Rationality of Descendent Series

Friday, April 3, 2020 4:50 PM

X = smooth projective surface

$\beta \in H^2(X, \mathbb{Z})$ effective

$\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)$

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O} \to Q \to 0$$

$$\operatorname{rank} Q = 0$$
$$c_1(Q) = \beta$$
$$\chi(Q) = n$$

$$\mathrm{Quot}_X(\mathbb{C}^1, 0, n) = \underline{X^{[n]}} = \mathrm{Hilb}(X, n)$$

Virtual Euler Characterics

$$T\,\mathrm{Quot} = \mathrm{Hom}(S, Q)$$
$$\mathrm{Obs} = \mathrm{Ext}^1(S, Q)$$
$$\mathrm{Ext}^2(S, Q) = \mathrm{Hom}(Q, S \otimes K)^\vee = 0$$

$$\mathrm{vdim} = \underline{\underline{Nn + \beta^2}}$$

<u>Remark</u> If $C \subset X$ is smooth cannonical curve

$$N = 1,\ \beta = 0 \qquad [X^{[n]}]^{vir} = (-1)^n [C^{[n]}]$$

<u>Define</u> $$e^{vir}(\mathrm{Quot}) \quad \int_{[\mathrm{Quot}]^{vir}} c(T^{vir}) \in \mathbb{Z}$$

$$Z_{\underline{X,N,\beta}} = \sum q^n e^{vir}(\mathrm{Quot}_X(\mathbb{C}^N, \beta, n))$$

Laurent series

Conj [Oprea Pandharipande '19]

$Z_{X,N,\beta}$ is a rational function q.

- Some cases in [OP '19] and also W. Lin

Thm 1 [J, O, P) '20

If X is simply connected, then $Z_{X,1,\beta}$ is rational.

Descendent Series

$\mathcal{Q}$ - universal quotient

$$\begin{tikzcd} & \mathrm{Quot}' \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Quot} & & X \end{tikzcd}$$

$\alpha \in K^0(X)$

$\alpha^{[n]} = R\pi_*(\mathcal{Q} \otimes \pi_2^* \alpha)$

"tautologic"

$$Z_{X,N,\beta}(\alpha_1, \underset{\substack{\uparrow \\ K^0(X)}}{\ldots}, \alpha_\ell \mid k_1, \ldots, k_\ell)$$

$$= \sum q^n \int_{[\mathrm{Quot}(\mathbb{C}^N, \beta, n)]^{vir}} ch_{k_1}(\alpha_1^{[n]}) \cdots ch_{k_\ell}(\alpha_\ell^{[n]})\, c(T^{vir})$$

Conj This is a rational function in q.

Thm 2 [JOP] $Z_{X,N,\beta=0}(\alpha_1, \ldots, \alpha_\ell \mid k_1, \ldots k_\ell)$

is rational.

Thm 3 [JOP] If X simply connected

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$$Z_{X,N=1,\beta}(\alpha_1,\dots,\alpha_\ell \mid k_1,\dots,k_\ell) \text{ is rational.}$$

Carlsson (2010) showed that non-virtual descendent series for Hilbert schemes ($N=1, \beta=0$) are quasimodular.

Conj (Pandharipande, Pixton)

$$\sum q^n \int_{[P_n(X,\beta)]^{vir}} \tau_{k_1}(\gamma_1)\cdots\tau_{k_\ell}(\gamma_\ell) \quad \text{is rational.}$$

$$\tau_i(\gamma) = \pi_{1*}(\pi_2^*\gamma \cdot ch_{2+i}(\mathbb{F}))$$

Stable pairs moduli space

3-fold

curve class

$\mathcal{O} \to F$ dim 1 support of class β

coker dim 0

$\chi(F) = n$

Universality [EGL '10, OP '19]

$$\sum q^n \int_{[Quot_X(\mathbb{C}^N, \beta=0, n)]^{vir?}} \prod_i \Phi_i(\alpha_i^{[n]}) \cdot \Psi(T\,Quot)$$

$$\Phi_i, \Psi : K^\cdot \to H^*$$

$$\Psi(V \oplus W) = \Psi(V)\Psi(W) \quad \text{e.g. Chern class, Segre class}$$

$$= \prod A_k^{m_k}$$

$m_k \in$ Chern numbers of α_i, TX

$c_1(\alpha_1)\cdot c_1(\alpha_2)$, $c_2(\alpha_1)$, $c_2(TX)$, ...

A_k does not depend on X or α_i.

- check finitely many convenient geometries.

of n, X

- check finitely many convenient geometries.

$$\left(S_1 \sqcup S_2\right)^{[n]} = \bigsqcup_{n_1+n_2=n} S_1^{[n_1]} \times S_2^{[n_2]}$$

Thm 1 $\mathrm{Quot}_X(\mathbb{C}^1, \beta, n) = X^{[n]} \times \mathrm{Hilb}_\beta$

X: simply connected

Hilb_β: divisors of class β

X simply connected $\Longrightarrow \mathrm{Hilb}_\beta = \mathbb{P} := \mathbb{P}(H^0(\beta))$

For remaining cases,

$$\mathrm{Obs} = \mathcal{O}(1)^{\oplus b} \oplus \underline{(M^{[n]})^\vee \otimes \mathcal{O}(1)} \oplus \mathcal{O}^{p_g(X)}$$

M = line bundle on X

$$e^{vir} = \int_{X^{[n]} \times \mathbb{P}} e(\mathrm{Obs}) \cdot \frac{c(TX^{[n]}) \otimes c(T\mathbb{P})}{c(\mathrm{Obs})}$$

$e^{vir} = 0$ unless $p_g = 0$

Integrate out $\mathbb{P}$, get terms of form

$$\int_{X^{[n]}} c_{n-a}(M^{[n]}) \cdot \alpha$$

$C \in H^0(M)$ $\qquad C^{[n]} \subset X^{[n]}$

$$c_n(M^{[n]}) = [C^{[n]}]$$

Pick $V \subset H^0(M)$ of reduced, irr. curves of dim $|V| = a$

$C \subset \mathcal{C} \to X \qquad C^{[n]} \subset \mathcal{C}^{[n]} \xrightarrow{i} X^{[n]}$

$$\begin{array}{ccc} C \subset \mathcal{C} \longrightarrow X & \qquad & C \subset \mathcal{C} \longrightarrow X \\ \big| \qquad \big| & & \big| \qquad \big| \\ p \in |V| & & p \in |V| \end{array}$$

$$\int_{X^{[n]}} c_{n-a}(\mathcal{M}^{[n]})\,\alpha = \int_{\mathcal{C}^{[n]}} i^*\alpha$$

$$\mathcal{C}^{[n]} \longrightarrow P_n(X,\beta)$$

$$I_q \subset \mathcal{O}_C \longmapsto [\mathcal{O}_C \to I_q^\vee]$$

$$\begin{array}{ccc} [\mathcal{O}\to F] & \quad & P_n(X,\beta) \\ \big\downarrow & & \big\downarrow \\ F\otimes \mathcal{L}^{-n} & & \mathcal{M} = \left\{ \begin{array}{l} \text{rank 1 torsion free} \\ \text{sheaves of degree 0} \\ \text{on the fibers of } \mathcal{C} \to |V| \end{array} \right\} \end{array}$$

for $n \gg 0$, this is a projective bundle.

$\mathcal{L} - \mathcal{C}$ line bundle degree 1 on fibers

Move the integrand to $\mathcal{M}$

$$\sum q^n \int_{\mathcal{M}} \alpha_n$$

<u>Check</u> $\int_{\mathcal{M}} \alpha_n = (-1)^n\left(p(n) + 2^n q(n)\right)$

polynomials

$$\text{answer} = \frac{r(q)}{(1-q)^a(1-2q)^b}$$

<u>Theorem 2</u>

- $Z = \sum q^n \int_{[\mathrm{Quot}(\mathbb{C}^N, n)]^{vir}} \mathrm{ch}_{k_1}(\alpha_1^{[n]}) \cdots \mathrm{ch}_{k_\ell}(\alpha_\ell^{[n]})\, c(T^{vir})$

is rational.

Pf Enough to check coef of $x_1^{k_1}\cdots x_\ell^{k_\ell}$ in

$$\cdot\ W = \sum q^n \int_{[\mathrm{Quot}]^{vir}} C_{x_1}(\alpha_1^{[n]}) \cdots C_{x_\ell}(\alpha_\ell^{[n]})\, c(T^{vir})$$

$$C_x = 1 + x c_1 + x^2 c_2 + \cdots$$

Check $\underline{\left(\frac{d}{dx_1}\right)^{k_1} \cdots \left(\frac{d}{dx_\ell}\right)^{k_\ell} W\Big|_{x_j=0}}$ is rational.

If $C \subset X$ is smooth cannonical curve

$$i_*[\mathrm{Quot}_C(\mathbb{C}^N, n)] = (-1)^n [\mathrm{Quot}_X(\mathbb{C}^N, n)]^{vir}$$

Universality $\Rightarrow$ only need work with curves

$\Rightarrow$ only need $C = \mathbb{P}^1$, $\alpha_i = \mathcal{O}(d_i)$

Localization Let $\mathbb{C}^*$ act on $\mathbb{C}^N$ with weights $w_1, \ldots, w_N$.

- gives action on $\mathrm{Quot}_{\mathbb{P}}(\mathbb{C}^N, n)$

The fixed loci

$$\bigoplus_{i=1}^{N} (\mathcal{O} \to \mathcal{O}_{Z_i}) \qquad \begin{array}{l} \mathrm{len}(Z_i) = n_i \\ \sum n_i = n \end{array}$$

$$\cong C^{[n_1]} \times \cdots \times C^{[n_N]}$$

$$\cong \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}$$

$$\int \alpha = \sum_{\substack{F \text{ fixed} \\ \text{loci}}} \int_F \frac{i^*\alpha}{e(N_F)}$$

$$W = \sum q^n \sum \int_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}} \Phi_1(h_1)^{n_1} \cdots \Phi_N(h_N)^{n_N}\, \underline{\Psi(h_1, \ldots, h_N)}$$

$$W = \sum' q^n \sum'_{n_1+\dots+n_N=n} \int_{\mathbb{P}^{n_1}\times\dots\times\mathbb{P}^{n_N}} \Phi_1(h_1)^{\dots} \cdots \Phi_N(h_N)\, \underline{\Psi(h_1,\dots,h_N)}$$

h_i hyperplane class on $\mathbb{P}^{n_i}$

rational function in h_1 and $\{x_j\}$ $\{w_j\}$

"$\int_{\mathbb{P}^{n_1}\times\dots\times\mathbb{P}^{n_N}}$" = "take coefficient of $h_1^{n_1}\cdots h_N^{n_N}$"

Multivariable Lagrange Bürman formula

Let $K = \prod_i \left(1 - h_i \dfrac{\Phi_i'(h_i)}{\Phi_i(h_i)}\right)$

Then $W = \dfrac{\Psi}{K}(h_1,\dots,h_N)$

where h_i is the unique solution to

$$q = \frac{h_i}{\Phi(h_i)} \quad \text{with } h_i(0)=0$$

h_i = series in q
coefs in $\{x_j\}$ and $\{w_j\}$

Observe $\dfrac{\Psi}{K}(g_1+w_1,\dots,g_N+w_N) = W$

is symmetric in g_i

Set $g_i = h_i - w_i$

Check Symmetric functions in g_i can set $x_j=0$, $w_j=0$ and get rational functions in q.

Check derivatives of $\dfrac{\Psi}{K}$ w.r.t. x_j are also symmetric in g_i.