COHOMOLOGY OF MODULI OF STABLE POINTED CURVES OF LOW GENUS. IV

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In this final talk I will mostly follow my paper [Petersen 2012], where I proved the following theorem that was announced in [Getzler 1997]: the even cohomology of $\overline{\mathcal{M}}_{1,n}$ is spanned additively by strata, and all relations are obtained by pulling back WDVV and Getzler's relation. To make the last part more precise, consider a dual graph Γ corresponding to a stratum $\mathcal{M}(\Gamma)$ of $\overline{\mathcal{M}}_{1,n}$. Choose a vertex and four incident half-edges. By pulling back the WDVV relation from $\overline{\mathcal{M}}_{0,4}$ (if this vertex has genus zero) or the Getzler relation from $\overline{\mathcal{M}}_{1,4}$ (if this vertex has genus one) one obtains a relation between strata classes of codimension one (resp. two) on $\overline{\mathcal{M}(\Gamma)}$.

In the end I discuss why there will be no such simple result when $g = 2$.

1. Résumé of genus zero

Suppose X is a filtered algebraic variety, $\cdots \subset X_p \subset X_{p+1} \subset \cdots = X$, each X_p closed in X_{p+1}. Then there is a spectral sequence for Borel–Moore homology,

$$E^1_{pq} = H_{p+q}(X_p \setminus X_{p-1}) \implies H_{p+q}(X).$$

We apply this to the filtration by topological type, so that $X_p \setminus X_{p-1}$ is the disjoint union of all p-dimensional strata:

$$X_p \setminus X_{p-1} = \coprod_\Gamma \mathcal{M}(\Gamma)$$

where

$$\mathcal{M}(\Gamma) = \prod_i \mathcal{M}_{0,n_i}.$$

Now $H^k(\overline{\mathcal{M}}_{0,n})$ is pure of weight $-k$ (since $\overline{\mathcal{M}}_{0,n}$ is smooth and projective). Then only the part of $H_k\mathcal{M}(\Gamma)$ that is pure of weight $-k$ can survive to E_∞. By Künneth this is the same as determining the pure part of $H_\bullet\mathcal{M}_{0,n}$, and we saw in the first talk that this was only the fundamental class; so the homology of $\overline{\mathcal{M}}_{0,n}$ is spanned by the cycle classes of strata. To figure out the relations we should determine when there's a differential giving a relation between different strata classes. Such a differential would need to start from $\mathfrak{gr}^W_{-i+1} H_i\mathcal{M}(\Gamma)$, so we need the *second lowest weight* part of the homology. (The 'second lowest weight' will be the whole theme of this last talk.)

We determined the weights in the homology of $\mathcal{M}_{0,n}$ and found the picture seen in Figure 1. We see that it degenerates after the first differential and that all relations come from codimension 1. In cohomology, this means that relations come from $H^1(\mathcal{M}_{0,n})$. Actually we found that $H^k(\mathcal{M}_{0,n})$ was pure of weight $2k$, so if we want cohomology that's of weight "pure+1" then we have to look in degree 1. We wanted to prove that all relations are pulled

2	·	·	−4	−4
1	·	−2	−2	−2
0	0	0	0	0
	0	1	2	3

Figure 1. Weights at E^1_{pq} of filtration spectral sequence for $\overline{\mathcal{M}}_{0,n}$.

back from $\mathcal{M}_{0,4}$ and this was the same as proving that $H^1(\mathcal{M}_{0,n})$ was spanned by classes pulled back from $H^1(\mathcal{M}_{0,4})$.

2. Genus one: statement of results

Now let's apply the same filtration to $\overline{\mathcal{M}}_{1,n}$. We find by the same arguments that generators for the homology come from $W_{-i}H_i\mathcal{M}(\Gamma)$ and relations from $\mathfrak{gr}^W_{-i+1}H_i\mathcal{M}(\Gamma)$. Thus we need to determine (let's now switch to cohomology) $W_iH^i(\mathcal{M}_{1,n})$ and $\mathfrak{gr}^W_{i+1}H^i(\mathcal{M}_{1,n})$, since we already know the weights in genus zero.

We will prove that

$$W_iH^i(\mathcal{M}_{1,n}) = \begin{cases} \text{fundamental class in degree zero} & i \text{ even} \\ \text{cusp form classes} & i \text{ odd} \end{cases}.$$

In fact we've already seen that the even cohomology of $\overline{\mathcal{M}}_{1,n}$ is spanned by strata and the odd cohomology of $\overline{\mathcal{M}}_{1,n}$ consists of cusp form classes, so this is not new. But we will see a different proof.

Then we also prove that

$$\mathfrak{gr}^W_{i+1}H^i(\mathcal{M}_{1,n}) = \begin{cases} \text{only in degree } i = 3 & i \text{ odd} \\ \text{don't really care} & i \text{ even} \end{cases}.$$

Note that only $\mathfrak{gr}^W_{i+1}H^i(\mathcal{M}_{1,n})$ for odd i gives relations between even classes, the other ones give relations between odd classes. This is clear if you think in terms of weights: the spectral sequence is compatible with weights, and we want to kill a class in even weights, so we need a 'relation' with even weight as well, so $i+1$ must be even.

The space $\mathfrak{gr}^W_4H^3(\mathcal{M}_{1,4})$ is 1-dimensional and gives a single codimension 2 relation on $\overline{\mathcal{M}}_{1,4}$, the Getzler relation. We will prove that $\mathfrak{gr}^W_4H^3(\mathcal{M}_{1,n})$ is in general spanned by classes pulled back from $\mathfrak{gr}^W_4H^3(\mathcal{M}_{1,4})$, so that all relations between strata are spanned (additively) by the pullbacks of the WDVV relation and Getzler's relation, applied to a vertex of a dual graph.

3. The fibration $g\colon \mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$.

We want to study the forgetful map $g\colon \mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$. The Leray spectral sequence just says

$$0 \to H^1(\mathcal{M}_{1,1}, \mathrm{R}^{q-1}g_*\mathbf{Q}) \to H^q(\mathcal{M}_{1,n}) \to H^0(\mathcal{M}_{1,1}, \mathrm{R}^q g_*\mathbf{Q}) \to 0.$$

Before we have considered variations of Hodge structure. The sheaves $\mathrm{R}^q g_*\mathbf{Q}$ are variations of *mixed* Hodge structure. The reason is that g is not proper, only smooth. This means that the local systems themselves have a weight filtration. For all i we have

$$\mathfrak{gr}_i^W \mathrm{R}^q g_*\mathbf{Q}$$

a PVHS of weight i, defined by the monodromy action on $\mathfrak{gr}_i^W H^q(\text{fiber})$. So the weight filtrations vary in a controlled way in families.

Now recall that if $\mathbb{V}$ is a PVHS of weight w, then $H^i(\mathbb{V})$ has weights $\geq w+i$. This tells us that we only need to consider the possible contributions from

$$\mathfrak{gr}_q^W \mathrm{R}^q g_*\mathbf{Q} \qquad \text{and} \qquad \mathfrak{gr}_{q+1}^W \mathrm{R}^q g_*\mathbf{Q}$$

to $\mathfrak{gr}_{i+1}^W H^i(\mathcal{M}_{1,n})$ — all other parts of the weight filtration of $\mathrm{R}^q g_*\mathbf{Q}$ will have cohomology with too high weight!

Let us now show that the first of these, $\mathfrak{gr}_q^W \mathrm{R}^q g_*\mathbf{Q}$, does not contribute. We can write this PVHS in terms of the local systems $\mathbb{V}_k$. I claim that there is no local system $\mathbb{V}_k$ with $\mathfrak{gr}_{i+k+1}^W H^i(\mathcal{M}_{1,1}, \mathbb{V}_k) \neq 0$. Indeed if $i=0$ then we are only considering the trivial local system and it is obvious. If $i=1$ then Eichler–Shimura says that $H^1(\mathcal{M}_{1,1}, \mathbb{V}_k)$ has a summand of weight $k+1$ (the pure part, from cusp forms) and a summand of weight $2k+2$ (the impure part, coming from Eisenstein series). If the impure part has weight 1 more than the pure part then $k=0$ and $\mathbb{V}_k$ is the trivial local system, impossible. (In other words there are no weight 2 Eisenstein series for $\mathrm{SL}(2,\mathbf{Z})$).

Thus only $\mathfrak{gr}_{q+1}^W \mathrm{R}^q g_*\mathbf{Q}$ will contribute nontrivially to $\mathfrak{gr}_{i+1}^W H^i(\mathcal{M}_{1,n})$. Moreover, only the trivial local system can occur if we demand that $i+1$ is even, since this is the only one with pure cohomology of even weight. So we need to understand

$$\mathfrak{gr}_{q+1}^W H^q(\text{fiber})^{\mathrm{SL}_2}.$$

This turns out not to be completely easy, which is why I postponed talking about this until the last lecture.

4. Totaro's dga

Recall one of the motivating examples of mixed Hodge theory from the first lecture, where we considered an embedding $U \hookrightarrow X$ of a smooth variety into a smooth compactification where the complement has strict normal crossings. We argued that the Leray spectral sequence should degenerate after the first differential by a weight argument: all intersections of boundary components are smooth and projective, so their cohomology is pure, and then all differentials go between cohomology groups of different weights.

Now proving Leray degeneration in this way would actually be circular, because the definition of a mixed Hodge structure on $H^\bullet(U)$ given in [Deligne 1971] uses the fact that this sequence degenerates after the first differential.

However, we could consider an analogous situation where $X \setminus U$ is not strict normal crossings, but where it's still true that all intersections of components are smooth and projective. Then this weight argument would imply degeneration of the spectral sequence after the first differential.

In particular, for a variety X we let $F(X, n)$ be the configuration space of n distinct ordered points in X. We consider the Leray spectral sequence for

$$F(X, n) \hookrightarrow X^n.$$

This is certainly not snc: the complement is not even a divisor unless X is a curve, and in any case the intersections of boundary divisors aren't transverse. Nevertheless this weight argument will prove that there is only a single nontrivial differential.

This was first noticed by Totaro in [Totaro 1996]. Moreover, he determined an explicit presentation of the first nontrivial page of the spectral sequence, considered as a differential graded algebra. This works more generally for an arbitrary oriented manifold X, say r-dimensional. We now define Totaro's dga $\Lambda_X(n)$. When X is a complex algebraic variety whose cohomology is pure, the cohomology of $\Lambda_X(n)$ is $H^\bullet(F(X, n))$.

We define

$$\Lambda_X(n) = H^\bullet(X^n)[\omega_{ij}]/(\text{relations})$$

where $1 \leq i \neq j \leq n$ and the relations are

$$(p_i^*\alpha - p_j^*\alpha)\omega_{ij} = 0,$$

(where p_i denotes projection onto the ith factor),

$$\omega_{ij}^2 = 0,$$

$$\omega_{ij} = (-1)^r \omega_{ji},$$

and the "Arnol'd relation"

$$\omega_{ij}\omega_{ik} + \omega_{jk}\omega_{ji} + \omega_{ki}\omega_{kj} = 0.$$

When $r = 2$ it is useful to think of ω_{ij} as analogous to the differential form $d \log(z_i - z_j)$ that appears in the cohomology of $F(\mathbf{C}, n)$. The differential on this algebra is defined by

$$d\alpha = 0 \text{ for } \alpha \in H^\bullet(X^n),$$

and

$$d\omega_{ij} = \Delta_{ij},$$

where Δ_{ij} is the class of the submanifold of X^n where the ith and jth coordinate coincide. This dga is bigraded, where $H^i(X^n)$ has degree $(i, 0)$ and ω_{ij} has degree $(0, r - 1)$.

We get a picture something like Figure 2. In the second row we sum over all $\binom{n}{2}$ variables ω_{ij}, equivalently, over all diagonals. In the third row we sum over all degree 2 monomials in the ω_{ij} (modulo Arnol'd relation). This is the same as summing over all intersections of two diagonals.

Let us now say that X is a smooth projective variety of dimension k (so $r = 2k$). Then we can give Totaro's dga a mixed Hodge structure by declaring each ω_{ij} to be of type $\mathbf{Q}(-k)$, and by taking the natural Hodge structure on $H^\bullet(X^n)$. Then the differential is compatible with the mixed Hodge structure on $\Lambda_X(n)$, and with the mixed Hodge structure on $H^\bullet(F(X, n))$. This is because Totaro's dga is just an explicit way of writing down the Leray spectral sequence, and the Leray spectral sequence is compatible with mixed Hodge theory.

$2(r-1)$	$\oplus H^0(X^{n-2})$	$\oplus H^1(X^{n-2})$	$\oplus H^2(X^{n-2})$	$\oplus H^3(X^{n-2})$
$r-1$	$\oplus H^0(X^{n-1})$	$\oplus H^1(X^{n-1})$	$\oplus H^2(X^{n-1})$	$\oplus H^3(X^{n-1})$
0	$H^0(X^n)$	$H^1(X^n)$	$H^2(X^n)$	$H^3(X^n)$
	0	1	2	3

Figure 2. Entries at E_r^{pq}. The differential d_r is the first nontrivial one.

Finally we apply this to the fibers of g. The fiber of g over the moduli point $[E]$ is $F(E,n)/E$, the configuration space of n points on E modulo the translation action. By translating the first point to the origin, this quotient is isomorphic to $F(E_0, n-1)$, where E_0 is the punctured elliptic curve. E_0 is smooth but not proper, however, its cohomology is still pure so we can apply Totaro's dga to compute the cohomology.

Observe that the bottom row of the spectral sequence, i.e. the part of Totaro's dga in bidegree $\Lambda_X^{\bullet,0}(n)$, will give the lowest weight part of the cohomology of $F(X,n)$, and the second row of the spectral sequence (i.e. bidegree $\Lambda_X^{\bullet,1}(n)$) gives the second lowest weight part of the cohomology of $F(X,n)$. Thus we should understand these two rows in the case $X = E_0$.

I proved the following propositions:

Proposition 4.1. $H^{p,0}\Lambda_{E_0}(n) \cong \binom{n}{p}\mathbb{V}_p$.

This implies in particular that the only pure even cohomology of $\mathcal{M}_{1,n}$ is the fundamental class. However we already knew that. The real reason for proving this is that this statement gets used in the proof of the next statement.

Proposition 4.2. $H^{p,1}\Lambda_{E_0}(n)^{\mathrm{SL}_2}$ *is nonzero only when* $p = 2$.

The proofs I gave in the lecture were identical to the ones given in [Petersen 2012], so I will not repeat them in these notes.

All in all, this tells us that all 'new' relations between strata classes in genus one must have codimension 2. (Whereas WDVV is a codimension 1 relation.) To conclude we just need to prove that $\mathfrak{gr}_4^W H^3(\mathcal{M}_{1,n})$ is spanned by classes pulled back from $\mathfrak{gr}_4^W H^3(\mathcal{M}_{1,4})$ (which we have already seen is 1-dimensional, spanned by Getzler's relation). Equivalently, that $H^{p,1}\Lambda_{E_0}(n)^{\mathrm{SL}_2}$ is spanned by classes pulled back from $H^{p,1}\Lambda_{E_0}(3)^{\mathrm{SL}_2}$ along forgetful maps.

For any injection $\{1,...,n\} \hookrightarrow \{1,...,n+k\}$ there is a point-forgetting map $F(X,n+k) \to F(X,n)$ and a pullback map in cohomology $H^\bullet(F(X,n)) \to H^\bullet(F(X,n+k))$. There is also an obvious embedding $\Lambda_X(n) \hookrightarrow \Lambda_X(n+k)$. These are compatible with each other. For any r-manifold X, it is very easy to see that the bidegree $(p, k(r-1))$ part of $\Lambda_X(n)$ is spanned by classes that are pulled back from $n = p + 2k$. Indeed a degree k monomial in the ω_{ij} can only involve $2k$ indices, and a degree p cohomology class in $H^\bullet(X^n)$ can only have p distinct non-identity components in a Künneth decomposition.

In our case this observation says that $H^{p,1}\Lambda_{E_0}(n)$ is spanned by classes pulled back from $H^{p,1}\Lambda_{E_0}(4)$. But we wanted to get classes from $H^{p,1}\Lambda_{E_0}(3)$! The reason for the mismatch is that in identifying the fibers of $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$ with $F(E_0, n-1)$ we've fixed one marked point, which we can not forget, so we've lost one of the point-forgetting maps.

In any case this is not a problem. What we have said so far tells us that all tautological relations in genus one are pulled back from $\overline{\mathcal{M}}_{1,5}$, rather than $\overline{\mathcal{M}}_{1,4}$. The space $\overline{\mathcal{M}}_{1,5}$ is small enough that one can verify with computer that there are no new relations here, and in fact this has already been done by several people: Pavel Belorousski, Stephanie Yang, Aaron Pixton, maybe others.

All in all this proves the result: the even cohomology of $\overline{\mathcal{M}}_{1,n}$ is spanned additively by strata, and all relations are obtained by pulling back WDVV and Getzler's relation.

5. Genus two

One expects the situation to become more complicated in genus two. As explained already in this course, there is in a sense a 'reasonable' description of the generators of the even cohomology of $\overline{\mathcal{M}}_{2,n}$, which would be the analogue of the first half of Getzler's result. But let's say we also wanted to understand the relations. We should consider $\mathfrak{gr}^W_{i+1} H^i(\mathcal{M}_{2,n})$ for odd i.

First of all, we would need to understand the cohomology of $F(X, n)$, where X is now a compact genus two curve. This adds two complications: the representation theory of $\mathrm{Sp}(4)$ is messier than the representation theory of $\mathrm{SL}(2)$, and it really simplified the situation a lot that we only considered configurations of points on a punctured curve. I don't know anything about what the bottom two rows of the Leray spectral sequence will look like in this case (although I haven't thought much about it, either).

There is also the matter that it's not clear whether or not the Leray spectral sequence for $\mathcal{M}_{2,n} \to \mathcal{M}_2$ degenerates or not (or more generally the Leray spectral sequence for $\mathcal{M}_{g,n} \to \mathcal{M}_g$). On a sidenote, I would be very interested if someone were to prove such a statement, or give an example of a nontrivial differential in one of these spectral sequences! Anyway, to avoid having to deal with this one should probably consider instead $\mathcal{M}^{\mathfrak{rt}}_{2,n} \to \mathcal{M}_2$, where degeneration is guaranteed, and then try to understand separately the relations on $\mathcal{M}^{\mathfrak{rt}}_{2,n}$.

However, a more serious issue is the following. In genus one, we found that $\mathfrak{gr}^W_q \mathrm{R}^q g_* \mathbf{Q}$ could be ignored when determining the relations. The reason is that there are local systems on $\mathcal{M}_{1,1}$ with impure cohomology, but that there are no local systems with cohomology exactly "pure weight $+1$". This is no longer true in genus two!

The first example is the local system $\mathbb{V}_{2,2}$ with $H^1(\mathcal{M}_2, \mathbb{V}_{2,2}) = \mathbf{Q}(-3)$. The cohomology of this local system was first computed by Orsola Tommasi. When she did that she realized it should correspond to a codimension three relation on $\overline{\mathcal{M}}_{2,4}$ and assigned the task of finding this relation to Nicola Pagani and Nicola Tarasca. What they eventually discovered is that this relation is just the pullback of the Belorousski–Pandharipande relation with a ψ-class! So it was not so new after all.

But there are more examples. After [Petersen 2013] we can see that $\mathfrak{gr}^W_{4a+6} H^3(\mathcal{M}_2, \mathbb{V}_{2a+1,2a+1})$ is of Tate type and its dimension is the dimension of the space of cusp forms of weight $4a+6$ for $\mathrm{SL}(2, \mathbf{Z})$. All these classes produce relations on $\overline{\mathcal{M}}_{2,n}$, where $n = 4a+2$. They will be in high codimension, too: these are codimension $2a+3$ relations.

The first of these (BP$\cdot\psi$) is in fact supported on $\overline{\mathcal{M}}_{2,4} \setminus \mathcal{M}^{\mathsf{rt}}_{2,4}$. The reason is that $H^1(\mathcal{M}_2, \mathbb{V}_{2,2})$ doesn't just define a class in $H^5(\mathcal{M}_{2,4})$ but it naturally defines a cohomology class in $H^5(\mathcal{M}^{\mathsf{rt}}_{2,4})$. For the second ones even more is true: these classes are in the image of the restriction map

$$\mathfrak{gr}^W_{4a+6} H^3(\mathcal{M}^{\mathsf{ct}}_2, \mathbb{V}_{2a+1,2a+1}) \to \mathfrak{gr}^W_{4a+6} H^3(\mathcal{M}_2, \mathbb{V}_{2a+1,2a+1})$$

which means that the corresponding relations are supported on $\overline{\mathcal{M}}_{2,n} \setminus \mathcal{M}^{\mathsf{ct}}_{2,n}$. This has the striking consequence that all these relations will produce relations in *genus one* Gromov–Witten theory, even though they are relations between tautological classes on $\overline{\mathcal{M}}_{2,n}$!!! It seems therefore that it would be very useful to be able to explicitly write down these tautological relations.

In any case, the conclusion is that there is no simple analogue of Getzler's result in genus two, in the sense of a finite list of tautological relations such that all others are obtained by pullback. Such a list would have to be infinite. In fact the above discussion only touched upon relations supported on $\overline{\mathcal{M}}_{2,n} \setminus \mathcal{M}^{\mathsf{rt}}_{2,n}$; even the ones supported away from compact type are infinitely generated. On the other hand it also makes sense to study only tautological relations on $\mathcal{M}^{\mathsf{rt}}_{2,n}$, or $\mathcal{M}^{\mathsf{ct}}_{2,n}$, say. Maybe *these* are generated by a finite list of relations — it seems likely that this is something one could prove by studying $\mathfrak{gr}^W_{k+1} H^k(F(X,n))^{\mathrm{Sp}(4)}$, as we did here.

Finally I should say that I am working here with a very restrictive notion of a 'new' relation. We should probably allow more operations on our relations than just pulling back, for instance pushing forward along forgetful maps or multiplying with something arbitrary. It is not clear whether either of these operations can be described in a nice way in the local systems picture. Already the fact that the relation obtained from $H^1(\mathcal{M}_2, \mathbb{V}_{2,2})$ was BP$\cdot\psi$ — whereas the BP relation comes from the trivial local system and $\mathfrak{gr}^W_4 H^3(F(X,n))^{\mathrm{Sp}(4)}$ — demonstrates a certain incompatibility of the multiplicative structure on the tautological ring with the techniques used in this mini-course.

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