W H^k$ can be represented as *relations between classes supported on the boundary.*

Remark 3.1. I can't resist mentioning a neat characterization of when the weight filtration on $H^1(C)$ splits. Give $\overline{C}$ the structure of an elliptic curve by taking one of the punctures to be the origin. Then $H^1(C)$ is a direct sum of two pure Hodge structure if and only if the other puncture is a torsion point on $\overline{C}$.

4. Cohomology of $\overline{\mathcal{M}}_{0,n}$

We now use mixed Hodge theory to re-prove known results on $H^\bullet(\overline{\mathcal{M}}_{0,n})$. In [Keel 1992] it is shown that it is generated as an algebra by boundary divisors, and the ideal of relations is spanned by the pullbacks of the WDVV relation from $\overline{\mathcal{M}}_{0,4}$. Here we prove a stronger statement: it is spanned *additively* by boundary strata, and the space of relations is *additively* generated by pullbacks of the WDVV relation. (The second half is obviously a stronger statement. The first half is actually equivalent, since it is easy to see that every boundary stratum in $\overline{\mathcal{M}}_{0,n}$ is the transverse intersection of the boundary divisors containing it.) This stronger statement was first proven in [Kontsevich and Manin 1994]. Our argument is similar to the one in [Getzler 1995], but we give a direct argument for why all relations are pulled back from $\overline{\mathcal{M}}_{0,4}$ whereas he deduces it from the knowledge of a quadratic presentation of the 'gravity operad'.

We begin by proving the first half, that the strata span the cohomology.

Theorem 4.1. *$H^\bullet(\overline{\mathcal{M}}_{0,n})$ is spanned by cycle classes of strata.*

Proof. Observe that we have an open immersion

$$\mathcal{M}_{0,n} \cong \{(x_1, \dots, x_{n-3}) \mid x_i \neq x_j, x_i \neq 0, x_i \neq 1\} \subset \mathbf{C}^{n-3}.$$

By what we said earlier, $W_k H^k(\mathbf{C}^{n-3}) \to W_k H^k(\mathcal{M}_{0,n})$ is surjective. This shows that the fundamental class in H^0 is the only pure part of the cohomology of $\mathcal{M}_{0,n}$.

Observe that $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n} = \bigcup D_i$ with each $D_i \cong \overline{\mathcal{M}}_{0,j+1} \times \overline{\mathcal{M}}_{0,n-j+1}$. The map

$$\coprod_i D_i \to \bigcup_i D_i$$

is a resolution of singularities. By Lemma 2.3 we have

$$\bigoplus_i H^{k-2}(\overline{\mathcal{M}}_{0,j+1} \times \overline{\mathcal{M}}_{0,n-j+1})(-1) \to H^k(\overline{\mathcal{M}}_{0,n}) \to W_k H^k(\mathcal{M}_{0,n})$$

and $W_k H^k(\mathcal{M}_{0,n}) = 0$ for $k > 0$, so $H^k(\overline{\mathcal{M}}_{0,n})$ is spanned by classes that are pushed forward from the boundary. But by induction on n these are going to be strata classes. □

Characterizing the relations is a bit trickier. We want to find the kernel of the pushforward map. We will need a slightly more detailed study of the mixed Hodge structure of $\mathcal{M}_{0,n}$. We observe that $\mathcal{M}_{0,n}$ is a complement of an arrangement of complex hyperplanes, according to the preceding proof, and the cohomology ring of such a complement has a classical description.

Theorem 4.2. *[Arnol'd 1969; Brieskorn 1973] Let U be the complement of the arrangement of hyperplanes $L_i = 0$ in $\mathbf{C}^N$. Then $H^\bullet(U)$ is generated as an algebra by $H^1(U)$, which is spanned by the classes*

$$\omega_i = \frac{1}{2\pi i} d\log(L_i).$$

It is easy to see from [Deligne 1971] that the classes ω_i are pure of weight 2 and in fact of type $(1,1)$. Thus each H^k is pure of weight $2k$ and of type (k,k). For a different proof of this fact see [Kim 1994; Shapiro 1993; Lehrer 1992] (they all independently found the same proof!)

Theorem 4.3. *All additive relations between strata are pullbacks of the WDVV relation on $\overline{\mathcal{M}}_{0,4}$.*

Proof. We will consider the spectral sequence of a filtration. Whenever you have a variety X with a sequence $\ldots \subset X_p \subset X_{p+1} \subset \ldots$ of closed subvarieties there is a spectral sequence in Borel–Moore homology

$$E^1_{pq} = H_{p+q}(X_p \setminus X_{p-1}) \implies H_{p+q}(X).$$

We take X_p to be the union of all strata of dimension $\leq p$ in the stratification of $\overline{\mathcal{M}}_{0,n}$ by topological type. Then we have

$$X_p \setminus X_{p-1} = \coprod_\Gamma \mathcal{M}(\Gamma),$$

where Γ is the dual graph of a p-dimensional stratum, and $\mathcal{M}(\Gamma)$ is the stratum:

$$\mathcal{M}(\Gamma) = \prod_{v \in \mathrm{Vert}(\Gamma)} \mathcal{M}_{0,n(v)}.$$

Observe that E^∞_{pq} is a subquotient of $H_{p+q}(\overline{\mathcal{M}}_{0,n})$, so it is pure of weight $-p-q$. So only the pure part of E^1_{pq} can survive to E_∞. But the only pure part of the homology of $\mathcal{M}_{0,n}$ is the fundamental class, and by our expression for $\mathcal{M}(\Gamma)$ the same holds for each stratum. We have thus given an alternative proof that the (co)homology of $\overline{\mathcal{M}}_{0,n}$ is spanned by strata.

But we can be more precise. Since $H^k(\mathcal{M}_{0,n})$ is pure of weight $2k$, the same is true for each $\mathcal{M}(\Gamma)$. Then $H_k(\mathcal{M}(\Gamma))$ is pure of weight $2d-2k$, where $d = \dim \mathcal{M}(\Gamma)$. Since each $X_p \setminus X_{p-1}$ is p-dimensional, we see that E^1_{pq} has pure weight $-2q$. We illustrate in a figure. Since each $\mathcal{M}(\Gamma)$ is affine it is actually a first quadrant spectral sequence.

It is thus clear that the Leray spectral sequence degenerates after the E_1-differential because of weights.

The fundamental classes of the strata are placed along the diagonal $p = q$ of the spectral sequence. There are no nonzero entries above the diagonal, so we obtain edge maps from the

2	·	·	−4	−4
1	·	−2	−2	−2
0	0	0	0	0
	0	1	2	3

Figure 2. Weights at E^1_{pq} of filtration spectral sequence for $\overline{\mathcal{M}}_{0,n}$.

diagonal to the homology of $\overline{\mathcal{M}}_{0,n}$; these are just the obvious map sending the fundamental class of a stratum to the class of that stratum in $H_\bullet(\overline{\mathcal{M}}_{0,n})$. Finding relations between these generators now amounts to determining the image of the E_1-differential $E^1_{p+1,p} \to E^1_{p,p}$.

Now $E^1_{p+1,p}$ is a sum of $H_{2p+1}(\mathcal{M}(\Gamma))$ over all $p+1$-dimensional strata, which is dual to $H^{2p+1}_c(\mathcal{M}(\Gamma))$, which is Poincaré dual to $H^1(\mathcal{M}(\Gamma))$, which therefore surjects onto the space of relations between the generators. We remark that we saw in the example with the twice punctured elliptic curve that classes in $\mathfrak{gr}^W_{k+1} H^k$ could be interpreted as relations between classes in the boundary, and that $H^1(\mathcal{M}(\Gamma))$ is pure of weight 2, so things make sense.

We want to prove that all relations are pulled back from $\overline{\mathcal{M}}_{0,4}$: we now see that we should prove that all classes in $H^1(\mathcal{M}_{0,n})$ are pulled back from $\mathcal{M}_{0,4}$. (We made the switch to cohomology to get compatibility with pullback maps.)

But this we can see from Arnol'd's theorem. If we take a class in $H^1(\mathcal{M}_{0,n})$ corresponding to the hyperplane $x_i = 0$ (resp. $x_i = 1$), then it is the pullback from $\mathcal{M}_{0,4}$ of the class of the hyperplane $x_1 = 0$ (resp. $x_1 = 1$). For the hyperplanes of the form $x_i = x_j$ we use the $\mathbb{S}_n$-action to put one of x_i or x_j at 0 or 1. This concludes the proof. □

5. Koszul duality for operads

I can't resist mentioning the following. We have seen in Example 1.3 a spectral sequence for an open immersion. If we apply it to the particular open immersion $\mathcal{M}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ then we are in the situation of the theorem and we get a spectral sequence

$$\bigoplus_{i_1 < \cdots < i_q} H^p(D_{i_1} \cap \cdots \cap D_{i_q}) \otimes \mathbf{Q}(-q) \implies H^{p+q}(\overline{\mathcal{M}}_{0,n}),$$

and it is easy to see that each intersection $D_{i_1} \cap \cdots \cap D_{i_q}$ is the closure of one of the strata in the stratification by topological type. Thus it is a product of the form $\prod_k \overline{\mathcal{M}}_{0,n_k}$.

But we also saw the spectral sequence associated to the filtration of $\overline{\mathcal{M}}_{0,n}$, in the last proof of the previous section. Now all terms were given by the homology of the *open strata*, so we had products of the form $\prod_k \mathcal{M}_{0,n_k}$, and it converged to the homology of $\overline{\mathcal{M}}_{0,n}$. Moreover, the combinatorial description of both spectral sequences are the same: the terms are described as identical sums over all dual graphs.

And even more: if we switch in the first spectral sequence to Borel–Moore homology, then we get rid of the Tate twist by $\mathbf{Q}(-q)$, and the differentials are given by the pushforward along the gluing maps,

$$H_\bullet(\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1}) \to H_\bullet(\overline{\mathcal{M}}_{0,i+j}).$$

In the second one the differentials are defined in terms of the connecting homomorphism

$$H_\bullet(\mathcal{M}_{0,i+1} \times \mathcal{M}_{0,j+1}) \leftarrow H_{\bullet-1}(\mathcal{M}_{0,i+j})$$

obtained by including $\mathcal{M}_{0,i+1} \times \mathcal{M}_{0,j+1}$ as a boundary stratum in the compactification of $\mathcal{M}_{0,i+j}$. So in both cases the spectral sequence is a sum over dual graphs, and the differential is a sum over edges in the dual graph. And both spectral sequences degenerate after the first differential.

What's going on here? Why is there a symmetry? In what sense are $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ dual to each other?

The simple answer is that $H_\bullet(\overline{\mathcal{M}}_{0,n})$ is an *operad*, and $H_{\bullet+1}(\mathcal{M}_{0,n})$ is a *co-operad*. These are often called the Hypercommutative operad and the Gravity co-operad, respectively. For any operad P there is the *bar construction* BP which is a co-operad, and for a co-operad Q the *cobar construction* ΩQ produces an operad. The co-operad BP is differential graded and is defined as a sum over trees with differential given by operadic composition in P and contracting along edges of trees. For $P = H_\bullet(\overline{\mathcal{M}}_{0,n})$ the bar construction is given exactly by the first of the above spectral sequences. The cobar construction is a similar sum over trees, with differential defined by co-operadic co-composition in Q and splitting of trees along edges. For $Q = H_{\bullet+1}(\mathcal{M}_{0,n})$ the cobar construction is given by the second of the above spectral sequences.

In general there are quasi-isomorphisms $\Omega BP \cong P$ and $B\Omega Q \cong Q$.

Now it turns out that the bar construction on the hypercommutative operad is quasi-isomorphic to its homology, which is the gravity co-operad. Similarly the cobar construction on the gravity co-operad is quasi-isomorphic to its homology, the hypercommutative operad. This is related to the fact that both these (co-)operads have natural chain level models which are *formal* in genus zero. Thus this really establishes a duality between the gravity and hypercommutative (co-)operads.

But we can go further. Let's dualize the co-operad $H_{\bullet+1}(\mathcal{M}_{0,n})$ to the operad $H_c^{\bullet+1}(\mathcal{M}_{0,n})$, which we call the gravity operad. The fact that $H_\bullet(\overline{\mathcal{M}}_{0,n})$ is generated by strata and all relations come from WDVV can be interpreted operadically as saying that the hypercommutative operad is *quadratic*. The same holds for the gravity operad. If we look at the spectral sequence associated to the filtration by topological type and consider the weights, we see that the rows in the spectral sequences are exact everywhere except the ends: the homology is concentrated along the diagonal (see Figure 2). In the language of operads this says exactly that the gravity operad is *Koszul*. Then the same holds for hypercommutative operad. (The result for hypercommutative operad can also be deduced in the same way, by looking at weights to see what part of the spectral sequence can survive to E_∞.) Hence Gravity and Hypercommutative operads are Koszul dual to each other.

I should also mention that in the above it is slightly nicer to think of both hypercommutative and gravity operads as cyclic operads, rather than ordinary operads.

The facts explained in this section can be found in [Getzler 1995]. The generalization of bar-cobar duality to higher genus is given by the Feynmann transform of [Getzler and Kapranov 1998], but the naive generalization of duality between hypercommutative and gravity fails: the cobar construction on the higher genus gravity operad is not quasi-isomorphic to the hypercommutative operad. This is related to the fact that we really should be working with the chain level version of the higher genus gravity operad, and that this operad is not formal. This also shows that there is no higher genus Koszul duality between the two operads.

A more down-to-earth explanation of this duality is in [Kimura, Stasheff, and Voronov 1996]. Koszul duality of operads is explained in Ginzburg and Kapranov's original paper or in [Loday and Vallette 2012].

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