COHOMOLOGY OF MODULI OF STABLE POINTED CURVES OF LOW GENUS. II

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In this lecture I first discuss briefly local systems and variations of Hodge structure. A good reference for this part is [Peters and Steenbrink 2008].

Then I move on to moduli of pointed genus one curves. I will prove the first half of the claims made by [Getzler 1997], proven in [Petersen 2012], that all even cohomology of $\overline{\mathcal{M}}_{1,n}$ consists of classes of strata.

However instead of also proving the second half I move on to genus two and apply similar methods as in the genus one case. Here we will follow the paper [Petersen and Tommasi 2012]. I almost prove in this lecture that the tautological ring of $\overline{\mathcal{M}}_{2,n}$ is not Gorenstein in general — all that will remain at the end is to prove that certain classes that are pushed forward from the boundary are zero. I will deal with these next time.

1. Variation of Hodge structure

Last time we had the guiding principle: if X is smooth and proper, then $H^i(X)$ is a 'pure object of weight i'. Today we will need the relative version: if $f : X \to Y$ is a smooth proper morphism, then $\mathrm{R}^i f_* \mathbf{Q}$ is a 'pure object of weight i'.

This raises the question: what kind of object is $\mathrm{R}^i f_* \mathbf{Q}$?

This is a sheaf on Y, and by proper base change we have $(\mathrm{R}^i f_* \mathbf{Q})_y = H^i(X_y, \mathbf{Q})$ for any point y of Y. Ehresmann's theorem says that all fibers X_y are diffeomorphic and f is a $\mathcal{C}^\infty$ fiber bundle. Finally there is the 'Gauss–Manin connection' which says that $\mathrm{R}^i f_* \mathbf{Q}$ is a local system.

A local system of rank n on a space X is a sheaf F such that there is an open cover $\{U_\alpha\}$ with $F|_{U_\alpha} \cong \mathbf{Q}^n$, that is, it really is locally isomorphic to a constant sheaf. An equivalent definition is that a local system is a functor

$$\rho : \Pi_1(X) \to \mathrm{Vect}_{\mathbf{Q}}$$

that assigns to a point a $\mathbf{Q}$-vector space, and to a homotopy class of paths between points an isomorphism between the vector spaces. Given F as above, define ρ by assigning to a point x the stalk F_x. For a path between points x and y, cover the path with open sets which trivialize F. On each open set the sheaf is constant which means that the stalks of any two points in the same connected component are canonically isomorphic. Applying this one open set at a time gives an isomorphism between F_x and F_y. Irrelevant exercise: understand why the inverse of this construction is given by the definition

$$F(U) = \lim \left(\Pi_1(U) \to \mathrm{Vect}_{\mathbf{Q}}\right).$$

Both definitions of local system arise naturally for us. The sheaf $\mathrm{R}^i f_* \mathbf{Q}$ given by the derived pushforward of the constant sheaf is locally constant in the first sense. But it will also be useful to think about it as a representation of the fundamental group of Y (if we pick a basepoint $y \in Y$ and assume Y connected). Then we have the monodromy action of $\pi_1(Y, y)$ on the vector space $H^i(X_y, \mathbf{Q})$, which contains the same information as $\mathrm{R}^i f_* \mathbf{Q}$.

We can also talk about the cohomology of a local system. When we think of it as a locally constant sheaf this is ordinary sheaf cohomology. There is also a definition of the cohomology of local systems in terms of representations of the fundamental groupoid. In the special case when Y is connected and aspherical (its universal cover is simply connected) we are in fact considering group cohomology:

$$H^k(Y, \mathrm{R}^i f_* \mathbf{Q}) \cong H^k(\pi_1(Y, y), H^i(X_y, \mathbf{Q})).$$

But in fact the sheaves $\mathrm{R}^i f_* \mathbf{Q}$ have more structure. They are the underlying local systems of a *(polarizable) variation of Hodge structure* (PVHS) of weight i. This means roughly that each stalk $H^i(X_y, \mathbf{Q})$ is a pure Hodge structure of weight i and that the Hodge structure varies in a controlled manner, I will not give the correct definition.

Theorem 1.1. *Let Y be a smooth variety and let $\mathbb{V}$ be a PVHS of weight i on Y. Then $H^k(Y, \mathbb{V})$ is canonically equipped with a mixed Hodge structure of weights $\geq k + i$. If Y is in addition proper then $H^k(Y, \mathbb{V})$ is pure of weight $k + i$. If we have a smooth proper map $f : X \to Y$ then the Leray spectral sequence*

$$E_2^{pq} = H^p(Y, \mathrm{R}^q f_* \mathbf{Q}) \implies H^{p+q}(X)$$

is compatible with mixed Hodge structure.

This is not an easy theorem. The case when Y is a punctured curve is in [Zucker 1979], and the general case requires (as far as I know) the far more general theory of mixed Hodge modules due to [Saito 1990]. In fact the reason I included smoothness assumption on Y is so that I can think of a PVHS as a pure Hodge module and apply Saito's theory.

I should also mention the following classical result.

Theorem 1.2. *[Deligne 1968] If $f : X \to Y$ is a smooth and projective morphism then the Leray spectral sequence degenerates.*

In particular the restriction map $H^k(X) \to H^k(X_y)$ surjects onto the monodromy invariant subspace of $H^k(X_y)$, since it is one of the edge maps of the spectral sequence (seeing as $H^k(X_y) = H^0(Y, \mathrm{R}^k f_* \mathbf{Q})$).

Projectivity can be weakened to properness in Deligne's result.

2. Local systems on the moduli of elliptic curves

Let $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$ be the universal elliptic curve. We define $\mathbb{V} = \mathrm{R}^1 \pi_* \mathbf{Q}$ and $\mathbb{V}_k = \mathrm{Sym}^k \mathbb{V}$. Each $\mathbb{V}_k$ is a PVHS of weight k on $\mathcal{M}_{1,1}$.

Lemma 2.1. *If k is odd, then $H^\bullet(\mathcal{M}_{1,1}, \mathbb{V}_k) = 0$.*

Proof. Consider the projection to the coarse space $\mu\colon \mathcal{M}_{1,1} \to \mathsf{M}_{1,1}$ and compute cohomology using Leray spectral sequence. The fiber of μ over a point $[E]$ is a single point, but this point has automorphism group $\mathrm{Aut}(E)$. So the cohomology of the fiber is group cohomology and

$$(\mathrm{R}^q\mu_*\mathbb{V}_k)_{[E]} = H^q(B\,\mathrm{Aut}(E), \mathrm{Sym}^k H^1(E)) = \begin{cases} (\mathrm{Sym}^k H^1(E))^{\mathrm{Aut}(E)} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

So we have no higher cohomology because the coefficients are $\mathbf{Q}$-vector spaces, and in degree zero we take the invariants. But inversion in the group structure of E acts as multiplication by $(-1)^k$ on $\mathrm{Sym}^k H^1(E)$, so there are no invariants if k is odd. □

An equivalent and maybe more standard way of formulating the above proof is in terms of the Hochschild–Serre spectral sequence for

$$1 \to \pm 1 \to \mathrm{SL}(2, \mathbf{Z}) \to \mathrm{PSL}(2, \mathbf{Z}) \to 1.$$

Here $\mathrm{SL}(2, \mathbf{Z}) = \pi_1(\mathcal{M}_{1,1})$ (in the stack sense).

Rahul mildly objected to the above proof as being unnecessarily complicated ("you can just use Čech cohomology!").

Our reason for introducing the sheaves $\mathbb{V}_k$ is that all irreducible representations of SL_2 are symmetric powers of the standard 2-dimensional one. So all tensor powers of $\mathbb{V}$ can be expressed in terms of the $\mathbb{V}_k$, e.g.

$$\begin{aligned} \mathbb{V}^{\otimes 2} &= \mathbb{V}_2 \oplus \mathbb{V}_0(-1) \\ \mathbb{V}^{\otimes 3} &= \mathbb{V}_3 \oplus 2\mathbb{V}_1(-1) \\ \mathbb{V}^{\otimes 4} &= \mathbb{V}_4 \oplus 3\mathbb{V}_2(-1) \oplus \mathbb{V}_0(-2) \end{aligned}$$

By $\mathbb{V}_k(-n)$ I mean an n-fold Tate twist of the PVHS $\mathbb{V}_k$. This is the same underlying local system but the mixed Hodge structure on its cohomology has been shifted by a Tate twist. $\mathbb{V}_k(-n)$ is a PVHS of weight $k + 2n$.

3. Pointed genus one curves

Now we consider $f\colon \mathcal{M}^{\mathsf{rt}}_{1,n} \to \mathcal{M}_{1,1}$. The superscript '$\mathsf{rt}$' means curves with *rational tails*. An n-pointed stable genus g curve has rational tails if it has a component of geometric genus g. This implies that all other components have genus zero and that the dual graph is a tree, so that we have attached 'tails' of rational curves to a smooth genus g curve.

Observe that f is a smooth projective morphism. Let F_n denote the fiber of f. Each $\mathrm{R}^i f_*\mathbf{Q}$ is a PVHS of weight i. Decomposing it into irreducible local systems is the same as decomposing $H^i(F_n)$ as a representation of SL_2. In this way one writes $\mathrm{R}^i f_*\mathbf{Q}$ as a direct sum $\bigoplus_j \mathbb{V}_{k_j}(-n_j)$, with $k_j + 2n_j = i$ for each j. (We are implicitly using that the category of PVHS of given weight is semisimple.)

In particular if i is odd/even, then $\mathrm{R}^i f_*\mathbf{Q}$ contains only the local systems $\mathbb{V}_k$ for k odd/even. So $\mathrm{R}^i f_*\mathbf{Q}$ has no cohomology for odd i, by Lemma 2.1. Moreover, since $\mathcal{M}_{1,1}$ is a non-compact curve, it can only have cohomology in degree 0 and 1 for any of the V_k. So the Leray spectral sequence for f has not so many nonzero entries, see Figure 1. It therefore collapses and we have proven:

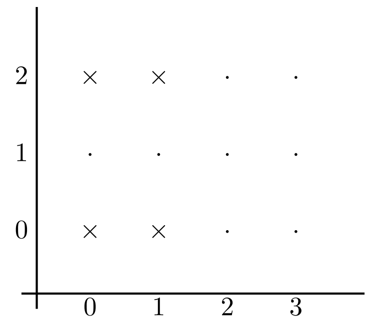

Figure 1. Leray spectral sequence for $\mathcal{M}_{1,n}^{\mathsf{rt}} \to \mathcal{M}_{1,1}$

Proposition 3.1. *For even k we have*

$$H^k(\mathcal{M}_{1,n}^{\mathsf{rt}}) = H^0(\mathcal{M}_{1,1}, \mathrm{R}^k f_* \mathbf{Q}) = H^k(F_n)^{\mathrm{SL}_2}$$

and for odd k we have

$$H^k(\mathcal{M}_{1,n}^{\mathsf{rt}}) = H^1(\mathcal{M}_{1,1}, \mathrm{R}^{k-1} f_* \mathbf{Q}).$$

To understand the even cohomology of $\mathcal{M}_{1,n}^{\mathsf{rt}}$ we should understand the cohomology of F_n as a representation of SL_2.

Observe that f can be factored as $\mathcal{M}_{1,n}^{\mathsf{rt}} \to \mathcal{E}^{n-1} \to \mathcal{M}_{1,1}$, where $\mathcal{E}^{n-1}$ denotes a fibered power of the universal curve. The first map contracts all rational components to get a pointed genus one curve where points may coincide. The second forgets all but one marking. In the same way we can think of F_n (the fiber over $[E]$) as a blow-up of E^{n-1}. To get the $\mathbb{S}_n$-action more canonically it is better to think of

$$E^{n-1} = E^n/E,$$

the quotient of E^n by E acting diagonally by translation. Then we also have

$$F_n = \mathsf{FM}(E, n)/E,$$

where $\mathsf{FM}(E, n)$ is the Fulton–MacPherson compactification [Fulton and MacPherson 1994] of n points on E.

In this way we can solve the problem in two steps. First we find the SL_2-invariants in $H^\bullet(E^{n-1}) = H^\bullet(E)^{\otimes n-1}$. Observe that H^0 and H^2 is invariant on each factor, so we should determine the invariants in $H^1(E)^{\otimes k}$. This is very classical invariant theory: for $k = 2$ the invariant subspace is exactly $\wedge^2 H^1(E)$, and for $k > 2$ even the invariant subspace is spanned by the subspace

$$\left(\wedge^2 H^1(E)\right)^{\otimes k/2} \subset H^1(E)^{\otimes k}$$

and its translates under the $\mathbb{S}_k$-action. Note that $\wedge^2 H^1(E)$ is spanned by the class of the diagonal in $H^2(E \times E)$ restricted to $H^1(E)^{\otimes 2}$.

What this tells us is that the subalgebra $H^\bullet(E^{n-1})^{\mathrm{SL}_2}$ is generated by the classes of diagonals and the generators of H^2, which can be taken to be loci of the form $x_i = e$, $i = 1, \ldots, n-1$, where e is the origin of the elliptic curve. If we consider more invariantly $H^\bullet(E^n/E)^{\mathrm{SL}_2}$ then the generators are just the diagonals.

Now $H^\bullet(\mathsf{FM}(E,n))$ is generated as an algebra over $H^\bullet(E^n)$ by the exceptional divisors. In other words the generators are fundamental classes of strata in the stratification by topological type of $\mathsf{FM}(E,n)$. In our case we blow up only in loci whose cohomology classes are SL_2-invariant. Thus the SL_2-action on $H^\bullet(\mathsf{FM}(E,n))$ is very simple: we extend the action on $H^\bullet(E^n)$ by giving all generators the trivial action. Hence the SL_2-invariants are generated as an algebra over the SL_2-invariants in $H^\bullet(E^n)$. The same holds for $H^\bullet(F_n)$ as an algebra over $H^\bullet(E^{n-1})$. All in all we have proven the following.

Proposition 3.2. *The invariant part $H^\bullet(F_n)^{\mathrm{SL}_2}$ is spanned by classes of strata.*

Corollary 3.3. *Let k be even. Then $H^k(\mathcal{M}^{\mathsf{rt}}_{1,n})$ is spanned by classes of strata.*

A stratum in $\mathcal{M}^{\mathsf{rt}}_{1,n}$ restricts to a stratum of F_n. Conversely, if we take a codimension i stratum in F_n and allow the moduli of the elliptic curve to vary we get a codimension i stratum of $\mathcal{M}^{\mathsf{rt}}_{1,n}$. These constructions are mutually inverse.

Now by a Lemma proven in Lecture 1 we have the exact sequence

$$H^{k-2}(\overline{\mathcal{M}}_{0,n+2})(-1) \to H^k(\overline{\mathcal{M}}_{1,n}) \to H^k(\mathcal{M}^{\mathsf{rt}}_{1,n}).$$

All classes on both left and right hand side are strata. We obtain the following result:

Proposition 3.4. *All even cohomology of $\overline{\mathcal{M}}_{1,n}$ is spanned by strata classes.*

This is the first half of Getzler's claims on the cohomology of $\overline{\mathcal{M}}_{1,n}$ that were announced in [Getzler 1997]. The first published proof was given in [Petersen 2012]. The second half concerns relations between generators and is trickier to prove. Maybe we will prove it later in the course.

4. The Eichler–Shimura theory and the odd cohomology

We can also say something about the odd cohomology of $\overline{\mathcal{M}}_{1,n}$. To do this we consider again

$$H^{k-2}(\overline{\mathcal{M}}_{0,n+2})(-1) \to H^k(\overline{\mathcal{M}}_{1,n}) \to H^k(\mathcal{M}^{\mathsf{rt}}_{1,n}).$$

Since $\overline{\mathcal{M}}_{0,n+2}$ has no cohomology and since we have a surjection on the right when we consider pure cohomology, we get that

$$H^k(\overline{\mathcal{M}}_{1,n}) \xrightarrow{\sim} W_k H^k(\mathcal{M}^{\mathsf{rt}}_{1,n}) = W_k H^1(\mathcal{M}_{1,1}, \mathrm{R}^{k-1} f_* \mathbf{Q})$$

for odd k.

The first cohomology of the local systems $\mathbb{V}_k$ is determined by the Hodge-theoretic interpretation of the Eichler–Shimura isomorphism. It is not so easy to find a reference that explains this in a way suitable for algebraic geometers interested in cohomology of moduli spaces (but who are not interested in number theory). In any case the theorem (in the case of $\mathcal{M}_{1,1}$) is that the weight filtration on $H^1(\mathcal{M}_{1,1}, \mathbb{V}_k)$ splits, so the cohomology is a sum of two pure Hodge structures. These are

$$\mathfrak{gr}^W_{k+1} H^1(\mathcal{M}_{1,1}, \mathbb{V}_k) = \mathsf{S}[k+2]$$

where I denote by $\mathsf{S}[n]$ the Hodge structure attached to cusp forms of weight n for $\mathrm{SL}_2(\mathbf{Z})$. For each such cusp eigenform there is a 2-dimensional subspace of $\mathsf{S}[n]$ of type $(n-1,0)$ and $(0,n-1)$.

We also have that $\mathfrak{gr}^W_{2k+2} H^1(\mathcal{M}_{1,1}, \mathbb{V}_k)$ is similarly 'attached' to Eisenstein series of weight $k+2$ for $\mathrm{SL}_2(\mathbf{Z})$. There is a 1-dimensional space of Eisenstein series of weight $k+2$ for any even $k \geq 2$, and this Hodge structure is just $\mathbf{Q}(-k-1)$.

Hence all odd cohomology of $\overline{\mathcal{M}}_{1,n}$ is given in this way in terms of cusp forms. The first nonzero cusp form for $\mathrm{SL}_2(\mathbf{Z})$ is the discriminant form Δ of weight 12, which appears in the cohomology of $\mathbb{V}_{10}$. This local system appears in the decomposition of the cohomology of F_n for the first time when $n = 11$, so we get cohomology in $\overline{\mathcal{M}}_{1,11}$ of type $(11, 0)$ and $(0, 11)$. In particular the variety $\overline{\mathcal{M}}_{1,11}$ is irrational. (Belorousski proved that it's rational for $n \leq 10$.)

Hence we have in a sense completely classified the cohomology of $\overline{\mathcal{M}}_{1,n}$. The even cohomology is strata and the odd cohomology is cusp forms.

5. Genus two

Let's try to apply the same methods to $\mathcal{M}^{\mathsf{rt}}_{2,n}$ and to $\overline{\mathcal{M}}_{2,n}$. Before we found that all even cohomology was tautological. But it is known from [Graber and Pandharipande 2003] that there is a non-tautological algebraic cycle on $\mathcal{M}_{2,20}$ giving a class in $H^{22}(\mathcal{M}_{2,20})$. Since it's the class of an algebraic cycle it extends to $H^{22}(\overline{\mathcal{M}}_{2,20})$. So it will not even be true that all even cohomology is tautological, or that all cohomology of (p,p)-type is tautological. Let's see how to understand this in terms of local systems.

Let's start off the same way as we did in genus one. We consider $f \colon \mathcal{M}^{\mathsf{rt}}_{2,n} \to \mathcal{M}_2$, a smooth projective morphism. Again $\mathrm{R}^q f_* \mathbf{Q}$ is a PVHS pure of weight q, and Lemma 2.1 holds also in this case: the odd local systems have no cohomology. (Use the same proof but the hyperelliptic involution instead of the elliptic involution: each genus two curve is hyperelliptic.)

The fibers of f have the form $\mathsf{FM}(C, n)$ where C is a smooth genus two curve, and FM as before denotes the Fulton–MacPherson compactification. Thus we should decompose $H^q(\mathsf{FM}(C, n))$ as an irreducible representation of $\pi_1(\mathcal{M}_2)$. But $\pi_1(\mathcal{M}_2)$ acts through the homomorphism $\pi_1(\mathcal{M}_2) \to \mathrm{Sp}_4(\mathbf{Z})$, where $\mathrm{Sp}_4(\mathbf{Z})$ acts on $H^1(C)$ (which is four-dimensional, and has a symplectic pairing via the cup product.) So we only need to decompose $H^q(\mathsf{FM}(C, n))$ as Sp_4-representation.

The irreducible representations of Sp_4 are indexed by their highest weights, which are integers $l \geq m \geq 0$. We write the corresponding local system as $\mathbb{V}_{l,m}$.

One can think about this is in terms of the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$. We have $\pi_1(\mathcal{A}_g) = \mathrm{Sp}_{2g}(\mathbf{Z})$, and for any representation of Sp_{2g} given by a highest weight $\lambda = l_1 \geq l_2 \geq \ldots \geq l_g \geq 0$ there is a local system $\mathbb{V}_\lambda$ on $\mathcal{A}_g$ (since a local system is just a representation of the fundamental group). It can naturally be considered a PVHS of weight $|\lambda| = l_1 + \ldots + l_g$, by identifying it inside of $\mathbb{V}^{\otimes |\lambda|}$ where $\mathbb{V}$ is the local system on $\mathcal{A}_g$ whose fiber over $[A]$ is $H^1(A, \mathbf{Q})$. Pulling back these local systems along the Torelli map gives also local systems $\mathbb{V}_\lambda$ on $\mathcal{M}_g$, which coincide with the $\mathbb{V}_{l,m}$ above since a curve and its Jacobian have canonically isomorphic H^1's.

Now we consider the Leray spectral sequence for f. We use that $\mathcal{M}_2$ is affine: indeed, up to $\mathbf{Z}/2$-action it can be identified with the quotient $\mathcal{M}_{0,6}/\mathbb{S}_6$, since a genus two curve is hyperelliptic, hence in a unique way a double cover of $\mathbf{P}^1$ branched over 6 unordered points.

But $\mathcal{M}_{0,6}$ is obviously affine and then so is the quotient by a finite group. This implies that $H^p(\mathcal{M}_2, \mathrm{R}^q f_* \mathbf{Q})$ vanishes for $p > 3$. So we have nonzero entries only in the four columns $p = 0, 1, 2, 3$ and only in the rows where q is even. This means that we get short exact sequences

$$0 \to H^2(\mathcal{M}_2, \mathrm{R}^{k-2} f_* \mathbf{Q}) \to H^k(\mathcal{M}_{2,n}^{\mathsf{rt}}) \to H^0(\mathcal{M}_2, \mathrm{R}^k f_* \mathbf{Q}) \to 0$$

for k even and

$$0 \to H^3(\mathcal{M}_2, \mathrm{R}^{k-3} f_* \mathbf{Q}) \to H^k(\mathcal{M}_{2,n}^{\mathsf{rt}}) \to H^1(\mathcal{M}_2, \mathrm{R}^{k-1} f_* \mathbf{Q}) \to 0$$

for k odd.

Here I am using that the Leray spectral sequence degenerates, because f is smooth and projective! This was the real reason to work with $\mathcal{M}_{2,n}^{\mathsf{rt}}$ rather than $\mathcal{M}_{2,n}$. (It's possible that the Leray spectral sequence for $\mathcal{M}_{2,n} \to \mathcal{M}_2$ degenerates but I don't know an argument.) In genus one we could equally well have considered $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$ and the arguments would actually have been slightly easier.

So let's first find the Sp_4-invariants in $H^\bullet(\mathsf{FM}(C,n))$. The argument is extremely similar: we can first consider the Sp_4-invariants in $H^\bullet(C^n)$. The same argument as before goes through. For the decomposition of $H^1(C)^{\otimes k}$ into irreducible representations of Sp_4 see [Fulton and Harris 1991], the part about "Weyl's construction of the irreducible representations of the symplectic group". One finds that $H^\bullet(C^n)^{\mathrm{Sp}_4}$ is the subalgebra generated by diagonals and the loci where one of the x_i is equal to some fixed point. Then $H^\bullet(\mathsf{FM}(C,n))^{\mathrm{Sp}_4}$ is generated by strata and the same loci. Now the class of a point in $H^2(C)$ is obviously equal to the canonical class, up to a scalar, which says that the class we get in $H^\bullet(\mathcal{M}_{2,n}^{\mathsf{rt}})$ in this way is a ψ-class.

We have found the following facts. The surjection

$$H^k(\mathcal{M}_{2,n}^{\mathsf{rt}}) \to H^k(\mathsf{FM}(C,n))^{\mathrm{Sp}_4}$$

for even k has a section, and the image of this section in $H^k(\mathcal{M}_{2,n}^{\mathsf{rt}})$ is exactly the tautological classes in $H^k(\mathcal{M}_{2,n}^{\mathsf{rt}})$. Indeed the tautological ring in genus two is spanned by strata and ψ-classes (there are no κ-classes). The non-tautological cohomology in $H^k(\mathcal{M}_{2,n}^{\mathsf{rt}})$ can be identified with $H^2(\mathcal{M}_2, \mathrm{R}^{k-2} f_* \mathbf{Q})$.

6. Cohomology of local systems on $\mathcal{M}_2$ and $\mathcal{A}_2$

After the previous section we know that the non-tautological even cohomology of $\mathcal{M}_{2,n}^{\mathsf{rt}}$ comes from $H^2(\mathcal{M}_2, \mathrm{R}^{k-2} f_* \mathbf{Q})$. We want to understand when this space is zero and in particular when there are pure classes in this subspace, since these are the classes that will extend to $\overline{\mathcal{M}}_{2,n}$. We need a higher genus generalization of the Eichler–Shimura theory.

In genus two, the Torelli map $\mathcal{M}_2 \to \mathcal{A}_2$ is an open immersion. The complement $\mathcal{A}_2 \setminus \mathcal{M}_2$ is the locus of products of elliptic curves, and we have a surjection $\mathcal{M}_{1,1} \times \mathcal{M}_{1,1} \to \mathcal{A}_2 \setminus \mathcal{M}_2$.

Now we are going to apply the Lemma that was shown in the previous class, but with nontrivial coefficients. (It's still true, with the same proof.) We find that

$$H^0(\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}, \mathbb{V}_{l,m})(-1) \to H^2(\mathcal{A}_2, \mathbb{V}_{l,m}) \to W_{2+l+m} H^2(\mathcal{M}_2, \mathbb{V}_{l,m}) \to 0.$$

The cohomology of the local systems $\mathbb{V}_{l,m}$ on $\mathcal{A}_2$ and their weight filtration/Hodge structure is known in any degree by work of myself (in preparation, will be on arXiv soon). But even before this the subspace of the *Eisenstein cohomology* was known due to [Harder 2012]. Combining Harder's work with the conjectural formulas of [Faber and van der Geer 2004] gives an expression for the cohomology in any degrees. Faber and van der Geer's conjecture was known for regular weights ($l > m > 0$) due to [Weissauer 2009]. In [Petersen and Tommasi 2012] we used these partial results, but our arguments can be simplified after one knows the whole cohomology.

In any case we have the following. Let $(l,m) \neq (0,0)$. (The trivial local system is uninteresting since $H^2(\mathcal{M}_2) = 0$.) Then

$$H^2(\mathcal{A}_2, \mathbb{V}_{l,m}) = \begin{cases} \bigoplus_f \mathbf{Q}(-1-2k) & l = m = 2k \\ 0 & \text{else} \end{cases}$$

where the sum is over the set of cusp eigenforms for $\mathrm{SL}_2(\mathbf{Z})$ of weight $4k+4$. So also here we have cohomology classes associated to modular forms, but note that we don't find the Hodge structures attached to cusp forms we saw before. All cohomology is of Tate type. This tells us that only the local systems $\mathbb{V}_{2k,2k}$ could give rise to a non-tautological class in $\mathcal{M}_{2,n}^{\mathrm{rt}}$, and assuming the Hodge conjecture, that all non-tautological even cohomology classes should be classes of algebraic cycles.

The dimensions of $H^2(\mathcal{A}_2, \mathbb{V}_{2k,2k})$ for $k > 0$ are $0, 1, 1, 1, 2, 2, 2, 3, 3, 3, \ldots$.

Now we consider $H^0(\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}, \mathbb{V}_{l,m})(-1)$. Note that H^0 is nonzero only for the trivial local system, so we want to find the multiplicity of the trivial local system when we pull back $\mathbb{V}_{l,m}$ to $\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$. This is the same as computing how irreducible representations of Sp_4 decompose into irreducibles when restricted to the subgroup $\mathrm{SL}_2 \times \mathrm{SL}_2$. This can also be found in [Fulton and Harris 1991] somewhere: the answer is that the trivial representation occurs with multiplicity 1 if $l = m$ and 0 otherwise.

So when we consider the local system $\mathbb{V}_{10,10}$, then the first term in the sequence

$$H^0(\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}, \mathbb{V}_{10,10})(-1) \to H^2(\mathcal{A}_2, \mathbb{V}_{10,10}) \to W_{22}H^2(\mathcal{M}_2, \mathbb{V}_{10,10}) \to 0$$

is 1-dimensional and the middle is 2-dimensional. Hence we must find a nonzero class in $W_{22}H^2(\mathcal{M}_2, \mathbb{V}_{10,10})$.

To summarize: pure classes in $H^2(\mathcal{M}_2, \mathbb{V}_{l,m})$ can come only from the local systems $\mathbb{V}_{2k,2k}$. Such classes exist for any $k \geq 5$ and possible a few lower values of k (but most likely $k = 5$ is the first case).

7. Putting it all together.

In the end of the last section we found a nonzero class in $W_{22}H^2(\mathcal{M}_2, \mathbb{V}_{10,10})$, giving rise to a class in $W_{22}H^{22}(\mathcal{M}_{2,20}^{\mathrm{rt}})$ (of Tate type). Note that the nontautological class found by [Graber and Pandharipande 2003] also was in H^{22} and of pure Tate type. They also transform according to the same $\S_{20}$-representation.

To simplify exposition, let's assume that $\mathbb{V}_{10,10}$ is the first local system for which $W_{2+l+m}H^2(\mathcal{M}_2, \mathbb{V}_{l,m}) \neq 0$.

Now if we write $\overline{\mathcal{M}}_{2,20} \setminus \mathcal{M}^{\mathrm{rt}}_{2,20}$ as a union of boundary divisors D_i, then we have

$$\bigoplus_i H^{k-2}(D_i)(-1) \to H^k(\overline{\mathcal{M}}_{2,20}) \to W_k H^k(\mathcal{M}^{\mathrm{rt}}_{2,20}) \to 0.$$

Let's say also k is even. The even pure cohomology of $\mathcal{M}^{\mathrm{rt}}_{2,20}$ is all tautological, except the funny class in H^{22} coming from $\mathbb{V}_{10,10}$ (since by our simplifying assumption this is the first local system with nonzero pure cohomology).

Each boundary divisor D_i is a product $\overline{\mathcal{M}}_{1,k+1} \times \overline{\mathcal{M}}_{1,20-k+1}$, and we already know that the even cohomology in genus one is tautological and the odd comes from cusp forms, and that the odd cohomology appears for the first time on $\overline{\mathcal{M}}_{1,11}$. This shows that all D_i have tautological even cohomology *except* those of the form $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$, in which case $H^{11}(\overline{\mathcal{M}}_{1,11}) \otimes H^{11}(\overline{\mathcal{M}}_{1,11})$ is nonzero and nontautological.

Hence there is possibility for non-tautological even cohomology on $\overline{\mathcal{M}}_{2,20}$ in two different degrees: there is the class in H^{22} coming from $\mathbb{V}_{10,10}$, and there is the pushforward of $H^{11}(\overline{\mathcal{M}}_{1,11}) \otimes H^{11}(\overline{\mathcal{M}}_{1,11})$ landing in H^{24}. Next time we will prove that the pushforward is actually zero for the specific case of 20 marked points, which means that the only non-tautological even cohomology on $\overline{\mathcal{M}}_{2,20}$ lies in H^{22}.

In particular this shows that the tautological ring of $\overline{\mathcal{M}}_{2,20}$ is not Gorenstein, i.e. it does not satisfy Poincaré duality. Indeed the even Betti numbers of $H^\bullet(\overline{\mathcal{M}}_{2,20})$ are symmetric about the middle degree by Poincaré duality. If there is non-tautological even cohomology below the middle degree and not above it, then the tautological part of the cohomology can not have this symmetry, so the image of the tautological ring in cohomology is not Gorenstein. But then the same is true for the tautological ring in Chow. This is spelled out in [Petersen and Tommasi 2012].

If our simplifying assumption was false — let's say that $\mathbb{V}_{6,6}$ is the first local system for which $W_{2+l+m}H^2(\mathcal{M}_2, \mathbb{V}_{l,m}) \neq 0$ — our argument would just be the same, but simpler. We could instead argue that the tautological ring of $\overline{\mathcal{M}}_{2,12}$ is not Gorenstein, and we would not even have to worry about products of cusp form classes from the boundary.

In genus one we had a kind of classification of the even and odd cohomology. The same is true in $g = 2$. We can say that there are three kinds of even cohomology on $\overline{\mathcal{M}}_{2,n}$:

- Tautological classes
- Classes from the local system $\mathbb{V}_{2k,2k}$ (non-tautological cohomology from the interior)
- Products of cusp form classes from the genus one boundary.

The second one appears when $n = 20$ and possibly earlier. The third one appears only for $n \geq 21$.

The odd cohomology can also be classified this way, let me just state the result without proof. It is

- Classes of Siegel modular forms
- Endoscopic lifts
- Cusp form classes from the genus one boundary.

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