COHOMOLOGY OF MODULI OF STABLE POINTED CURVES OF LOW GENUS. III

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Today we shall study the cohomology of $\mathcal{M}^{\mathsf{ct}}_{2,n}$. We will fiber it over $\mathcal{M}^{\mathsf{ct}}_{2}$ which requires the decomposition theorem. In this talk I mostly treat this as a black box. A very nice introduction to this theory is [de Cataldo and Migliorini 2009]. In the process I will finish the proof that $\overline{\mathcal{M}}_{2,n}$ has a non-Gorenstein tautological ring for $n \geq 20$ and prove that the same is true for the tautological rings of $\mathcal{M}^{\mathsf{ct}}_{2,n}$ for $n \geq 8$.

1. Last time

In the lecture I began by recalling material from the second lecture. Since there are already notes from the second lecture online, I will allow myself to be brief.

Last time, we considered $f\colon \mathcal{M}^{\mathsf{rt}}_{2,n} \to \mathcal{M}_2$. The Leray spectral sequence for f degenerates and is compatible with mixed Hodge structure. Each $\mathrm{R}^q f_* \mathbf{Q}$ is a PVHS of weight q, and is a direct sum of local systems $\mathbb{V}_{l,m}$ (up to Tate twist). We are mostly interested in the pure even cohomology of $\mathcal{M}^{\mathsf{rt}}_{2,n}$. Since only local systems of even weight have nonzero cohomology, we should consider the even pure cohomology of $\mathbb{V}_{l,m}$. There is the exact sequence

$$\mathfrak{gr}^W_{k+l+m} H^{k-2}(\mathrm{Sym}^2 \mathcal{M}_{1,1}, \mathbb{V}_{l,m})(-1) \to \mathfrak{gr}^W_{k+l+m} H^k(\mathcal{M}^{\mathsf{ct}}_2, \mathbb{V}_{l,m}) \to \mathfrak{gr}^W_{k+l+m} H^k(\mathcal{M}_2, \mathbb{V}_{l,m}) \to 0,$$

and we know the cohomology groups $H^k(\mathcal{M}^{\mathsf{ct}}_2, \mathbb{V}_{l,m})$ with their weight filtration for any k, l and m by the results of [Petersen 2013] (fresh off the presses!). Perhaps this is the right point to state that there is an isomorphism (of stacks!) $\mathcal{M}^{\mathsf{ct}}_2 \cong \mathcal{A}_2$, the moduli space of principally polarized abelian surfaces. A principally polarized abelian surface is either a Jacobian or a product of elliptic curves. The Jacobian locus is isomorphic to $\mathcal{M}_2$ and the locus of products of elliptic curves is isomorphic to $\mathrm{Sym}^2 \mathcal{M}_{1,1} = \mathcal{M}^{\mathsf{ct}}_2 \setminus \mathcal{M}_2$.

In any case, from [Petersen 2013] we saw that (if we restrict ourselves to nontrivial local systems) then $\mathfrak{gr}^W_{k+l+m} H^k(\mathcal{M}^{\mathsf{ct}}_2, \mathbb{V}_{l,m})$ is only[1] nonzero for $k = 2$ and $l = m = 2a$, in which case $H^2(\mathcal{M}^{\mathsf{ct}}_2, \mathbb{V}_{2a,2a})$ is pure of Tate type and its dimension is the dimension of the space of cusp forms of weight $4k+4$ for $\mathrm{SL}(2, \mathbf{Z})$.

Moreover, we identified the tautological cohomology of $\mathcal{M}^{\mathsf{rt}}_{2,n}$ with the part coming from the trivial local system in the Leray spectral sequence

$$H^p(\mathcal{M}_2, \mathrm{R}^q f_* \mathbf{Q}) \implies H^{p+q}(\mathcal{M}^{\mathsf{rt}}_{2,n});$$

equivalently, since $\mathcal{M}_2$ has the rational cohomology of a point, with the monodromy invariants $H^\bullet(\text{fiber})^{\mathrm{Sp}(4)}$.

[1] This is a small lie. As explained in the paper, there could in addition be a contribution in H^4 coming from cusp forms for $\mathrm{SL}(2, \mathbf{Z})$ of weight 0 (mod 4) with vanishing central value. But conjecturally such cusp forms don't exist and they have been computed not to exist for local systems of weight at most 200.

Finally we exploited (relative) Poincaré duality in the form

$$\mathrm{R}^{n+q} f_* \mathbf{Q} \cong \mathrm{R}^{n-q} f_* \mathbf{Q}(-q),$$

since f is a smooth proper morphism. This shows first of all that the trivial local system has the same multiplicity in R^{n+q} and R^{n-q}. Since the trivial local system has cohomology only in degree 0, this proves that the tautological cohomology satisfies $RH^{n+q}(\mathcal{M}^{\mathsf{rt}}_{2,n}) \cong RH^{n-q}(\mathcal{M}^{\mathsf{rt}}_{2,n})$ in accordance with the fact that the tautological ring of $\mathcal{M}^{\mathsf{rt}}_{2,n}$ is Gorenstein [Tavakol 2011].

Then we could consider the rest of the pure cohomology of $\mathcal{M}^{\mathsf{rt}}_{2,n}$, the contribution from the $\mathbb{V}_{2a,2a}$. Since this local system has pure cohomology in degree 2 only, the local systems of this kind contribute equally to $H^{n+2+q}(\mathcal{M}^{\mathsf{rt}}_{2,n})$ and $H^{n+2-q}(\mathcal{M}^{\mathsf{rt}}_{2,n})$. Since these classes are pure they lift to the cohomology of $\overline{\mathcal{M}}_{2,n}$ where we obtain in this way more non-tautological classes *below* the middle degree than *above*.

2. Some consequences of the decomposition theorem

Today we are going to play a similar game but we consider instead the Leray spectral sequence for the morphism

$$f\colon \mathcal{M}^{\mathsf{ct}}_{2,n} \to \mathcal{M}^{\mathsf{ct}}_{2}.$$

The obvious issue one runs into here is that f is no longer smooth and proper. Before we had compatibility with Hodge structure, we had Poincaré duality (in the form $\mathrm{R}^{n+q} f_* \mathbf{Q} \cong \mathrm{R}^{n-q} f_* \mathbf{Q}(-q)$), and we had degeneration of the Leray spectral sequence. A priori there is no reason for any of this to be true.

However, we can still use that f is proper. For a proper morphism of schemes there is the *decomposition theorem.* I will not state it precisely since I don't want to talk about perverse sheaves. But the claim is that there is a funny t-structure on the derived category of sheaves; we call the objects in the heart of this t-structure *perverse sheaves.* An informal slogan summarizing the decomposition theorem is that if all cohomology is computed with respect to this funny t-structure, then an arbitrary proper morphism behaves just as nicely as if it were a smooth proper morphism.

Now on a smooth variety the constant sheaf $\mathbf{Q}$ is perverse (up to a degree shift). In particular we can apply the decomposition theorem to $\mathrm{R}f_*\mathbf{Q}$. One can prove that for this particular f, all perverse cohomology sheaves of the complex $\mathrm{R}f_*\mathbf{Q}$ are actually ordinary sheaves. This holds more generally for $\mathcal{M}^{\mathsf{ct}}_{g,n} \to \mathcal{M}^{\mathsf{ct}}_{g}$. In other words, the decomposition theorem holds (in this case) both in the perverse and in the classical t-structure. So we get the conclusion of the decomposition theorem also for the usual derived pushforwards $\mathrm{R}^q f_*\mathbf{Q}$ and for the usual Leray spectral sequence.

What we find is the following.

- The (usual, not just the perverse!) Leray spectral sequence for f degenerates.
- Each sheaf $\mathrm{R}^q f_*\mathbf{Q}$ is a direct sum of polarized variations of Hodge structure on $\mathcal{M}^{\mathsf{ct}}_2$ and sheaves of the form $j_*\mathbb{V}$ (with $\mathbb{V}$ a PVHS of weight q on $\mathrm{Sym}^2(\mathcal{M}_{1,1})$).
- The Leray spectral sequence is compatible with mixed Hodge structure.
- There is Poincaré duality, but with a funny degree shift (and usually one would call this Verdier duality instead) — but we'll come to this in a second.

Let me spell out what the third point above means. PVHS's on $\mathcal{M}_2^{\mathsf{ct}}$ have a natural mixed Hodge structure on their cohomology. Summands of the form $j_*\mathbb{V}$ obtain a mixed Hodge structure on their cohomology via the isomorphism

$$H^\bullet(\mathcal{M}_2^{\mathsf{ct}}, j_*\mathbb{V}) = H^\bullet(\operatorname{Sym}^2 \mathcal{M}_{1,1}, \mathbb{V}).$$

It is this mixed Hodge structure that is compatible with the Leray spectral sequence.

Let us write

$$\mathrm{R}^\bullet f_*\mathbf{Q} = A^\bullet \oplus B^\bullet$$

where $A^\bullet$ consists of PVHS's on $\mathcal{M}_2^{\mathsf{ct}}$ and $B^\bullet$ consists of pushforwards of PVHS's on $\operatorname{Sym}^2 \mathcal{M}_{1,1}$. The Poincaré–Verdier duality for $\mathrm{R}f_*\mathbf{Q}$ says now that there are isomorphisms

$$A^{n+q} \cong A^{n-q}(-q)$$

and

$$B^{n+1+q} \cong B^{n+1-q}(-q)$$

for all q. This is the funny degree shift I mentioned. It comes from the codimension of $\operatorname{Sym}^2 \mathcal{M}_{1,1}$ in $\mathcal{M}_2^{\mathsf{ct}}$.

Example 2.1. Let us look at $n = 1$. Clearly,

$$\mathrm{R}^0 f_*\mathbf{Q} = \mathbf{Q},$$

the trivial local system. Slightly more interesting is

$$\mathrm{R}^1 f_*\mathbf{Q} = \mathbb{V}_{1,0}.$$

A priori it's maybe not so clear (even though all fibers have 4-dimensional H^1) that $\mathrm{R}^1 f_*\mathbf{Q}$ will be a local system since f is not a fiber bundle. What you can say though is that for any curve C of compact type, $H^1(C) = H^1(\operatorname{Jac}(C))$. So we can compute $\mathrm{R}^1 f_*\mathbf{Q}$ also as a derived pushforward from the universal abelian surface over $\mathcal{M}_2^{\mathsf{ct}} \cong \mathcal{A}_2$, which *is* a fiber bundle (it's smooth and projective).

Finally we should also understand $\mathrm{R}^2 f_*\mathbf{Q}$. Since the fibers over $\operatorname{Sym}^2 \mathcal{M}_{1,1}$ have 2-dimensional H^2's, we should find a jump in the rank here. In fact

$$\mathrm{R}^2 f_*\mathbf{Q} = \mathbf{Q}(-1) \oplus j_*\mathbb{L}$$

where $\mathbb{L}$ is a local system on $\operatorname{Sym}^2 \mathcal{M}_{1,1}$. This local system is the one associated with the sign representation of $\mathbb{S}_2$, which sits in an obvious way in the orbifold fundamental group of $\operatorname{Sym}^2 \mathcal{M}_{1,1} = (\mathcal{M}_{1,1})^2/\mathbb{S}_2$. The reason is that H^2 of the fibers over $\operatorname{Sym}^2(\mathcal{M}_{1,1})$ is spanned by the *sum* of the fundamental classes of the two components (which transforms according to the trivial representation of the fundamental group) and the *difference* (which transforms according to the sign representation).

Example 2.2. Let's consider $n = 2$. The space $\mathcal{M}_{2,2}^{\mathsf{ct}}$ is 'almost' the fibered product of $\mathcal{M}_{2,1}^{\mathsf{ct}}$ with itself over $\mathcal{M}_2^{\mathsf{ct}}$, so let's start by computing the tensor product of the complex $\mathrm{R}^\bullet f_*\mathbf{Q}$ from the previous example with itself. This gives the following contributions in various degrees:

0	$\mathbf{Q}$	
1	$2\mathbb{V}_{1,0}$	
2	$\mathbb{V}_2 \oplus \mathbb{V}_{1,1} \oplus 3\mathbf{Q}(-1)\oplus$	$2j_*\mathbb{L}$
3	$2\mathbb{V}_{1,0}(-1)\oplus$	$2j_*\mathbb{L} \otimes \mathbb{V}_{1,0}$
4	$\mathbf{Q}(-2)\oplus$	$2j_*\mathbb{L}(-1) \oplus j_*\mathbb{L} \otimes \mathbb{L}$.

Clearly $\mathbb{L}\otimes\mathbb{L} = \mathbf{Q}(-2)$ so the last summand can be simplified to $j_*\mathbf{Q}(-2)$. Finally, the difference between this fibered product and $\mathcal{M}_{2,2}^{\mathsf{ct}}$ is that we need to blow-up the locus where both marked points are placed on the node. In terms of the cohomology of the fibers we get an exceptional divisor which produces an extra class in H^2 of the fibers over $\mathrm{Sym}^2\mathcal{M}_{1,1}$. All in all we find that $\mathrm{R}^\bullet f_*\mathbf{Q}$ is given by

0	$\mathbf{Q}$	
1	$2\mathbb{V}_{1,0}$	
2	$\mathbb{V}_2\oplus\mathbb{V}_{1,1}\oplus 3\mathbf{Q}(-1)\oplus$	$2j_*\mathbb{L}\oplus j_*\mathbf{Q}(-1)$
3	$2\mathbb{V}_{1,0}(-1)\oplus$	$2j_*\mathbb{L}\otimes\mathbb{V}_{1,0}$
4	$\mathbf{Q}(-2)\oplus$	$2j_*\mathbb{L}(-1)\oplus j_*\mathbf{Q}(-2)$.

Here the first column corresponds to what I called $A^\bullet$ earlier and the second column to $B^\bullet$. So we see the Poincaré duality with the degree shift in both these examples.

3. The pure cohomology of $\mathcal{M}_{2,n}^{\mathsf{ct}}$

We are now going to be interested in the pure part of the cohomology of $\mathcal{M}_{2,n}^{\mathsf{ct}}$, and therefore in the pure part of the cohomology of local systems. On $\mathcal{M}_2^{\mathsf{ct}}$ we already know the cohomology of the local systems, as stated. We only need to consider the trivial local system which has

$$H^0(\mathcal{M}_2^{\mathsf{ct}}) = \mathbf{Q},\qquad H^2(\mathcal{M}_2^{\mathsf{ct}}) = \mathbf{Q}(-1).$$

Let's in addition restrict ourselves to the even cohomology. Then there is in addition the $\mathbb{V}_{2a,2a}$ with $H^2(\mathcal{M}_2^{\mathsf{ct}},\mathbb{V}_{2a,2a})$ pure of Tate type and with dimension given by the dimension of the space of cusp forms of weight $4k+4$ for $\mathrm{SL}(2,\mathbf{Z})$, and this is all.

There are also local systems on $\mathrm{Sym}^2\mathcal{M}_{1,1}$. The trivial local system is easy:

$$H^\bullet(\mathrm{Sym}^2\mathcal{M}_{1,1}) = \mathbf{Q}.$$

In general, the cohomology of local systems on $\mathrm{Sym}^2\mathcal{M}_{1,1}$ can be expressed in terms of cohomology of local systems on $\mathcal{M}_{1,1}$ in a way explained in detail in [Petersen 2010]. Since the only pure cohomology of local systems on $\mathcal{M}_{1,1}$ was the trivial local system and the cusp form classes (in odd weight), the only even pure cohomology of local systems on $\mathrm{Sym}^2\mathcal{M}_{1,1}$ is in H^0 (coming from the trivial local system) and in H^2 (coming from products of cusp form classes in the Künneth theorem).

The latter ones appear for the first time when considering local systems on $\mathrm{Sym}^2\mathcal{M}_{1,1}$ of weight 20: cusp forms appear in the cohomology of $\mathcal{M}_{1,1}$ for the first time in a weight 10 local system, and we need two of those.

3.1. **Failure of Poincaré duality for the pure cohomology.** Consider the contribution of the trivial local system on $\mathcal{M}_2^{\mathsf{ct}}$ to the pure cohomology of $\mathcal{M}_{2,n}^{\mathsf{ct}}$. The cohomology of the trivial local system on $\mathcal{M}_2^{\mathsf{ct}}$ is symmetric about degree 1, and it appears with the same multiplicity in $\mathrm{R}^{n+q}f_*\mathbf{Q}$ and in $\mathrm{R}^{n-q}f_*\mathbf{Q}$. All in all we see that we get equal contributions to $H^{n+1+q}(\mathcal{M}_{2,n}^{\mathsf{ct}})$ and $H^{n+1-q}(\mathcal{M}_{2,n}^{\mathsf{ct}})$.

Now consider instead the trivial local system on $\mathrm{Sym}^2\mathcal{M}_{1,1}$. Its cohomology is symmetric about degree 0, and it appears with the same multiplicity in $\mathrm{R}^{n+1+q}f_*\mathbf{Q}$ and in $\mathrm{R}^{n+1-q}f_*\mathbf{Q}$. This local system, too, constributes equally to $H^{n+1+q}(\mathcal{M}_{2,n}^{\mathsf{ct}})$ and $H^{n+1-q}(\mathcal{M}_{2,n}^{\mathsf{ct}})$.

However, the local system $\mathbb{V}_{2a,2a}$ for $a > 0$ has pure cohomology symmetric about degree 2, so its contribution is equal in $H^{n+2+q}(\mathcal{M}^{\mathsf{ct}}_{2,n})$ and $H^{n+2-q}(\mathcal{M}^{\mathsf{ct}}_{2,n})$.

The local systems on $\operatorname{Sym}^2 \mathcal{M}_{1,1}$ with cohomology coming from products of cusp forms is even worse: we get equal contributions to $H^{n+3+q}(\mathcal{M}^{\mathsf{ct}}_{2,n})$ and $H^{n+3-q}(\mathcal{M}^{\mathsf{ct}}_{2,n})$.

All in all, we see that the pure even cohomology of $\mathcal{M}^{\mathsf{ct}}_{2,n}$ satisfies Poincaré duality if and only if the only contributions come from the trivial local systems. As soon as other local systems appear, Poincaré duality fails. This happens for the first time for $\mathbb{V}_{4,4}$ which is a summand in $\mathrm{R}^8 f_* \mathbf{Q}$ when $n = 8$; here there is a class in H^2 associated with the famous discriminant cusp form Δ of weight 12.

In general the tautological cohomology is contained in the pure cohomology, since the class of an algebraic cycle is pure. We will see later that the pure cohomology of $\mathcal{M}^{\mathsf{ct}}_{2,8}$ is all tautological, so we get a Gorenstein counterexample in this case.

3.2. **Tautological and non-tautological cohomology in terms of local systems.** Before proving that the pure cohomology of $\mathcal{M}^{\mathsf{ct}}_{2,8}$ is all tautological, let me explain more generally how to see what cohomology classes on $\mathcal{M}^{\mathsf{ct}}_{2,n}$ are/aren't tautological, in terms of local systems.

Recall that in the pure even cohomology of $\mathcal{M}^{\mathsf{rt}}_{2,n}$, there were two kinds of classes: those coming from the trivial local system, and those coming from the cohomology of $\mathbb{V}_{2a,2a}$ on $\mathcal{M}_2$. The former are exactly the tautological ones and the latter exactly the non-tautological ones.

The normalization of the boundary $\overline{\mathcal{M}}_{2,n} \setminus \mathcal{M}^{\mathsf{rt}}_{2,n}$ has only two kinds of even cohomology: there is the tautological cohomology, and then there are the summands of the form

$$H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,n-k+1}) \otimes H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,k+1})$$

which are manifestly non-tautological. Now of course the pushforward of a non-tautological class α from the boundary may be tautological, but in that case we can write $\alpha = \beta + \gamma$ where β is tautological and γ pushes forward to zero. Since $H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,n-k+1}) \otimes H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,k+1})$ contains no tautological classes, we see that the only way for a class in this subspace to push forward to something tautological is if it pushes forward to zero. Thus the image of these 'product of cusp form'-classes in $H^\bullet(\overline{\mathcal{M}}_{2,n})$ is all non-tautological.

This tells us also something about the pure even cohomology of $\mathcal{M}^{\mathsf{ct}}_{2,n}$. First of all, the trivial local systems on $\mathcal{M}^{\mathsf{ct}}_2$ and $\operatorname{Sym}^2 \mathcal{M}_{1,1}$ gives cohomology which is either associated the the trivial local system on $\mathcal{M}_2$ or which is pushed forward from the genus one boundary and *not* of the 'product of cusp form' type. So all of this part is tautological.

The local systems on $\operatorname{Sym}^2 \mathcal{M}_{1,1}$ with nontrivial H^2 coming from products of cusp forms give exactly the cohomology which is pushed forward from summands of the type

$$H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,n-k+1}) \otimes H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,k+1}).$$

So all of this part is non-tautological.

Finally we consider $\mathbb{V}_{2a,2a}$. We have the short exact sequence

$$H^0(\operatorname{Sym}^2 \mathcal{M}_{1,1}, \mathbb{V}_{2a,2a})(-1) \to H^2(\mathcal{M}^{\mathsf{ct}}_2, \mathbb{V}_{2a,2a}) \to H^2(\mathcal{M}_2, \mathbb{V}_{2a,2a}).$$

Here $H^0(\mathrm{Sym}^2 \mathcal{M}_{1,1}, \mathbb{V}_{2a,2a})$ is pure of Tate type (obviously) and 1-dimensional for any $a > 0$ (use the branching formula proven in [Petersen 2010]). Thus $H^2(\mathcal{M}_2^{\mathsf{ct}}, \mathbb{V}_{2a,2a})$ has a subspace of dimension at most 1, consisting of classes pushed forward from $\mathrm{Sym}^2 \mathcal{M}_{1,1}$. Classes in this subspace give tautological cohomology classes in $H^\bullet(\mathcal{M}_{2,n}^{\mathsf{ct}})$. Indeed, they are pushed forward to $\mathcal{M}_{2,n}^{\mathsf{ct}}$ from $\mathcal{M}_{2,n}^{\mathsf{ct}} \setminus \mathcal{M}_{2,n}^{\mathsf{rt}}$, and on the boundary $\mathcal{M}_{2,n}^{\mathsf{ct}} \setminus \mathcal{M}_{2,n}^{\mathsf{rt}}$ these classes are not products of cusp form classes and thus tautological. Conversely, the image of $H^2(\mathcal{M}_2^{\mathsf{ct}}, \mathbb{V}_{2a,2a}) \to H^2(\mathcal{M}_2, \mathbb{V}_{2a,2a})$ produces non-tautological cohomology classes in $H^\bullet(\mathcal{M}_{2,n}^{\mathsf{ct}})$ (since they restrict to non-tautological classes on $\mathcal{M}_{2,n}^{\mathsf{rt}}$).

Combining the reasoning in this subsection with that in the previous one we see that the tautological cohomology of $\mathcal{M}_{2,n}^{\mathsf{ct}}$ fails to be Gorenstein as soon as the pushforward map

$$H^0(\mathrm{Sym}^2 \mathcal{M}_{1,1}, \mathbb{V}_{2a,2a})(-1) \to H^2(\mathcal{M}_2^{\mathsf{ct}}, \mathbb{V}_{2a,2a})$$

has non-zero image for some $a > 0$. In [Petersen and Tommasi 2012] we conjectured that this is true for any $a \geq 2$. The result of the next section is that this is true when $a = 2$, so we get Gorenstein failure in this case. Thus for any $n \geq 8$, the ranks of $RH^\bullet(\mathcal{M}_{2,n}^{\mathsf{ct}})$ are greater above the middle degree than below it.

Finally, there are also consequences for the even cohomology of $\overline{\mathcal{M}}_{2,n}$. It consists of tautological classes, classes associated to the local system $\mathbb{V}_{2a,2a}$ on $\mathcal{M}_2$, and products of cusp form classes. We now see that the $\mathbb{V}_{2a-2a}$-classes contribute equally to $H^{n+2+q}(\overline{\mathcal{M}}_{2,n})$ and $H^{n+2-q}(\overline{\mathcal{M}}_{2,n})$, and that the products of cusp form classes contribute equally to $H^{n+3+q}(\overline{\mathcal{M}}_{2,n})$ and $H^{n+3-q}(\overline{\mathcal{M}}_{2,n})$ (so that they *are* symmetric about the middle degree). Thus for any $n \geq 20$ (since $a = 5$ is the first case when we are sure that $\mathbb{V}_{2a,2a}$ has non-zero cohomology on $\mathcal{M}_2$) we see similarly that the ranks of $RH^\bullet(\overline{\mathcal{M}}_{2,n})$ are greater above the middle degree than below it.

In particular we have given an independent proof that the product of cusp form classes vanish when $n = 20$. The only products of cusp form in the boundary come from

$$H^{11}(\overline{\mathcal{M}}_{1,11}) \otimes H^{11}(\overline{\mathcal{M}}_{1,11})$$

and should therefore land in cohomological degree 24. But this is strictly above the middle degree, and we have just proven that contributions from products of cusp forms should be symmetric about the middle degree. So the pushforward map has to vanish in this case.

4. The pure cohomology of $\mathcal{M}_{2,8}^{\mathsf{ct}}$ is tautological

We wish to prove that the (even) pure cohomology of $\mathcal{M}_{2,8}^{\mathsf{ct}}$ is tautological. (In fact there is no odd cohomology, as one sees also from [Petersen 2013].) As stated in the preceding section, this amounts to proving that the pushforward map

$$H^0(\mathrm{Sym}^2 \mathcal{M}_{1,1}, \mathbb{V}_{2a,2a})(-1) \to H^2(\mathcal{M}_2^{\mathsf{ct}}, \mathbb{V}_{2a,2a})$$

has non-zero image in the particular case $a = 2$. I don't know how to do this. In general this problem can be stated as evaluating the generalized modular symbol corresponding to $\mathrm{SL}_2 \times \mathrm{SL}_2 \subset \mathrm{Sp}_4$ on a particular residual Eisenstein cohomology class, and checking whether or not the answer is zero. My secret hope is that it is possible in general to evaluate these generalized modular symbols.

Fortunately one can give a more direct geometric proof that all pure cohomology is tautological instead. This uses a rational parametrization of $\mathcal{M}_{2,8}$. This is a general technique, maybe *the* general technique, for proving that the Chow ring of some moduli space consists of tautological classes. The same method can be used (and is even a bit easier) to prove that pure cohomology consists of tautological classes, since pure cohomology shares several formal properties with Chow rings.

In particular if we have X smooth, $Z \subset X$ closed, $\widehat{Z} \to Z$ proper and surjective, then there is a short exact sequence of Chow groups

$$A_\bullet(\widehat{Z}) \to A_\bullet(X) \to A_\bullet(X \setminus Z) \to 0,$$

the "localization sequence". If X and $\widehat{Z}$ are smooth this can be dualized to an exact sequence of Chow rings (with a degree shift on the first term given by the codimension of the map $\widehat{Z} \to X$). As we learned in the first talk of this mini-course there is a completely analogous localization sequence for the pure part of Borel–Moore homology, which similarly if X and $\widehat{Z}$ are smooth can be dualized to an exact sequence of ordinary cohomology groups.

Finally we can observe some simplifications. First of all, the only potentially non-tautological cohomology on $\mathcal{M}^{\mathsf{ct}}_{2,8}$ comes from the local system $\mathbb{V}_{4,4}$ on $\mathcal{M}_2$ and is therefore supported on the interior: it will suffice to prove that the pure cohomology of $\mathcal{M}_{2,8}$ is tautological. Moreover, the only possibly non-tautological cohomology is in degree 10, so it suffices to prove that the cohomology in this particular degree is tautological. Thus it suffices to prove that the cohomology ring is generated below degree 10.

Let U be the open subset of $\mathcal{M}_{2,8}$ consisting of $(C, p_1, \ldots, p_8)$ such that $|p_i + p_j|$ is not the canonical $\mathfrak{g}^2_1$ for any $1 \le i < j \le 8$.

Let $\mathcal{M}^0_{2,7}$ be the open subset of $\mathcal{M}_{2,7}$ consisting of $(C, p_1, \ldots, p_7)$ such that $i(p_7) \neq p_i$ for $i = 1, \ldots, 7$, where i is the hyperelliptic involution. There is a natural map

$$g\colon \mathcal{M}^0_{2,7} \to \mathcal{M}_{2,8}$$

defined by

$$(C, p_1, \ldots, p_7) \mapsto (C, p_1, \ldots, p_7, i(p_8)).$$

In this way, the disjoint union of $\binom{8}{2}$ copies of $\mathcal{M}^0_{2,7}$ maps surjectively onto $\mathcal{M}_{2,8} \setminus U$ and is a resolution of singularities. We therefore have an exact sequence

$$\bigoplus_{i=1}^{\binom{8}{2}} W_k \left(H^{k-2}(\mathcal{M}^0_{2,7})(-1)\right) \to W_k H^k(\mathcal{M}_{2,8}) \to W_k H^k(U) \to 0.$$

Observe first of all that $W_k H^k(\mathcal{M}_{2,7}) = 0$ for any $k > 2$, since we already know that the pure part of cohomology is tautological for $n < 8$, and the tautological ring vanishes above degree 2. Then the same vanishing holds for the pure cohomology of $\mathcal{M}^0_{2,7}$). Since the only possibly non-tautological class in $H^\bullet(\mathcal{M}_{2,8})$ was in degree 10, it will thus suffice to show that the pure cohomology of U is tautological. We will prove that the pure cohomology is generated in degree 2.

For this we need to set up some notation. Let $\mathbf{P}V$ be the projective space of $(3,2)$-curves in $\mathbf{P}^1 \times \mathbf{P}^1$ (so $V \cong \mathbf{C}^{12}$). For any point of $\mathbf{P}^1 \times \mathbf{P}^1$, the set of $(3,2)$-curves passing through it is a hyperplane in $\mathbf{P}V$.

Let $W \subset (\mathbf{P}^1 \times \mathbf{P}^1)^8$ be set of all configurations $(p_1, \ldots, p_8)$ such that

(1) no two points have the same second projections,
(2) no collection of four points have the same first projection,
(3) $p_1 = (0,0)$, $p_2 = (0,1)$, $p_3 = (0,\infty)$ and $p_4 = (\infty, y)$ for some $y \in \mathbf{P}^1 \setminus \{0,1,\infty\}$.

Let $I \subset \mathbf{P}V \times W$ be the incidence variety of all pairs $(f, p_1, \ldots, p_8)$ such that $f(p_i) = 0$ for $i = 1, \ldots, 8$. Finally let $I^{\mathrm{sm}} \subset I$ be the Zariski open subset where $V(f)$ is nonsingular. There is a free $\mathbb{G}_m$-action on I preserving I^{sm} which rescales the x-coordinate.

Lemma 4.1. *There is an isomorphism $U \cong I^{\mathrm{sm}}/\mathbb{G}_m$.*

Proof. Take a point $(C, p_1, \ldots, p_8)$ of U and map C to $\mathbf{P}^1 \times \mathbf{P}^1$ using the pairs of linear systems $|p_1 + p_2 + p_3|$ and $|K_C|$. Since we chose a point of U the former linear system will not contain the canonical one, so the image is a smooth $(3,2)$-curve, and the images of p_1, p_2 and p_3 in the second copy of $\mathbf{P}^1$ are distinct. We can therefore choose coordinates so that they are 0, 1 and ∞. On the first copy of $\mathbf{P}^1$ we can choose coordinates so that p_1, p_2 and p_3 map to 0 and p_4 is mapped to ∞. It is now clear that we have obtained a point of $I^{\mathrm{sm}}/\mathbb{G}_m$.

Conversely, there is an obvious map $I^{\mathrm{sm}}/\mathbb{G}_m \to \mathcal{M}_{2,8}$, since $V(f)$ will be a nonsingular genus 2 curve with 8 ordered markings, whose isomorphism type is invariant under the $\mathbb{G}_m$-action. Condition (1) in the definition of W ensures that we land inside $U \subset \mathcal{M}_{2,8}$. □

Lemma 4.2. *Suppose given a configuration $(p_1, \ldots, p_8) \in W$. Then the corresponding eight hyperplanes in $\mathbf{P}V$ are in general position with respect to each other.*

Proof. We need to find, for $k = 1, \ldots, 7$, a $(3,2)$-curve passing through $p_1, \ldots, p_k$ but none of $p_{k+1}, \ldots, p_8$. By condition (2) in the definition of W we can assume (after reordering the last four points) that p_7 and p_8 do not pass through the vertical line through p_4. For $k = 1, 2$ and 3, use three horizontal lines (and implicitly two vertical lines passing through none of the points). For $k = 4$, 5 and 6 take the vertical line through p_1, p_2 and p_3, and three horizontal lines. Finally take the two vertical lines through p_1—p_3 and p_4 respectively, and use horizontal lines for the rest. □

Proposition 4.3. *The pure cohomology of U is generated by divisors.*

Proof. We do this in smaller steps. First of all W is Zariski open in an affine space, so its pure cohomology is in fact trivial. Lemma 4.2 shows that $I \to W$ is a $\mathbf{P}^3$-bundle, and the projective bundle formula shows that the pure cohomology of I is isomorphic to $H^\bullet(\mathbf{P}^3)$. Since I^{sm} is open in I, its pure cohomology is a quotient of $H^\bullet(\mathbf{P}^3)$ and therefore generated by divisors (most likely it is trivial, but we don't care). The $\mathbb{G}_m$-action on I^{sm} is free, so the cohomology of $U = I^{\mathrm{sm}}/\mathbb{G}_m$ is the same as the $\mathbb{G}_m$-equivariant cohomology of I^{sm}. By the following lemma the pure cohomology of U is then generated in degree 2, since the same is true both for the pure cohomology of I^{sm} and the ring $H^\bullet_{\mathbb{G}_m}(\mathrm{pt}) = \mathbf{Q}[c_1]$. □

Lemma 4.4. *Let X be a smooth algebraic variety with an action by the algebraic group G. There is a (noncanonical) surjective algebra homomorphism from the pure part of $H^\bullet(X) \otimes H^\bullet_G(\mathrm{pt})$ to the pure part of $H^\bullet_G(X)$.*

Proof. Since G is connected, the Leray spectral sequence for $EG \times_{BG} X \to BG$ reads

$$H_G^p(\mathrm{pt}) \otimes H^q(X) \implies H_G^{p+q}(X).$$

It has a multiplicative structure. We have the spectral sequence for the equivariant cohomology of any space X:

$$H_G^p(\mathrm{pt}) \otimes H^q(X) \implies H_G^{p+q}(X).$$

This spectral sequence is compatible with the multiplications and with the weight filtrations on both sides. Moreover $H_G^p(\mathrm{pt})$ is pure for any p and any algebraic group G [Deligne 1974]. If X is smooth then this shows that $E_2^{p,q}$ has weights at least $p+q$. Thus in this case, the pure part of $H_G^\bullet(\mathrm{pt}) \otimes H^\bullet(X)$ is a subalgebra of the E_2-page, the differential d_2 vanishes on this subalgebra, d_3 vanishes on the image of the subalgebra in E_3, et cetera. Thus the pure part of $H_G^\bullet(X)$ is a quotient of the pure part of $H_G^\bullet(\mathrm{pt}) \otimes H^\bullet(X)$. □

In a way it is overkill to bring in equivariant cohomology here. Instead of dividing by $\mathbb{G}_m$ by rescaling the first coordinate, one could separate into cases (i.e. stratify the open set U): if the first coordinate of p_5 is not ∞ then we get one stratum where we fix coordinates by taking this first coordinate to be 1, if the first coordinate of p_5 is ∞ but the first coordinate of p_6 isn't then we get another where p_6 has first coordinate 1, and if the first coordinate of both p_5 and p_6 is ∞ we get a third stratum where p_7 has this first coordinate. Then each stratum could be approached in the same way that we studied the cohomology of U.

The mixed Hodge structure on equivariant cohomology is constructed in [Deligne 1974], although Deligne never states this explicitly. If G acts on X then there is a simplicial object whose realization is a model of the Borel construction $EG \times_{BG} X$. This simplicial object is given by

$$X, \qquad G \times X, \qquad G \times G \times X, \qquad G \times G \times G \times X, \ldots$$

in a way explained in []. Now Deligne does define the mixed Hodge structure on a simplicial algebraic variety, so we can define a mixed Hodge structure on $H_G^\bullet(X)$ by identifying it with the cohomology of the realization of the above simplicial Borel construction.

References

de Cataldo, Mark Andrea A. and Migliorini, Luca (2009). "The decomposition theorem, perverse sheaves and the topology of algebraic maps". *Bull. Amer. Math. Soc. (N.S.)* **46** (4), 535–633.

Deligne, Pierre (1974). "Théorie de Hodge. III". *Inst. Hautes Études Sci. Publ. Math.* (44), 5–77.

Petersen, Dan (2010). "Cohomology of local systems on loci of d-elliptic abelian surfaces". To appear in *Michigan Math. J.* Preprint. arXiv:1004.5462.

Petersen, Dan (2013). "The ℓ-adic cohomology of local systems on $\mathcal{A}_2$". Preprint. arXiv:1310.2508.

Petersen, Dan and Tommasi, Orsola (2012). "The Gorenstein conjecture fails for the tautological ring of $\overline{M}_{2,n}$". To appear in *Invent. Math.* Preprint. arXiv:1210.5761.

Tavakol, Mehdi (2011). "The tautological ring of the moduli space $M_{2,n}^{rt}$". Preprint. arXiv:1101.5242.

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