The virtual K-theory of Quot schemes of surfaces

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$\boxed{1}$ Introduction

We consider pairs $(M_n, \mathcal{V}_n)$ where $\mathcal{V}_n \longrightarrow M_n$ is K-theory class.

We form the generating series

$$Z = \sum_n q^n \chi(M_n, \mathcal{V}_n).$$

Examples to keep in mind:

$\boxed{\mathrm{I}}$ $M_n = \mathbb{P}^n, \quad \mathcal{V}_n = \mathcal{O}(a)$

$$Z = \sum q^n \chi(\mathbb{P}^n, \mathcal{O}(a)) = \frac{1}{(1-q)^{a+1}}.$$

rational function

$\boxed{\mathrm{II}}$ $M_n = \mathbb{P}^a, \quad \mathcal{V}_n = \mathcal{O}(n)$

$$Z = \sum q^n \chi(\mathbb{P}^a, \mathcal{O}(n)) = \frac{1}{(1-q)^{a+1}}.$$

rational function

Note the symmetry

$$\chi(\mathbb{P}^n, \mathcal{O}(a)) = \chi(\mathbb{P}^a, \mathcal{O}(n)).$$

A more interesting example

$\boxed{\mathrm{III}}$ $M_n = X^{[n]}$, X smooth projective surface

We could take $\mathcal{V}_n = \mathcal{O}$, but better $\mathcal{V}_n = \alpha^{[n]}$, $\alpha \in K(X)$:

$$\alpha^{[n]} = p_!(q^*\alpha \cdot \mathcal{O}_Z) \quad \text{where} \quad \begin{tikzcd} Z \arrow[r, hook] & X^{[n]} \times X \arrow[dl, "p"] \arrow[dr, "q"] & \\ X^{[n]} & & X \end{tikzcd}$$

Question : Compute K-theoretic invariants of $X^{[n]}$?

Some examples

$\boxed{\text{I}}$ $$\sum q^n \chi(X^{[n]}, \alpha^{[n]}) = \chi(\alpha) \cdot \frac{q}{(1-q)^{\chi(\mathcal{O}_X)}}$$ rational

[Ellingsrud – Göttsche – Lehn].

$\boxed{\text{II}}$ $$\sum q^n \chi(X^{[n]}, \wedge^k \alpha^{[n]}) = \binom{\chi(\alpha)}{k} \cdot \frac{q^k}{(1-q)^{\chi(\mathcal{O}_X)}}$$ rational

[Danila, Scala, Arbesfeld] , rk $\alpha = 1$.

also rk $\alpha = -1$.

$\boxed{\text{III}}$ $$\sum q^n \chi(X^{[n]}, \det \alpha^{[n]}) . = \text{Verlinde series}$$

When rk $\alpha = 0$, the answer is $\dfrac{1}{(1-q)^{\chi(\det \alpha)}}$ rational

Answer known rk $\alpha = \pm 1$ [Ellingsrud – Göttsche – Lehn].

Answer conjectured for rk $\alpha = \pm 2, \pm 3$ [Marian – O – Pandharipande]

This is based on a conjectural Segre / Verlinde correspondence.

[Johnson, Marian – O – Pandharipande, Göttsche – Kool]

Segre – Verlinde correspondence

$$V_\alpha^{\mathrm{Hilb}} = \sum q^n \chi\left(X^{[n]}, \det \alpha^{[n]}\right).$$

$$S_\alpha^{\mathrm{Hilb}} = \sum q^n \int_{X^{[n]}} s\left(\alpha^{[n]}\right)$$

$$V_\alpha^{\mathrm{Hilb}} \longleftarrow\!\!-\!\!-\!\!\longrightarrow S_{\tilde{\alpha}}^{\mathrm{Hilb}}$$

after an explicit change of variables, & for pairs $(\alpha, \tilde{\alpha})$ related explicitly.

For rank $\alpha = 2$, this gives the conjecture

$$V_\alpha^{\mathrm{Hilb}} = A_1^{\chi(\det \alpha)} \cdot A_2^{\chi(\mathcal{O}_X)} \cdot A_3^{c_1(\alpha).K_X} \cdot A_4^{K_X^2}$$

A_i are algebraic functions.

$$q = t(1+t)^3$$

$$A_1 = 1+t$$

$$A_2 = \frac{(1+t)^2}{(1+4t)^{\frac{1}{2}}}$$

$$A_3 = \frac{1+\sqrt{1+4t}}{2(1+t)}$$

$$A_4 = (1+4t)^{\frac{1}{2}} \cdot \left(\frac{1+\sqrt{1+4t}}{2}\right)^{-3}.$$

2 Higher rank sheaves

We wish to generalize. Two possible directions:

I. replace $X^{[n]}$ by moduli M_n of higher rank sheaves

II. replace $X^{[n]}$ by a more general Quot scheme.

Results for moduli of sheaves

a. X rational surface, say $X = \mathbb{P}^2$

$$M_n = M_{\mathbb{P}^2}(rk = 2,\ c_1 = 0,\ c_2 = n).$$

$$w \longrightarrow rk\, w = 0,\quad c_1(w) = k H,\quad \chi(v \cdot w) = 0.$$

$$\Theta_w \longrightarrow M_n, \quad \text{determinant line bundle.}$$

$$\Theta_w = \det R p_! (\underbrace{\mathcal{U}}_{\text{universal sheaf}} \cdot q^* w)^{-1}.$$

universal sheaf $\mathcal{U} \longrightarrow M_n \times X$.

Conjecture (Göttsche)

$$\sum q^n \chi(M_n, \Theta_w) = \frac{P_k(q)}{(1-q)^{\binom{k+2}{2}}}$$

- verified for $k \leq 11$ & several other cases.

5 Virtual K-theoretic invariants

Conjecture (Göttsche – Kool – Williams)

X simply connected, $p_g > 0$

Seiberg-Witten invariant.

$$\chi^{vir}(M_n, \Theta_w) \rightsquigarrow \frac{2^{1-p_g+K^2}}{(1-g^2)^{\alpha}} \cdot \sum_a SW(a) \cdot \left(\frac{1+g}{1-g}\right)^{\beta}$$

rank 2.

no strictly semistables

$$\alpha = \frac{(c_1(w)-K)^2_X}{2} + \chi(\mathcal{O}_X).$$

$$\beta = \left(\frac{K_X}{2} - a\right)\left(c_1(w) - K\right)_X$$

$\boxed{3}$ Quot schemes over surfaces

Fix N, β, n. The Quot scheme $\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)$.

$$0 \longrightarrow S \longrightarrow \mathbb{C}^N \otimes \mathcal{O}_X \longrightarrow Q \longrightarrow 0$$

$$\mathrm{rk}\, Q = 0, \quad c_1(Q) = \beta, \quad \chi(Q) = n.$$

Example $N = 1$, $\mathrm{Quot}(\mathbb{C}^1, \beta, n) \cong \underbrace{X^{[m]}}_{\text{points}} \times \underbrace{\mathrm{Hilb}_\beta}_{\text{curves}}$

where $S = I_Z(-C) \hookrightarrow \mathcal{O}_X$ for $Z \in X^{[m]}$, $C \in \mathrm{Hilb}_\beta$.

Fact (Marian - O - Pandharipande)

$\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)$ admits a 2-term perfect obstruction theory.

rank 0.

$$\mathrm{Ext}^2(S, Q) = \mathrm{Ext}^0(Q, S \otimes K_X)^\vee = 0$$

$$\mathrm{vdim} = \chi(S, Q) = Nn + \beta^2.$$

Fact' (O - Pandharipande, Schultheis) If $p_g = 0$, the virtual fundamental class exists even if $\mathrm{rk}\, Q > 0$.

Results in cohomology

$$\int_{[\mathrm{Quot}]^{vir}} c(T^{vir}\,\mathrm{Quot}).$$

$$\boxed{\mathrm{I}} \quad \sum_n q^n \, e^{vir}\left(\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)\right) \qquad [\text{O- Pandharipande}]$$

$$\boxed{\mathrm{II}} \quad \sum_n q^n \, \chi_{-y}^{vir}\left(\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)\right) \qquad [\text{Lim}]$$

$\boxed{\mathrm{III}}$ descendant series. Fix $\alpha_1, \ldots, \alpha_\ell$, $k_1, \ldots, k_\ell$

$$\sum_n q^n \int_{[\mathrm{Quot}(\mathbb{C}^N, \beta, n)]^{vir}} c(T^{vir}\,\mathrm{Quot})\, \mathrm{ch}_{k_1} \alpha_1^{[n]} \cdots \mathrm{ch}_{k_\ell} \alpha_\ell^{[n]}.$$

[Johnson – O – Pandharipande]

Conjecture These are given by rational functions.

Known cases

$\boxed{\mathrm{I}}$ $\boxed{\mathrm{II}}$ $\boxed{\mathrm{III}}$ when $\beta = 0$ [OP, L, JOP]

$\boxed{\mathrm{I}}$ $\boxed{\mathrm{II}}$ — for $p_g > 0$ [OP, L]

$\boxed{\mathrm{I}}$ — $\boxed{\mathrm{III}}$ for $p_g = 0$, $N = 1$ [JOP].

4 Virtual K-theory

S scheme with 2-term perfect obstruction theory.

$\mathcal{O}_S^{vir}$ virtual structure sheaf.

$$\chi^{vir}(S, W) = \chi(S, W \otimes \mathcal{O}_S^{vir}).$$

Example S smooth, Obs = obstruction bundle

$$\mathcal{O}_S^{vir} = \Lambda^{\bullet}_{-1} \mathrm{Obs}^{\vee}$$

Notation $\Lambda_t W = \sum_k t^k \Lambda^k W.$

When $S = X^{[n]}$,

$$\mathrm{Obs} = \mathrm{Ext}^1(I_Z, \mathcal{O}_Z) = \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$$

$$= \mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X)^{\vee}$$

$$= K_X^{[n]\vee}$$

$$\Rightarrow \mathcal{O}_{X^{[n]}}^{vir} = \sum_k (-1)^k \Lambda^k K_X^{[n]}$$

$\Rightarrow \chi^{vir}(X^{[n]}, \_\_)$ are related to $\chi(X^{[n]}, \_\_)$.

Rationality in K-theory

Fix $\alpha \in K(X)$. Define

$$\alpha^{[n]} = p_! \left(q^* \alpha \cdot \mathcal{Q} \right) \quad \text{where}$$

$$\begin{tikzcd} & \mathcal{Q} \arrow[d] & \text{universal quotient} \\ & X \times \mathrm{Quot} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Quot} & & X \end{tikzcd}$$

For $\alpha_1, \ldots, \alpha_\ell$, $k_1, \ldots, k_\ell$ define

$$Z_{X,N,\beta} = \sum q^n \chi^{vir}\left(\mathrm{Quot}_X(\mathbb{C}^N, \beta, n), \wedge^{k_1} \alpha_1^{[n]} \otimes \cdots \otimes \wedge^{k_\ell} \alpha_\ell^{[n]} \right)$$

Conjecture $Z_{X,N,\beta}$ is given by a rational function.

Results

Theorem A Conjecture is true for $p_g > 0$.

Theorem B When $N = 1$, $p_g > 0$, the shifted series

$q^{\beta \cdot K_X} Z$ has pole only at $q = 1$ of order $\leq 2(k_1 + \cdots + k_\ell)$.

Theorem C If $p_g = 0$, the series

$$Z = \sum q^n \chi^{vir}(\mathrm{Quot}, \alpha^{[n]}) \text{ is a rational function}$$

$\boxed{\text{i}}$ $N = 1$, X simply connected

$\boxed{\text{ii}}$ $N = 2$, $\mathrm{rk}\, Q = 1$, $\beta = 0$

Examples

$$\sum q^n \chi^{vir}(\mathrm{Quot}_X(\mathbb{C}^N, \beta, n), \alpha^{[n]}) :$$

Example $\beta = 0$

$$\sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(X^{[n]}, \alpha^{[n]}) = -\mathrm{rk}\ \alpha \cdot K_X^2 \cdot \frac{q^2}{(1-q)^2} - \langle K_X, c_1(\alpha)\rangle \cdot \frac{q}{1-q}.$$

$$\sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, n), \alpha^{[n]}) = N \sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(X^{[n]}, \alpha^{[n]}).$$

Example $X \to C$ elliptic, β fiber class

$$\sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \alpha^{[n]}) = \mathsf{sw}_\beta \cdot \left(-N \cdot \langle K_X, c_1(\alpha)\rangle \cdot \frac{q}{1-q} + \langle \beta, c_1(\alpha)\rangle \cdot \frac{1}{1-q}\right)$$

$$\mathsf{sw}_\beta = \sum_{\beta_1+\ldots+\beta_N=\beta} \mathsf{SW}(\beta_1)\cdots\mathsf{SW}(\beta_N).$$ Seiberg-Witten invariants

Example X minimal general type, $p_g > 0$, $\beta = K_X$

$$\sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, K_X, n), \alpha^{[n]}) = \mathsf{SW}(K_X) \cdot \left(-\frac{N}{1-q}\right)^{K_X^2} \left(\sum_{i=1}^{N} z_i^{-K_X^2} \cdot \mathsf{p}_\alpha(z_i)\right)$$

$$\mathsf{p}_\alpha(z) = \langle K_X, c_1(\alpha)\rangle \left(\frac{1-(N+1)q}{1-q} - z\right)$$

$z_1 \ldots z_N$ solve $z^N - q(z-1)^N = 0$.

[5] Segre - Verlinde correspondence & symmetry

Recall

$$\left.\begin{aligned} S_\alpha^{Hilb} &= \sum q^n \int_{X^{[n]}} s(\alpha^{[n]}). \\ V_\alpha^{Hilb} &= \sum q^n \chi(X^{[n]}, \det \alpha^{[n]}). \end{aligned}\right\} \text{ non-virtual geometry}$$

$$S_\alpha^{Hilb} \longleftarrow\!\!-\!\!-\!\!\rightarrow V_{\tilde{\alpha}}^{Hilb} \quad \text{for certain pairs } (\alpha, \tilde{\alpha})$$

change of variables.

Virtual Segre - Verlinde correspondence

$$S_{N,\alpha}^{vir} = \sum q^n \int_{[\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)]^{vir}} s(\alpha^{[n]}).$$

$$V_{N,\alpha}^{vir} = \sum q^n \chi^{vir}(\mathrm{Quot}_X(\mathbb{C}^N, \beta, n), \det \alpha^{[n]}).$$

Theorem D

$$S_{N,\alpha}^{vir}((-1)^N q) = V_{N,\alpha}^{vir}(q) \quad \text{when}$$

[i] $\beta = 0$, any N

[ii] $N = 1$, $p_g > 0$, any β

[iii] $X \longrightarrow C$ minimal elliptic, β supported on fibers

Symmetry

Theorem E For $\beta = 0$, let $\alpha, \tilde{\alpha}$ such that

$$\operatorname{rk} \alpha = r, \quad \operatorname{rk} \tilde{\alpha} = N, \quad \frac{c_1(\alpha)}{\operatorname{rk} \alpha} \cdot K_X = \frac{c_1(\tilde{\alpha})}{\operatorname{rk} \tilde{\alpha}} \cdot K_X$$

the Segre series has the $r - N$ symmetry

$$S_{N,\alpha}\left((-1)^N q\right) = S_{r,\tilde{\alpha}}\left((-1)^r q\right).$$

$\rightsquigarrow r$.(rank) $\qquad \rightsquigarrow N$. (rank)

This symmetry also holds for the Verlinde series.

6 Cosection localization

Recall $\mathcal{O}^{vir}_{X^{[n]}} = \sum (-1)^k \Lambda^k K_X^{[n]}$

Question What is $\mathcal{O}^{vir}$ for $\mathrm{Quot}_X(\mathbb{C}^N, n)$, $\beta = 0$?

Assume $C \hookrightarrow X$ smooth canonical curve, $T = N_{C/X}$.

theta characteristic.

$$i : \mathrm{Quot}_C(\mathbb{C}^N, n) \longrightarrow \mathrm{Quot}_X(\mathbb{C}^N, n)$$

virtual normal bundle

$$N^{vir} = i^* T^{vir} \mathrm{Quot}_X - T\,\mathrm{Quot}_C.$$

Theorem F $\quad \mathcal{O}^{vir}_{\mathrm{Quot}_X(\mathbb{C}^N, n)} = (-1)^n i_* \mathcal{D}_n$ where

$\boxed{i}$ $\mathcal{D}_n$ is a square root of $\det N^{vir}$, and also

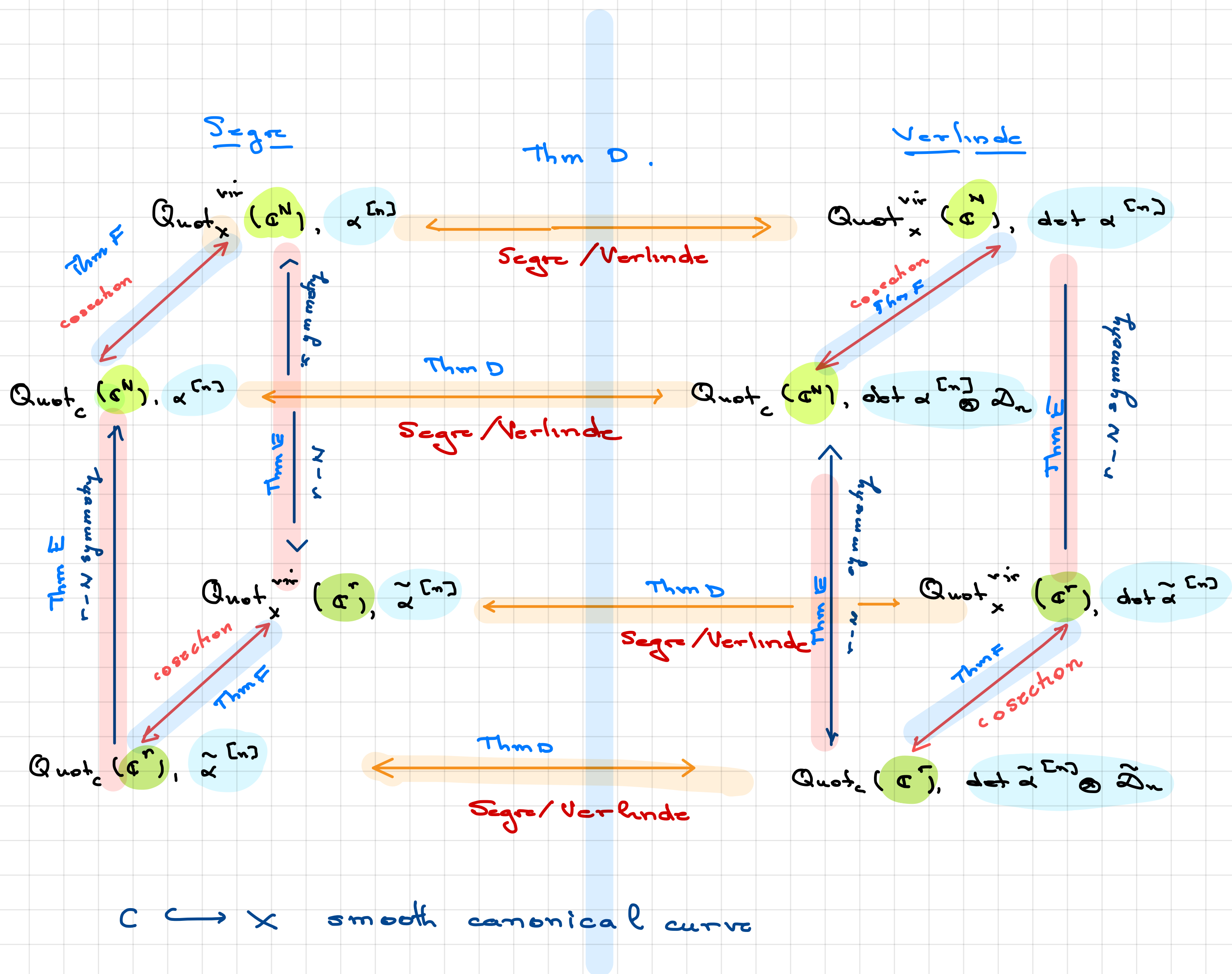
$\boxed{ii}$ $\mathcal{D}_n = p^* \det T^{[n]}$ where

$$p : \mathrm{Quot}_C(\mathbb{C}^N, n) \longrightarrow C^{[n]}$$
$$Q \longmapsto \text{support of } Q.$$

When $N = 1$ this can be checked directly.

$\boxed{7}$ **8-fold equivalence**

Using Theorems D, E, F we match 8 Segre / Verlinde invariants over surfaces (virtually) & curves.



$C \hookrightarrow X$ smooth canonical curve

Two corners: If C is a smooth projective curve,

$$\alpha = L^{\oplus r}, \quad \tilde{\alpha} = L^{\oplus N}, \quad L \longrightarrow C \text{ line bundle}$$

$$\chi\left(\mathrm{Quot}_C(\mathbb{C}^N, n), \left(\det L^{[n]}\right)^{\otimes r} \otimes \mathcal{D}_n\right) =$$

$$= \chi\left(\mathrm{Quot}_C(\mathbb{C}^r, n), \left(\det L^{[n]}\right)^{\otimes N} \otimes \mathcal{D}_n\right)$$

where $\mathcal{D}_n$ is the pullback of $\det \underbrace{T}_{\text{theta characteristic.}}^{[n]} \longrightarrow C^{[n]}$.

Question What is the geometric meaning of this symmetry?