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HOLOMORPHIC ANOMALY EQUATIONS FOR THE FORMAL QUINTIC

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ABSTRACT. We define a formal Gromov-Witten theory of the quintic 3-fold via localization on $\mathbb{P}^4$. Our main result is a direct geometric proof of holomorphic anomaly equations for the formal quintic in precisely the same form as predicted by B-model physics for the true Gromov-Witten theory of the quintic 3-fold. The results suggest that the formal quintic and the true quintic theories should be related by transformations which respect the holomorphic anomaly equations. Such a relationship has been recently found by Q. Chen, S. Guo, F. Janda, and Y. Ruan via the geometry of new moduli spaces.

CONTENTS

0. INTRODUCTION

0.1. **GW/SQ.** Let $X_5 \subset \mathbb{P}^4$ be a nonsingular quintic Calabi-Yau 3-fold. The moduli space of stable maps to the quintic of genus g and degree d,

$$\overline{M}_g(X_5, d) \subset \overline{M}_g(\mathbb{P}^4, d)\,,$$

has virtual dimension 0. The Gromov-Witten invariants,

$$N^{\mathsf{GW}}_{g,d} = \langle 1 \rangle^{\mathsf{GW}}_{g,d} = \int_{[\overline{M}_g(X_5,d)]^{vir}} 1\,, \tag{1}$$

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have been studied for more than 20 years, see [12, 13, 19] for an introduction to the subject.

The theory of stable quotients developed in [24] was partially inspired by the question of finding a geometric approach to a higher genus linear sigma model. The moduli space of stable quotients for the quintic,

$$\overline{Q}_g(X_5, d) \subset \overline{Q}_g(\mathbb{P}^4, d)\,,$$

was defined in [24, Section 9]. The existence of a natural obstruction theory on $\overline{Q}_g(X_5, d)$ and a virtual fundamental class $[\overline{Q}_g(X_5, d)]^{vir}$ is easily seen[1] in genus 0 and 1. A proposal in higher genus for the obstruction theory and virtual class was made in [24] and was carried out in significantly greater generality in the setting of quasimaps in [9]. The associated integral theory is defined by

$$N^{\mathsf{SQ}}_{g,d} \;=\; \langle 1\rangle^{\mathsf{SQ}}_{g,d} \;=\; \int_{[\overline{Q}_g(X_5,d)]^{vir}} 1\,. \tag{2}$$

In genus 0 and 1, the invariants (2) were calculated in [11] and [18] respectively. The answers on the stable quotient side *exactly* match the string theoretic B-model for the quintic in genus 0 and 1.

A relationship in every genus between the Gromov-Witten and stable quotient invariants of the quintic has been proven by Ciocan-Fontanine and Kim [7].[2] Let $H \in H^2(X_5, \mathbb{Z})$ be the hyperplane class of the quintic, and let

$$\mathcal{F}^{\mathsf{GW}}_{g,n}(Q) \;=\; \langle \underbrace{H, \ldots, H}_{n} \rangle^{\mathsf{GW}}_{g,n} \;=\; \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,n}(X_5,d)]^{vir}} \prod_{i=1}^{n} \mathrm{ev}_i^*(H)\,,$$

$$\mathcal{F}^{\mathsf{SQ}}_{g,n}(q) \;=\; \langle \underbrace{H, \ldots, H}_{n} \rangle^{\mathsf{SQ}}_{g,n} \;=\; \sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_{g,n}(X_5,d)]^{vir}} \prod_{i=1}^{n} \mathrm{ev}_i^*(H)$$

be the Gromov-Witten and stable quotient series respectively (involving the pointed moduli spaces and the evaluation morphisms at the markings). Let

$$I_0(q) = \sum_{d=0}^{\infty} q^d \frac{(5d)!}{(d!)^5}\,, \quad I_1(q) = \log(q) I_0(q) + 5 \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left(\sum_{r=d+1}^{5d} \frac{1}{r} \right)\,.$$

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[1] For stability, marked points are required in genus 0 and positive degree is required in genus 1.

[2] A second proof (in most cases) can be found in [10].

The mirror map is defined by

$$Q(q) = \exp\left(\frac{I_1(q)}{I_0(q)}\right) = q \cdot \exp\left(\frac{5\sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5}\left(\sum_{r=d+1}^{5d}\frac{1}{r}\right)}{\sum_{d=0}^{\infty} q^d \frac{(5d)!}{(d!)^5}}\right).$$

The relationship between the Gromov-Witten and stable quotient invariants of the quintic in case

$$2g-2+n>0$$

is given by the following result [7]:

$$\mathcal{F}^{\mathsf{GW}}_{g,n}(Q(q)) = I_0(q)^{2g-2+n} \cdot \mathcal{F}^{\mathsf{SQ}}_{g,n}(q)\,. \tag{3}$$

The transformation (3) shows the stable quotient theory matches the string theoretic B-model series for the quintic X_5.

0.2. **Formal quintic invariants.** The (conjectural) holomorphic anomaly equation is a beautiful property of the string theoretic B-model series which has been used effectively since [3]. Since the stable quotients invariants provide a geometric proposal for the B-model series, we should look for the geometry of the holomorphic anomaly equation in the moduli space of stable quotients.

A particular twisted theory on $\mathbb{P}^4$ is related to the quintic 3-fold. Let the algebraic torus

$$\mathsf{T} = (\mathbb{C}^*)^5$$

act with the standard linearization on $\mathbb{P}^4$ with weights $\lambda_0, \ldots, \lambda_4$ on the vector space $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$. Let

$$\mathsf{C} \to \overline{M}_g(\mathbb{P}^4, d)\,, \quad f: \mathsf{C} \to \mathbb{P}^4\,, \quad \mathsf{S} = f^*\mathcal{O}_{\mathbb{P}^4}(-1) \to \mathsf{C} \tag{4}$$

be the universal curve, the universal map, and the universal bundle over the moduli space of stable maps — all equipped with canonical T-actions. We define the *formal quintic invariants* following [22] by[3]

$$\widetilde{N}^{\mathsf{GW}}_{g,d} = \int_{[\overline{M}_g(\mathbb{P}^4,d)]^{vir}} e(R\pi_*(\mathsf{S}^{-5}))\,, \tag{5}$$

where $e(R\pi_*(\mathsf{S}^{-5}))$ is the equivariant Euler class defined *after* localization. More precisely, on each T-fixed locus of $\overline{M}_g(\mathbb{P}^4, d)$, both

$$R^0\pi_*(\mathsf{S}^{-5}) \quad \text{and} \quad R^1\pi_*(\mathsf{S}^{-5})$$

are vector bundles with moving weights, so

$$e(R\pi_*(\mathsf{S}^{-5})) = \frac{c_{\text{top}}(R^0\pi_*(\mathsf{S}^{-5}))}{c_{\text{top}}(R^1\pi_*(\mathsf{S}^{-5}))}$$

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[3]The negative exponent denotes the dual: S is a line bundle and $\mathsf{S}^{-5} = (\mathsf{S}^\star)^{\otimes 5}$.

is well-defined. The integral (5) is homogeneous of degree 0 in localized equivariant cohomology,

$$\int_{[\overline{M}_g(\mathbb{P}^4,d)]^{vir}} e(R\pi_*(\mathsf{S}^{-5})) \in \mathbb{Q}(\lambda_0,\ldots,\lambda_4)\,,$$

and defines a rational number $\widetilde{N}^{\mathsf{GW}}_{g,d} \in \mathbb{Q}$ after the specialization[4]

$$\lambda_i = \zeta^i$$

for a primitive fifth root of unity $\zeta^5 = 1$.

Our main result here is that the holomorphic anomaly equations conjectured for the true quintic theory (1) are satisfied by the formal quintic theory (5). In particular, the formal quintic theory and the true quintic theory should be related by transformations which respect the holomorphic anomaly equations. In a recent breakthrough by Q. Chen, S. Guo, F. Janda, and Y. Ruan, precisely such a transformation was found via the virtual geometry of new moduli spaces intertwining the formal and true theories of the quintic.

0.3. Holomorphic anomaly for the formal quintic.

We state here the precise form of the holomorphic anomaly equations for the formal quintic.

Let $H \in H^2(\mathbb{P}^4,\mathbb{Z})$ be the hyperplane class on $\mathbb{P}^4$, and let

$$\begin{aligned}
\widetilde{\mathcal{F}}^{\mathsf{GW}}_g(Q) &= \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_g(\mathbb{P}^4,d)]^{vir}} e(R\pi_*(\mathsf{S}^{-5}))\,,\\
\widetilde{\mathcal{F}}^{\mathsf{SQ}}_g(Q) &= \sum_{d=0}^{\infty} Q^d \int_{[\overline{Q}_g(\mathbb{P}^4,d)]^{vir}} e(R\pi_*(\mathsf{S}^{-5}))
\end{aligned}$$

be the formal Gromov-Witten and formal stable quotient series respectively (involving the evaluation morphisms at the markings). The relationship between the formal Gromov-Witten and formal stable quotient invariants of quintic in case of $2g-2+n>0$ follows from [8]:

$$\widetilde{\mathcal{F}}^{\mathsf{GW}}_g(Q(q)) = I_0(q)^{2g-2}\cdot \widetilde{\mathcal{F}}^{\mathsf{SQ}}_g(q) \tag{6}$$

[4]Since the formal quintic theory is homogeneous of degreee 0, the specialization $\lambda_i = \zeta^i\lambda_0$ could also be taken as in [22]. However, for other specializations, we expect the finite generation of Theorem 1 and the holomorphic anomaly equation of Theorem 2 to take different forms. A study of the dependence on specialization will appear in [21]. For local $\mathbb{P}^2$ considered in [22] and $\mathbb{C}^3/\mathbb{Z}_3$ considered in [23], the theories are independent of specialization.

with respect to the true quintic mirror map

$$Q(q) = \exp\left(\frac{I_1(q)}{I_0(q)}\right) = q \cdot \exp\left(\frac{5\sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5}\left(\sum_{r=d+1}^{5d} \frac{1}{r}\right)}{\sum_{d=0}^{\infty} q^d \frac{(5d)!}{(d!)^5}}\right).$$

In order to state the holomorphic anomaly equations, we require several series in q. First, let

$$L(q) = (1-5^5 q)^{-\frac{1}{5}} = 1 + 625q + 117185q^2 + \dots .$$

Let $\mathsf{D} = q\frac{d}{dq}$, and let

$$C_0(q) = I_0\,, \quad C_1(q) = \mathsf{D}\left(\frac{I_1}{I_0}\right),$$

where I_0 and I_1 are the hypergeometric series appearing in the mirror map for the true quintic theory. We define

$$\begin{aligned}
K_2(q) &= -\frac{1}{L^5}\frac{\mathsf{D}C_0}{C_0}\,,\\
A_2(q) &= \frac{1}{L^5}\left(-\frac{1}{5}\frac{\mathsf{D}C_1}{C_1} - \frac{2}{5}\frac{\mathsf{D}C_0}{C_0} - \frac{3}{25}\right),\\
A_4(q) &= \frac{1}{L^{10}}\Bigg(-\frac{1}{25}\left(\frac{\mathsf{D}C_0}{C_0}\right)^2 - \frac{1}{25}\left(\frac{\mathsf{D}C_0}{C_0}\right)\left(\frac{\mathsf{D}C_1}{C_1}\right)\\
&\quad +\frac{1}{25}\mathsf{D}\left(\frac{\mathsf{D}C_0}{C_0}\right) + \frac{2}{25^2}\Bigg),\\
A_6(q) &= \frac{1}{31250L^{15}}\Bigg(4 + 125\mathsf{D}\left(\frac{\mathsf{D}C_0}{C_0}\right) + 50\left(\frac{\mathsf{D}C_0}{C_0}\right)\left(1 + 10\mathsf{D}\left(\frac{\mathsf{D}C_0}{C_0}\right)\right)\\
&\quad -5L^5\left(1 + 10\left(\frac{\mathsf{D}C_0}{C_0}\right) + 25\left(\frac{\mathsf{D}C_0}{C_0}\right)^2 + 25\mathsf{D}\left(\frac{q\frac{d}{dq}C_0}{C_0}\right)\right)\\
&\quad +125\mathsf{D}^2\left(\frac{\mathsf{D}C_0}{C_0}\right) - 125\left(\frac{\mathsf{D}C_0}{C_0}\right)^2\left(\left(\frac{\mathsf{D}C_1}{C_1}\right) - 1\right)\Bigg).
\end{aligned}$$

Let T be the standard coordinate mirror to $t = \log(q)$,

$$T = \frac{I_1(q)}{I_0(q)}\,.$$

Then $Q(q) = \exp(T)$ is the mirror map.

Define a new series

$$\widetilde{\mathcal{F}}_g^{\mathsf{B}} = I_0^{2g-2} \cdot \widetilde{\mathcal{F}}_g^{\mathsf{SQ}}$$

motivated by (6). The superscript B here is for the B-model. Let

$$\mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2]$$

be the free polynomial ring over $\mathbb{C}[L^{\pm 1}]$.

Theorem 1. *For the series $\widetilde{\mathcal{F}}_g^{\mathsf{B}}$ associated to the formal quintic,*

(i) $\widetilde{\mathcal{F}}_g^{\mathsf{B}}(q) \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2]$ *for* $g \geq 2$,

(ii) $\frac{\partial^k \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial T^k}(q) \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2]$ *for* $g \geq 1$, $k \geq 1$,

(iii) $\frac{\partial^k \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial T^k}$ *is homogeneous with respect to* C_1^{-1} *of degree* k.

We follow here the *canonical lift* convention of [22, Section 0.4]. When we write

$$\widetilde{\mathcal{F}}_g^{\mathsf{B}}(q) \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2]\,,$$

we mean that the series $\widetilde{\mathcal{F}}_g^{\mathsf{B}}(q)$ has a canonical lift to the free algebra. The question of uniqueness of the lift has to do with the algebraic independence of the series

$$A_2(q)\,,\ A_4(q)\,,\ A_6(q)\,,\ C_0^{\pm}(q)\,,\ C_1^{-1}(q)\,,\ K_2(q)$$

which we do not address nor require.

Theorem 2. *The holomorphic anomaly equations for the series $\widetilde{\mathcal{F}}_g^{\mathsf{B}}$ associated to the formal quintic hold for $g \geq 2$:*

$$\frac{1}{C_0^2 C_1^2}\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_2} - \frac{1}{5C_0^2C_1^2}\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_4}K_2 + \frac{1}{50C_0^2C_1^2}\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_6}K_2^2 = \frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial \widetilde{\mathcal{F}}_{g-i}^{\mathsf{B}}}{\partial T}\frac{\partial \widetilde{\mathcal{F}}_i^{\mathsf{B}}}{\partial T} + \frac{1}{2}\frac{\partial^2 \widetilde{\mathcal{F}}_{g-1}^{\mathsf{B}}}{\partial T^2}\,,$$

$$\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial K_2} = 0\,.$$

The equality of Theorem 2 holds in the ring

$$\mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2]\,.$$

Theorem 2 exactly matches[5] the conjectural holomorphic anomaly equations [1, (2.52)] for the true quintic theory $I_0^{2g-2} \cdot \mathcal{F}_g^{\mathsf{SQ}}$.

The first holomorphic anomaly equation of Theorem 2 was announced in our paper [22] in February 2017 where a parallel study of the toric Calabi-Yau $K\mathbb{P}^2$ was developed. In January 2018 at the *Workshop on*

[5]Our functions K_2 and A_{2k} are normalized differently with respect to C_0 and C_1. The dictionary to exactly match the notation of [1, (2.52)] is to multiply our K_2 by $(C_0C_1)^2$ and our A_{2k} by $(C_0C_1)^{2k}$.

higher genus at ETH Zürich, Shuai Guo of Peking University informed us that our same argument also yields the second holomorphic anomaly equation

$$\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial K_2} = 0\,. \tag{7}$$

In fact, we had incorrectly thought (7) would fail in the formal theory and would *not* have included (7) without communication with Guo, so Guo should be credited with the first proof of (7).

0.4. **Constants.** Theorem 2 and a few observations determine $\widetilde{\mathcal{F}}_g^{\mathsf{B}}$ from the lower genus data

$$\left\{ h < g \,\middle|\, \widetilde{\mathcal{F}}_h^{\mathsf{B}} \right\}$$

and finitely many constants of integration. The additional observations[6] required are:

(i) The proof of part (i) of Theorem 1 shows that

$$\widetilde{\mathcal{F}}_g^{\mathsf{B}} \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2]$$

does *not* depend on C_1^{-1}. Hence, every term (both on the left and right) in the first holomorphic anomaly equation of Theorem 2 is of degree 2 in C_1^{-1}. After multiplying by C_1^2, no C_1 dependence remains.

(ii) The proof of Theorem 1 shows that all terms in the first equation are homogeneous of degree $2g-4$ with respect to C_0. After dividing by C_0^{2g-4}, no C_0 dependence remains.

Therefore, the first holomorphic anomaly equation of Theorem 2 may be viewed as holding in $\mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, K_2]$.

Since the second holomorphic anomaly equation (7) implies $\widetilde{\mathcal{F}}_g^{\mathsf{B}}$ has no K_2 dependence, the first holomorphic anomaly equation determines the each of the three derivatives

$$\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_2}\,, \quad \frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_4}\,, \quad \frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_6}\,.$$

Hence, Theorem 2 determines

$$C_0^{2-2g} \cdot \widetilde{\mathcal{F}}_g^{\mathsf{B}}$$

uniquely as a polynomial in A_2, A_4, A_6 up to a constant term in $\mathbb{C}[L^{\pm 1}]$. In fact, the degree of the constant term can be bounded (as will be seen

[6]See Remark 27.

in Section 6.5). So Theorem 2 determines $\widetilde{\mathcal{F}}_g^{\mathsf{B}}$ from lower genus data together with finitely many constants of integration.

The constants of integration for the formal quintic can be effectively computed via the localization formula, but whether there exists a closed formula determining the constants is an interesting open question.

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1. Localization graphs

1.1. **Torus action.** Let $\mathsf{T} = (\mathbb{C}^*)^{m+1}$ act diagonally on the vector space $\mathbb{C}^{m+1}$ with weights

$$-\lambda_0, \ldots, -\lambda_m .$$

Denote the T-fixed points of the induced T-action on $\mathbb{P}^m$ by

$$p_0, \ldots, p_m .$$

The weights of T on the tangent space $T_{p_j}(\mathbb{P}^m)$ are

$$\lambda_j - \lambda_0, \ldots, \widehat{\lambda_j - \lambda_j}, \ldots, \lambda_j - \lambda_m .$$

There is an induced T-action on the moduli space $\overline{Q}_{g,n}(\mathbb{P}^m, d)$. The localization formula of [17] applied to the virtual fundamental class $[\overline{Q}_{g,n}(\mathbb{P}^m, d)]^{vir}$ will play a fundamental role our paper. The T-fixed loci are represented in terms of dual graphs, and the contributions of the T-fixed loci are given by tautological classes. The formulas here are standard. We precisely follow the notation of [22, Section 2].

1.2. **Graphs.** Let the genus g and the number of markings n for the moduli space be in the stable range

$$2g-2+n>0\,. \tag{8}$$

We can organize the T-fixed loci of $\overline{Q}_{g,n}(\mathbb{P}^m,d)$ according to decorated graphs. A *decorated graph* $\Gamma \in \mathsf{G}_{g,n}(\mathbb{P}^m)$ consists of the data $(\mathsf{V},\mathsf{E},\mathsf{N},\mathsf{g},\mathsf{p})$ where

(i) V is the vertex set,
(ii) E is the edge set (including possible self-edges),
(iii) $\mathsf{N}:\{1,2,...,n\}\to\mathsf{V}$ is the marking assignment,
(iv) $\mathsf{g}:\mathsf{V}\to\mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$g=\sum_{v\in V}\mathsf{g}(v)+h^1(\Gamma)$$

and for which $(\mathsf{V},\mathsf{E},\mathsf{N},\mathsf{g})$ is stable graph[7],
(v) $\mathsf{p}:\mathsf{V}\to(\mathbb{P}^m)^{\mathsf{T}}$ is an assignment of a T-fixed point $\mathsf{p}(v)$ to each vertex $v\in\mathsf{V}$.

The markings $\mathsf{L}=\{1,\ldots,n\}$ are often called *legs*.

To each decorated graph $\Gamma\in\mathsf{G}_{g,n}(\mathbb{P}^m)$, we associate the set of T-fixed loci of

$$\sum_{d\geq 0}\left[\overline{Q}_{g,n}(\mathbb{P}^m,d)\right]^{\mathrm{vir}}q^d$$

with elements described as follows:

(a) If $\{v_{i_1},\ldots,v_{i_k}\}=\{v\,|\,\mathsf{p}(v)=p_i\}$, then $f^{-1}(p_i)$ is a disjoint union of connected stable curves of genera $\mathsf{g}(v_{i_1}),\ldots,\mathsf{g}(v_{i_k})$ and finitely many points.
(b) There is a bijective correspondence between the connected components of $C\setminus D$ and the set of edges[8] and legs of Γ respecting vertex incidence where C is domain curve and D is union of all subcurves of C which appear in (a).

We write the localization formula as

$$\sum_{d\geq 0}\left[\overline{Q}_{g,n}(\mathbb{P}^m,d)\right]^{\mathrm{vir}}q^d=\sum_{\Gamma\in\mathsf{G}_{g,n}(\mathbb{P}^m)}\mathrm{Cont}_\Gamma\,.$$

While $\mathsf{G}_{g,n}(\mathbb{P}^m)$ is a finite set, each contribution Cont_Γ is a series in q obtained from an infinite sum over all edge possibilities (b).

[7]Corresponding to a stratum of the moduli space of stable curves $\overline{M}_{g,n}$.
[8]Self-edges correspond to loops of T-invariant rational curves.

1.3. **Unstable graphs.** The moduli spaces of stable quotients

$$\overline{Q}_{0,2}(\mathbb{P}^m, d) \quad \text{and} \quad \overline{Q}_{1,0}(\mathbb{P}^m, d)$$

for $d > 0$ are the only[9] cases where the pair (g, n) does *not* satisfy the Deligne-Mumford stability condition (8).

An appropriate set of decorated graphs $\mathsf{G}_{0,2}(\mathbb{P}^m)$ is easily defined: The graphs $\Gamma \in \mathsf{G}_{0,2}(\mathbb{P}^m)$ all have 2 vertices connected by a single edge. Each vertex carries a marking. All of the conditions (i)-(v) of Section 1.2 are satisfied except for the stability of $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \gamma)$. The localization formula holds,

$$\sum_{d \geq 1} \left[\overline{Q}_{0,2}(\mathbb{P}^m, d)\right]^{\text{vir}} q^d \;=\; \sum_{\Gamma \in \mathsf{G}_{0,2}(\mathbb{P}^m)} \text{Cont}_\Gamma \,, \tag{9}$$

For $\overline{Q}_{1,0}(\mathbb{P}^m, d)$, the matter is more problematic — usually a marking is introduced to break the symmetry.

2. Basic correlators

2.1. **Overview.** We review here basic generating series in q which arise in the genus 0 theory of quasimap invariants. The series will play a fundamental role in the calculations of Sections 3 - 6 related to the holomorphic anomaly equation for formal quintic invariants.

We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^5$ on $\mathbb{P}^4$ with weights[10]

$$-\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$$

on the vector space $\mathbb{C}^5$. The T-weight on the fiber over p_i of the canonical bundle

$$\mathcal{O}_{\mathbb{P}^4}(5) \to \mathbb{P}^4 \tag{10}$$

is $5\lambda_i$.

For our formal quintic theory, we will use the specialization

$$\lambda_i = \zeta^i \tag{11}$$

[9]The moduli spaces $\overline{Q}_{0,0}(\mathbb{P}^m, d)$ and $\overline{Q}_{0,1}(\mathbb{P}^m, d)$ are empty by the definition of a stable quotient.

[10]The associated weights on $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$ are $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and so match the conventions of Section 0.2.

where ζ is the primitive fifth root of unity. Of course, we then have

$$\begin{aligned} \lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4 &= 0\,, \\ \sum_{i\neq j}\lambda_i\lambda_j &= 0\,, \\ \sum_{i\neq j\neq k}\lambda_i\lambda_j\lambda_k &= 0\,, \\ \sum_{i\neq j\neq k\neq l}\lambda_i\lambda_j\lambda_k\lambda_l &= 0\,. \end{aligned}$$

2.2. **First correlators.** We will require several correlators defined via the Euler class[11],

$$e(\mathrm{Obs}) = e(R\pi_*(\mathsf{S}^{-5}))\,, \tag{12}$$

associated to the formal quintic geometry on the moduli space $\overline{Q}_{g,n}(\mathbb{P}^4,d)$. The first two are obtained from standard stable quotient invariants. For $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}^4)$, let

$$\begin{aligned} \Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle^{\mathsf{SQ}}_{g,n,d} &= \int_{[\overline{Q}_{g,n}(\mathbb{P}^4,d)]^{\mathrm{vir}}} e(\mathrm{Obs})\cdot\prod_{i=1}^{n}\mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i}\,, \\ \Big\langle\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle\Big\rangle^{\mathsf{SQ}}_{0,n} &= \sum_{d\geq 0}\sum_{k\geq 0}\frac{q^d}{k!}\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n},t,\ldots,t\Big\rangle^{\mathsf{SQ}}_{0,n+k,d}, \end{aligned}$$

where, in the second series, $t\in H^*_{\mathsf{T}}(\mathbb{P}^4)$. We will systematically use the quasimap notation $0+$ for stable quotients,

$$\begin{aligned} \Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle^{0+}_{g,n,d} &= \Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle^{\mathsf{SQ}}_{g,n,d} \\ \Big\langle\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle\Big\rangle^{0+}_{0,n} &= \Big\langle\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle\Big\rangle^{\mathsf{SQ}}_{0,n}\,. \end{aligned}$$

2.3. **Light markings.** Moduli of quasimaps can be considered with n ordinary (weight 1) markings and k light (weight ϵ) markings[12],

$$\overline{Q}^{0+,0+}_{g,n|k}(\mathbb{P}^4,d)\,.$$

Let $\gamma_i\in H^*_{\mathsf{T}}(\mathbb{P}^4)$ be equivariant cohomology classes, and let

$$\delta_j \in H^*_{\mathsf{T}}([\mathbb{C}^5/\mathbb{C}^*])$$

be classes on the stack quotient. Following the notation of [18], we define series for the formal quintic geometry,

[11] Equation (12) is the definition of $e(\mathrm{Obs})$. The right side of (12) is defined after localization as explained in Section 0.2.

[12] See Sections 2 and 5 of [6].

$$\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n};\delta_1,\ldots,\delta_k\Big\rangle_{g,n|k,d}^{0+,0+} = \int_{[\overline{Q}^{0+,0+}_{g,n|k}(\mathbb{P}^4,d)]^{\text{vir}}} e(\text{Obs})\cdot\prod_{i=1}^{n}\text{ev}_i^*(\gamma_i)\psi_i^{a_i}\cdot\prod_{j=1}^{k}\widehat{\text{ev}}_j^*(\delta_j)\,,$$

$$\Big\langle\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle\Big\rangle_{0,n}^{0+,0+} = \sum_{d\geq 0}\sum_{k\geq 0}\frac{q^d}{k!}\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n};t,\ldots,t\Big\rangle_{0,n|k,d}^{0+,0+},$$

where, in the second series, $t\in H^*_{\mathsf{T}}([\mathbb{C}^5/\mathbb{C}^*])$.

For each T-fixed point $p_i\in\mathbb{P}^4$, let

$$e_i=\frac{e(T_{p_i}(\mathbb{P}^4))}{5\lambda_i}$$

be the equivariant Euler class of the tangent space of $\mathbb{P}^4$ at p_i with twist by $\mathcal{O}_{\mathbb{P}^4}(5)$. Let

$$\phi_i=\frac{\prod_{j\neq i}(H-\lambda_j)}{5\lambda_i e_i},\quad \phi^i=e_i\phi_i \quad \in H^*_{\mathsf{T}}(\mathbb{P}^4)$$

be cycle classes. Crucial for us are the series

$$\begin{aligned}\mathbb{S}_i(\gamma) &= e_i\Big\langle\Big\langle\frac{\phi_i}{z-\psi},\gamma\Big\rangle\Big\rangle_{0,2}^{0+,0+}\ ,\\ \mathbb{V}_{ij} &= \Big\langle\Big\langle\frac{\phi_i}{x-\psi},\frac{\phi_j}{y-\psi}\Big\rangle\Big\rangle_{0,2}^{0+,0+}\ .\end{aligned}$$

Unstable degree 0 terms are included by hand in the above formulas. For $\mathbb{S}_i(\gamma)$, the unstable degree 0 term is $\gamma|_{p_i}$. For $\mathbb{V}_{ij}$, the unstable degree 0 term is $\frac{\delta_{ij}}{e_i(x+y)}$.

We also write

$$\mathbb{S}(\gamma)=\sum_{i=0}^{4}\phi_i\mathbb{S}_i(\gamma)\,.$$

The series $\mathbb{S}_i$ and $\mathbb{V}_{ij}$ satisfy the basic relation

$$e_i\mathbb{V}_{ij}(x,y)e_j=\frac{\sum_{k=0}^{4}\mathbb{S}_i(\phi_k)|_{z=x}\,\mathbb{S}_j(\phi^k)|_{z=y}}{x+y} \tag{13}$$

proven[13] in [8].

Associated to each T-fixed point $p_i \in \mathbb{P}^4$, there is a special T-fixed point locus,

$$\overline{Q}^{0+,0+}_{0,k|m}(\mathbb{P}^4,d)^{\mathsf{T},p_i} \subset \overline{Q}^{0+,0+}_{0,k|m}(\mathbb{P}^4,d)\,, \tag{14}$$

where all markings lie on a single connected genus 0 domain component contracted to p_i. Let Nor denote the equivariant normal bundle of $Q^{0+,0+}_{0,n|k}(\mathbb{P}^4,d)^{\mathsf{T},p_i}$ with respect to the embedding (14). Define

$$\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n};\delta_1,...,\delta_k\Big\rangle^{0+,0+,p_i}_{0,n|k,d} =$$

$$\int_{[\overline{Q}^{0+,0+}_{0,n|k}(\mathbb{P}^4,d)^{\mathsf{T},p_i}]} \frac{e(\mathrm{Obs})}{e(\mathrm{Nor})}\cdot\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i}\cdot\prod_{j=1}^k \widehat{\mathrm{ev}}_j^*(\delta_j)\,,$$

$$\Big\langle\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\Big\rangle\Big\rangle^{0+,0+,p_i}_{0,n} =$$

$$\sum_{d\geq 0}\sum_{k\geq 0}\frac{q^d}{k!}\Big\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n};t,\ldots,t\Big\rangle^{0+,0+,p_i}_{0,n|k,\beta}\,.$$

2.4. **Graph spaces and I-functions.**

2.4.1. *Graph spaces.* The big I-function is defined in [6] via the geometry of weighted quasimap graph spaces. We briefly summarize the constructions of [6] in the special case of $(0+,0+)$-stability. The more general weightings discussed in [6] will not be needed here.

As in Section 2.3, we consider the quotient

$$\mathbb{C}^5/\mathbb{C}^*$$

associated to $\mathbb{P}^4$. Following [6], there is a $(0+,0+)$-*stable quasimap graph space*

$$\mathsf{QG}^{0+,0+}_{g,n|k,d}([\mathbb{C}^5/\mathbb{C}^*])\,. \tag{15}$$

A $\mathbb{C}$-point of the graph space is described by data

$$((C,\mathbf{x},\mathbf{y}),(f,\varphi): C \longrightarrow [\mathbb{C}^5/\mathbb{C}^*]\times[\mathbb{C}^2/\mathbb{C}^*]).$$

By the definition of stability, φ is a regular map to

$$\mathbb{P}^1 = \mathbb{C}^2/\!\!/\mathbb{C}^*$$

[13]In Gromov-Witten theory, a parallel relation is obtained immediately from the WDDV equation and the string equation. Since the map forgetting a point is not always well-defined for quasimaps, a different argument is needed here [8]

of class 1. Hence, the domain curve C has a distinguished irreducible component C_0 canonically isomorphic to $\mathbb{P}^1$ via φ. The *standard* $\mathbb{C}^*$-action,

$$t \cdot [\xi_0, \xi_1] = [t\xi_0, \xi_1], \quad \text{for } t \in \mathbb{C}^*,\ [\xi_0, \xi_1] \in \mathbb{P}^1, \tag{16}$$

induces a $\mathbb{C}^*$-action on the graph space.

The $\mathbb{C}^*$-equivariant cohomology of a point is a free algebra with generator z,

$$H^*_{\mathbb{C}^*}(\mathrm{Spec}(\mathbb{C})) = \mathbb{Q}[z]\,.$$

Our convention is to define z as the $\mathbb{C}^*$-equivariant first Chern class of the tangent line $T_0\mathbb{P}^1$ at $0 \in \mathbb{P}^1$ with respect to the action (16),

$$z = c_1(T_0\mathbb{P}^1)\,.$$

The T-action on $\mathbb{C}^5$ lifts to a T-action on the graph space (15) which commutes with the $\mathbb{C}^*$-action obtained from the distinguished domain component. As a result, we have a $\mathsf{T} \times \mathbb{C}^*$-action on the graph space and $\mathsf{T} \times \mathbb{C}^*$-equivariant evaluation morphisms

$$\begin{aligned} \mathrm{ev}_i : \mathsf{QG}^{0+,0+}_{g,n|k,\beta}([\mathbb{C}^5/\mathbb{C}^*]) &\to \mathbb{P}^4, & i &= 1, \ldots, n\,, \\ \widehat{\mathrm{ev}}_j : \mathsf{QG}^{0+,0+}_{g,n|k,\beta}([\mathbb{C}^5/\mathbb{C}^*]) &\to [\mathbb{C}^5/\mathbb{C}^*], & j &= 1, \ldots, k\,. \end{aligned}$$

Since a morphism

$$f : C \to [\mathbb{C}^5/\mathbb{C}^*]$$

is equivalent to the data of a principal $\mathbf{G}$-bundle P on C and a section u of $P \times_{\mathbb{C}^*} \mathbb{C}^5$, there is a natural morphism

$$C \to E\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^5$$

and hence a pull-back map

$$f^* : H^*_{\mathbb{C}^*}([\mathbb{C}^5/\mathbb{C}^*]) \to H^*(C).$$

The above construction applied to the universal curve over the moduli space and the universal morphism to $[\mathbb{C}^5/\mathbb{C}^*]$ is T-equivariant. Hence, we obtain a pull-back map

$$\widehat{\mathrm{ev}}_j^* : H^*_{\mathsf{T}}(\mathbb{C}^5, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z] \to H^*_{\mathsf{T}\times\mathbb{C}^*}(\mathsf{QG}^{0+,0+}_{g,n|k,\beta}([\mathbb{C}^5/\mathbb{C}^*]), \mathbb{Q})$$

associated to the evaluation map $\widehat{\mathrm{ev}}_j$.

2.4.2. I-*functions.* The description of the fixed loci for the $\mathbb{C}^*$-action on

$$\mathsf{QG}^{0+,0+}_{g,0|k,d}([\mathbb{C}^5/\mathbb{C}^*])$$

is parallel to the description in [5, §4.1] for the unweighted case. In particular, there is a distinguished subset $\mathsf{M}_{k,d}$ of the $\mathbb{C}^*$-fixed locus for which all the markings and the entire curve class d lie over $0 \in \mathbb{P}^1$. The locus $\mathsf{M}_{k,d}$ comes with a natural *proper* evaluation map $ev_\bullet$ obtained from the generic point of $\mathbb{P}^1$:

$$\mathrm{ev}_\bullet : \mathsf{M}_{k,d} \to \mathbb{C}^5/\!\!/\mathbb{C}^* = \mathbb{P}^4.$$

We can explicitly write

$$\mathsf{M}_{k,d} \cong \mathsf{M}_d \times 0^k \subset \mathsf{M}_d \times (\mathbb{P}^1)^k,$$

where M_d is the $\mathbb{C}^*$-fixed locus in $\mathsf{QG}^{0+}_{0,0,d}([\mathbb{C}^5/\mathbb{C}^*])$ for which the class d is concentrated over $0 \in \mathbb{P}^1$. The locus M_d parameterizes quasimaps of class d,

$$f : \mathbb{P}^1 \longrightarrow [\mathbb{C}^5/\mathbb{C}^*]\,,$$

with a base-point of length d at $0 \in \mathbb{P}^1$. The restriction of f to $\mathbb{P}^1 \setminus \{0\}$ is a constant map to $\mathbb{P}^4$ defining the evaluation map $\mathrm{ev}_\bullet$.

As in [4, 5, 9], we define the big $\mathbb{I}$-function as the generating function for the push-forward via $ev_\bullet$ of localization residue contributions of $\mathsf{M}_{k,d}$. For $\mathbf{t} \in H^*_{\mathsf{T}}([\mathbb{C}^5/\mathbb{C}^*], \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z]$, let

$$\begin{aligned}\mathrm{Res}_{\mathsf{M}_{k,d}}(\mathbf{t}^k) &= \prod_{j=1}^{k} \widehat{\mathrm{ev}}_j^*(\mathbf{t}) \cap \mathrm{Res}_{\mathsf{M}_{k,d}}[\mathsf{QG}^{0+,0+}_{g,0|k,d}([\mathbb{C}^5/\mathbb{C}^*])]^{\mathrm{vir}} \\ &= \frac{\prod_{j=1}^{k} \widehat{\mathrm{ev}}_j^*(\mathbf{t}) \cap [\mathsf{M}_{k,d}]^{\mathrm{vir}}}{\mathrm{e}(\mathrm{Nor}^{\mathrm{vir}}_{\mathsf{M}_{k,d}})},\end{aligned}$$

where $\mathrm{Nor}^{\mathrm{vir}}_{\mathsf{M}_{k,d}}$ is the virtual normal bundle.

Definition 3. *The big* $\mathbb{I}$*-function for the* $(0+, 0+)$*-stability condition, as a formal function in* $\mathbf{t}$*, is*

$$\mathbb{I}(q, \mathbf{t}, z) = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \mathrm{ev}_{\bullet *}\Big(\mathrm{Res}_{\mathsf{M}_{k,d}}(\mathbf{t}^k)\Big)\,.$$

2.4.3. *Evaluations.* Let $\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^5/\mathbb{C}^*])$ and $H \in H^*_{\mathsf{T}}(\mathbb{P}^4)$ denote the respective hyperplane classes. The $\mathbb{I}$-function of Definition 3 is evaluated in [6].

Proposition 4. *For the restriction* $\mathbf{t} = t\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^5/\mathbb{C}^*], \mathbb{Q})$,

$$\mathbb{I}(t) = \sum_{d=0}^{\infty} q^d e^{t(H+dz)/z} \frac{\prod_{k=0}^{5d}(5H+kz)}{\prod_{i=0}^{4}\prod_{k=1}^{d}(H-\lambda_i+kz)} .$$

We return now to the functions $\mathbb{S}_i(\gamma)$ defined in Section 2.3. Using Birkhoff factorization, an evaluation of the series $\mathbb{S}(H^j)$ can be obtained from the $\mathbb{I}$-function, see [18]:

$$\begin{aligned}
\mathbb{S}(1) &= \frac{\mathbb{I}}{\mathbb{I}|_{t=0,H=1,z=\infty}} \,, \\
\mathbb{S}(H) &= \frac{z\frac{d}{dt}\mathbb{S}(1)}{z\frac{d}{dt}\mathbb{S}(1)|_{t=0,H=1,z=\infty}} \,, \\
\mathbb{S}(H^2) &= \frac{z\frac{d}{dt}\mathbb{S}(H)}{z\frac{d}{dt}\mathbb{S}(H)|_{t=0,H=1,z=\infty}} \,, \\
\mathbb{S}(H^3) &= \frac{z\frac{d}{dt}\mathbb{S}(H^2)}{z\frac{d}{dt}\mathbb{S}(H^2)|_{t=0,H=1,z=\infty}} \,, \\
\mathbb{S}(H^4) &= \frac{z\frac{d}{dt}\mathbb{S}(H^3)}{z\frac{d}{dt}\mathbb{S}(H^3)|_{t=0,H=1,z=\infty}} \,, \\
\mathbb{S}(1) &= \frac{z\frac{d}{dt}\mathbb{S}(H^4)}{z\frac{d}{dt}\mathbb{S}(H^4)|_{t=0,H=1,z=\infty}} \,.
\end{aligned} \tag{17}$$

For a series $F \in \mathbb{C}[[\frac{1}{z}]]$, the specialization $F|_{z=\infty}$ denotes constant term of F with respect to $\frac{1}{z}$.

2.4.4. *Further calculations.* Define small I-function

$$\overline{\mathbb{I}}(q) \in H^*_{\mathsf{T}}(\mathbb{P}^4, \mathbb{Q})[[q]]$$

by the restriction

$$\overline{\mathbb{I}}(q) = \mathbb{I}(q,t)|_{t=0} \,.$$

Define differential operators

$$\mathsf{D} = q\frac{d}{dq} \,, \quad M = H + z\mathsf{D} \,.$$

Applying $z\frac{d}{dt}$ to $\mathbb{I}$ and then restricting to $t = 0$ has same effect as applying M to $\overline{\mathbb{I}}$

$$\left[\left(z\frac{d}{dq}\right)^k \mathbb{I}\right]|_{t=0} = M^k\overline{\mathbb{I}}\,.$$

The function $\overline{\mathbb{I}}$ satisfies the following Picard-Fuchs equation

$$\big(M^5 - 1 - q(5M+z)(5M+2z)(5M+3z)(5M+4z)(5M+5z)\big)\,\overline{\mathbb{I}} = 0$$

implied by the Picard-Fuchs equation for $\mathbb{I}$,

$$\left(\left(z\frac{d}{dt}\right)^5 - 1 - q\prod_{k=1}^{5}\left(5\left(z\frac{d}{dt}\right) + kz\right)\right)\mathbb{I} = 0\,.$$

The restriction $\overline{\mathbb{I}}|_{H=\lambda_i}$ admits the following asymptotic form

$$\overline{\mathbb{I}}|_{H=\lambda_i} = e^{\mu\lambda_i/z}\left(R_0 + R_1\left(\frac{z}{\lambda_i}\right) + R_2\left(\frac{z}{\lambda_i}\right)^2 + \ldots\right) \tag{18}$$

with series $\mu, R_k \in \mathbb{C}[[q]]$.

A derivation of (18) is obtained in [27] via the Picard-Fuchs equation for $\overline{\mathbb{I}}|_{H=\lambda_i}$. The series μ and R_k are found by solving differential equations obtained from the coefficient of z^k. For example,

$$\begin{aligned} 1 + \mathsf{D}\mu &= L\,, \\ R_0 &= L\,, \\ R_1 &= \frac{3}{20}(L - L^5)\,, \\ R_2 &= \frac{9L}{800}(1 - L^4)^2\,, \end{aligned}$$

where $L(q) = (1 - 5^5 q)^{-1/5}$. The specialization (11) is used for these results.

Define the series C_i by the equations

$$C_0 = \mathbb{I}|_{z=\infty,t=0,H=1}\,, \tag{19}$$

$$C_i = z\frac{d}{dt}\mathbb{S}(H^{i-1})|_{z=\infty,t=0,H=1}\,, \quad \text{for}\, i = 1,2,3,4\,. \tag{20}$$

The following relations were proven in [27],

$$C_0C_1C_2C_3C_4 = L^5\,,$$

$$C_i = C_{4-i} \ \text{ for } \ i = 0,1,2,3,4.$$

From the equations (17) and (18), we can show the series

$$\overline{\mathbb{S}}_i(H^k) = \mathbb{S}(H^k)|_{H=\lambda_i,t=0}$$

have the following asymptotic expansion:

$$\begin{aligned}
\overline{\mathbb{S}}_i(1) &= e^{\frac{\mu\lambda_i}{z}}\frac{1}{C_0}\Big(R_{00}+R_{01}\big(\frac{z}{\lambda_i}\big)+R_{02}\big(\frac{z}{\lambda_i}\big)^2+\dots\Big)\,,\\
\overline{\mathbb{S}}_i(H) &= e^{\frac{\mu\lambda_i}{z}}\frac{L\lambda_i}{C_0C_1}\Big(R_{10}+R_{11}\big(\frac{z}{\lambda_i}\big)+R_{12}\big(\frac{z}{\lambda_i}\big)^2+\dots\Big)\,, \qquad (21)\\
\overline{\mathbb{S}}_i(H^2) &= e^{\frac{\mu\lambda_i}{z}}\frac{L^2\lambda_i^2}{C_0C_1C_2}\Big(R_{20}+R_{21}\big(\frac{z}{\lambda_i}\big)+R_{22}\big(\frac{z}{\lambda_i}\big)^2+\dots\Big)\,,\\
\overline{\mathbb{S}}_i(H^3) &= e^{\frac{\mu\lambda_i}{z}}\frac{L^3\lambda_i^3}{C_0C_1C_2C_3}\Big(R_{30}+R_{31}\big(\frac{z}{\lambda_i}\big)+R_{32}\big(\frac{z}{\lambda_i}\big)^2+\dots\Big)\,,\\
\overline{\mathbb{S}}_i(H^4) &= e^{\frac{\mu\lambda_i}{z}}\frac{L^4\lambda_i^4}{C_0C_1C_2C_3C_4}\Big(R_{40}+R_{41}\big(\frac{z}{\lambda_i}\big)+R_{42}\big(\frac{z}{\lambda_i}\big)^2+\dots\Big)\,.
\end{aligned}$$

We follow here the normalization of [27]. Note

$$R_{0k} = R_k\,.$$

As in [27, Theorem 4], we obtain the following constraints.

Proposition 5. (Zagier-Zinger [27]) *For all $k \geq 0$, we have*

$$R_k \in \mathbb{C}[L^{\pm 1}]\,.$$

Define generators

$$\mathcal{X} = \frac{\mathsf{D}C_0}{C_0}\,, \quad \mathcal{X}_1 = \mathsf{D}\mathcal{X}\,, \quad \mathcal{X}_2 = \mathsf{D}\mathcal{X}_1\,, \quad \mathcal{Y} = \frac{\mathsf{D}C_1}{C_1}\,.$$

From (17), we obtain the following result.

Lemma 6. *For $k \geq 0$ wee have*

$$\begin{aligned}
R_{1\,k+1} =& R_{0\,k+1} + \frac{\mathsf{D}R_{0k}}{L} - \frac{\mathcal{X}}{L}R_{0k}\,,\\
R_{2\,k+1} =& R_{1\,k+1} + \frac{\mathsf{D}R_{1k}}{L} - \frac{\mathcal{X}}{L}R_{1k} - \frac{\mathcal{Y}}{L}R_{1k} + \frac{\mathsf{D}L}{L^2}R_{1k}\,,\\
R_{3\,k+1} =& R_{2\,k+1} + \frac{\mathsf{D}R_{2k}}{L} + \frac{\mathcal{X}}{L}R_{2k} + \frac{\mathcal{Y}}{L}R_{2k} - 3\frac{\mathsf{D}L}{L^2}R_{2k}\,, \qquad (22)\\
R_{4\,k+1} =& R_{3\,k+1} + \frac{\mathsf{D}R_{3k}}{L} + \frac{\mathcal{X}}{L}R_{3k} - 2\frac{\mathsf{D}L}{L^2}R_{3k}\,,\\
R_{0\,k+1} =& R_{4\,k+1} + \frac{\mathsf{D}R_{4k}}{L} - \frac{\mathsf{D}L}{L^2}R_{4k}\,.
\end{aligned}$$

Applying Lemma 6 for $k = 0, 1$, we obtain the following two equations among above generators which were also proven in [26, Section

3.1]. First,

$$\mathsf{D}\mathcal{Y} = \frac{2}{5}(L^5-1)+2(L^5-1)\mathcal{X}-2\mathcal{X}^2-4\mathcal{X}_1 + (L^5-1)\mathcal{Y}-\mathcal{Y}^2-2\mathcal{X}\mathcal{Y}\,. \tag{23}$$

For the second equation, define[14]

$$B_1 = -5\mathcal{X}\,,$$
$$B_2 = 5^2(\mathcal{X}_1+\mathcal{X}^2)\,,$$
$$B_3 = -5^3(\mathcal{X}_2+3\mathcal{X}\mathcal{X}_1+\mathcal{X}^3)\,,$$
$$B_4 = 5^4(\mathsf{D}\mathcal{X}_2+4\mathcal{X}\mathcal{X}_2+3\mathcal{X}_1^2+6\mathcal{X}^2\mathcal{X}_1+\mathcal{X}^4)\,.$$

Then, we have

$$B_4 = -(L^5-1)(10B_3-35B_2+50B_1-24)\,. \tag{24}$$

For the proof of first holomorphic anomaly equation, we will require the following generalization of Proposition 5.

Proposition 7. *For all $k \geq 0$, we have*

(i) $R_{1k} \in \mathbb{C}[L^{\pm 1}][\mathcal{X}]$,
(ii) $R_{2k} = Q_{2k} - \frac{R_{1\,k-1}}{L}\mathcal{Y}$, *with* $Q_{2k} \in \mathbb{C}[L^{\pm 1}][\mathcal{X},\mathcal{X}_1]$,
(iii) $R_{3k}, R_{4k} \in \mathbb{C}[L^{\pm 1}][\mathcal{X},\mathcal{X}_1,\mathcal{X}_2]$.

Proof. (i) Using Lemma 6, we can calculate

$$R_{1\,k+1} = \frac{\mathsf{D}R_{0k}}{L} + R_{0\,k+1} - \frac{R_{0k}}{L}\mathcal{X}\,.$$

(ii) Using Lemma 6 and relations (23), we can calculate

$$R_{2\,k+2} = \frac{\mathsf{D}^2R_{0k}}{L^2} - \frac{R_{0\,k+1}}{5L} + \frac{L^4R_{0\,k+1}}{5} + \frac{2\mathsf{D}R_{0\,k+1}}{L} + R_{0\,k+2} - \frac{2\mathsf{D}R_{0k}\mathcal{X}}{L^2} - \frac{2R_{0\,k+1}\mathcal{X}}{L} + \frac{R_{0k}\mathcal{X}^2}{L^2} - \frac{R_{0k}\mathcal{X}_1}{L^2} + \frac{-\mathsf{D}R_{0k}-LR_{0\,k+1}+R_{0k}\mathcal{X}}{L^2}\mathcal{Y}\,.$$

(iii) We can also explicitly calculate R_{3k} and R_{4k} in terms of

$$R_{0k}\,,\ R_{0\,k-1}\,,\ R_{0\,k-2}\,,\ \mathcal{X}\,,\ \mathcal{X}_1\,,\ \mathcal{X}_2\,,\ \mathcal{Y}$$

using Lemma 6 and relations (23) and (24). We can check (iii) using these explicit calculations and Proposition 5. We leave the details to the reader.

□

[14] We follow here the notation of [26] for B_k.

For the proof of second holomorphic anomaly equation, we will require the following result.

Proposition 8. *For all $k \geq 0$, we have*

(i) $R_{1k} = P_{0k} - \frac{R_{0\,k-1}}{L}\mathcal{X}$, *with* $P_{0k} \in \mathbb{C}[L^{\pm 1}]$,
(ii) $R_{2k} \in \mathbb{C}[L^{\pm 1}][A_2, A_4]$,
(iii) $R_{3k} = P_{3k} - \frac{R_{2\,k-1}}{L}\mathcal{X}$ *with* $P_{3k} \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6]$,
(iv) $R_{4k} \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6]$.

Proof. The proof follows from the explicit calculations in the proof of Proposition 7 and the definition of A_2, A_4, A_6. □

3. Higher genus series on $\overline{M}_{g,n}$

3.1. **Intersection theory on $\overline{M}_{g,n}$.** We review here the now standard method used by Givental [15, 16, 20] to express genus g descendent correlators in terms of genus 0 data. We refer the reader to [22, Section 4.1] for a more leisurely treatment.

Let $t_0, t_1, t_2, \ldots$ be formal variables. The series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \ldots$$

in the additional variable c plays a basic role. The variable c will later be replaced by the first Chern class ψ_i of a cotangent line over $\overline{M}_{g,n}$,

$$T(\psi_i) = t_0 + t_1\psi_i + t_2\psi_i^2 + \ldots\,,$$

with the index i depending on the position of the series T in the correlator.

Let $2g - 2 + n > 0$. For $a_i \in \mathbb{Z}_{\geq 0}$ and $\gamma \in H^*(\overline{M}_{g,n})$, define the correlator

$$\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \,|\, \gamma \rangle\rangle_{g,n} \;=\; \sum_{k\geq 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma\, \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{i=1}^{k} T(\psi_{n+i})\,.$$

Here, γ also denotes the pull-back of γ via the morphism

$$\overline{M}_{g,n+k} \to \overline{M}_{g,n}$$

defined by forgetting the last k points. In the above summation, the $k = 0$ term is

$$\int_{\overline{M}_{g,n}} \gamma\, \psi_1^{a_1} \cdots \psi_n^{a_n}\,.$$

We also need the following correlator defined for the unstable case,

$$\langle\langle 1, 1 \rangle\rangle_{0,2} = \sum_{k>0} \frac{1}{k!} \int_{\overline{M}_{0,2+k}} \prod_{i=1}^{k} T(\psi_{2+i})\,.$$

For formal variables $x_1, \ldots, x_n$, we also define the correlator

$$\left\langle\left\langle \frac{1}{x_1-\psi}, \ldots, \frac{1}{x_n-\psi} \,\Big|\, \gamma \right\rangle\right\rangle_{g,n} \tag{25}$$

in the standard way by expanding $\frac{1}{x_i-\psi}$ as a geometric series.

Denote by $\mathbb{L}$ the differential operator

$$\mathbb{L} = \frac{\partial}{\partial t_0} - \sum_{i=1}^{\infty} t_i \frac{\partial}{\partial t_{i-1}} = \frac{\partial}{\partial t_0} - t_1 \frac{\partial}{\partial t_0} - t_2 \frac{\partial}{\partial t_1} - \ldots .$$

The string equation yields the following result.

Lemma 9. *For $2g-2+n>0$ and $\gamma \in H^*(\overline{M}_{g,n})$, we have*

$$\mathbb{L}\langle\langle 1, \ldots, 1 \,|\, \gamma \rangle\rangle_{g,n} = 0\,,$$

$$\mathbb{L}\left\langle\left\langle \frac{1}{x_1-\psi}, \ldots, \frac{1}{x_n-\psi} \,\Big|\, \gamma \right\rangle\right\rangle_{g,n} = \left(\frac{1}{x_1} + \ldots + \frac{1}{x_n}\right)\left\langle\left\langle \frac{1}{x_1-\psi}, \ldots \frac{1}{x_n-\psi} \,\Big|\, \gamma \right\rangle\right\rangle_{g,n} .$$

We consider $\mathbb{C}(t_1)[t_2, t_3, ...]$ as $\mathbb{Z}$-graded ring over $\mathbb{C}(t_1)$ with

$$\deg(t_i) = i-1 \quad \text{for } i \geq 2\ .$$

Define a subspace of homogeneous elements by

$$\mathbb{C}\left[\frac{1}{1-t_1}\right][t_2, t_3, \ldots]_{\mathrm{Hom}} \subset \mathbb{C}(t_1)[t_2, t_3, ...]\,.$$

After the restriction $t_0 = 0$ and application of the dilaton equation, the correlators are expressed in terms of finitely many integrals (by the dimension constraints). From this, we easily see

$$\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \,|\, \gamma \rangle\rangle_{g,n}\,|_{t_0=0} \ \in\ \mathbb{C}\left[\frac{1}{1-t_1}\right][t_2, t_3, \ldots]_{\mathrm{Hom}}\,.$$

Using the leading terms (of lowest degree in $\frac{1}{(1-t_1)}$), we obtain the following result.

Lemma 10. *The set of genus 0 correlators*

$$\Big\{ \langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}\,|_{t_0=0} \Big\}_{n\geq 4}$$

freely generate the ring $\mathbb{C}(t_1)[t_2, t_3, ...]$ over $\mathbb{C}(t_1)$.

Definition 11. *For $\gamma \in H^*(\overline{M}_{g,k})$, let*

$$\mathsf{P}^{a_1,\ldots,a_n,\gamma}_{g,n}(s_0,s_1,s_2,\ldots) \in \mathbb{Q}(s_0,s_1,..)$$

be the unique rational function satisfying the condition

$$\langle\langle \psi^{a_1},\ldots,\psi^{a_n} \mid \gamma \rangle\rangle_{g,n}|_{t_0=0} = \mathsf{P}^{a_1,a_2,\ldots,a_n,\gamma}_{g,n}|_{s_i=\langle\langle 1,\ldots,1\rangle\rangle_{0,i+3}|_{t_0=0}}\,.$$

By applying Lemma 9, we obtain the two following results, see [22, Section 4.1].

Proposition 12. *For $2g-2+n>0$, we have*

$$\langle\langle 1,\ldots,1 \mid \gamma \rangle\rangle_{g,n} = \mathsf{P}^{0,\ldots,0,\gamma}_{g,n}|_{s_i=\langle\langle 1,\ldots,1\rangle\rangle_{0,i+3}}\,.$$

Proposition 13. *For $2g-2+n>0$,*

$$\left\langle\left\langle \frac{1}{x_1-\psi_1},\ldots,\frac{1}{x_n-\psi_n}\,\Big|\,\gamma \right\rangle\right\rangle_{g,n} = \\ e^{\langle\langle 1,1\rangle\rangle_{0,2}(\sum_i \frac{1}{x_i})} \sum_{a_1,\ldots,a_n} \frac{\mathsf{P}^{a_1,\ldots,a_n,\gamma}_{g,n}|_{s_i=\langle\langle 1,\ldots,1\rangle\rangle_{0,i+3}}}{x_1^{a_1+1}\cdots x_n^{a_n+1}}.$$

$$\mathbb{L}\langle\langle 1,1\rangle\rangle_{0,2} = 1\,, \qquad \langle\langle 1,1\rangle\rangle_{0,2}|_{t_0=0} = 0\,.$$

The definition given in (25) of the correlator is valid in the stable range

$$2g-2+n>0\,.$$

The unstable case $(g,n)=(0,2)$ plays a special role. We define

$$\left\langle\left\langle \frac{1}{x_1-\psi_1},\frac{1}{x_2-\psi_2} \right\rangle\right\rangle_{0,2}$$

by adding the degenerate term

$$\frac{1}{x_1+x_2}$$

to the terms obtained by the expansion of $\frac{1}{x_i-\psi_i}$ as a geometric series. The degenerate term is associated to the (unstable) moduli space of genus 0 with 2 markings. By [22, Section 4.2], we have.

Proposition 14. *We have*

$$\left\langle\left\langle \frac{1}{x_1-\psi_1},\frac{1}{x_2-\psi_2} \right\rangle\right\rangle_{0,2} = e^{\langle\langle 1,1\rangle\rangle_{0,2}\left(\frac{1}{x_1}+\frac{1}{x_2}\right)} \left(\frac{1}{x_1+x_2}\right)\,.$$

3.2. **Local invariants and wall crossing.** The torus T acts on the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^4,d)$ and $\overline{Q}_{g,n}(\mathbb{P}^4,d)$. We consider here special localization contributions associated to the fixed points $p_i \in \mathbb{P}^4$.

Consider first the moduli of stable maps. Let

$$\overline{M}_{g,n}(\mathbb{P}^4,d)^{\mathsf{T},p_i} \subset \overline{M}_{g,n}(\mathbb{P}^4,d)$$

be the union of T-fixed loci which parameterize stable maps obtained by attaching T-fixed rational tails to a genus g, n-pointed Deligne-Mumford stable curve contracted to the point $p_i \in \mathbb{P}^4$. Similarly, let

$$\overline{Q}_{g,n}(\mathbb{P}^4,d)^{\mathsf{T},p_i} \subset \overline{Q}_{g,n}(\mathbb{P}^4,d)$$

be the parallel T-fixed locus parameterizing stable quotients obtained by attaching base points to a genus g, n-pointed Deligne-Mumford stable curve contracted to the point $p_i \in \mathbb{P}^4$.

Let Λ_i denote the localization of the ring

$$\mathbb{C}[\lambda_0^{\pm 1},\ldots,\lambda_4^{\pm 1}]$$

at the five tangent weights at $p_i \in \mathbb{P}^4$. Using the virtual localization formula [17], there exist unique series

$$S_{p_i} \in \Lambda_i[\psi][[Q]]$$

for which the localization contribution of the T-fixed locus $\overline{M}_{g,n}(\mathbb{P}^4,d)^{\mathsf{T},p_i}$ to the equivariant Gromov-Witten invariants of formal quintic can be written as

$$\sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,n}(\mathbb{P}^4,d)^{\mathsf{T},p_i}]^{\mathrm{vir}}} \frac{e(\mathrm{Obs})}{e(\mathrm{Nor})} \psi_1^{a_1}\cdots\psi_n^{a_n} =$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \mathsf{H}_g^{p_i}\, \psi_1^{a_1}\cdots\psi_n^{a_n} \prod_{j=1}^{k} S_{p_i}(\psi_{n+j})\,.$$

Here, $\mathsf{H}_g^{p_i}$ is the standard vertex class,

$$\frac{e(\mathbb{E}_g^* \otimes T_{p_i}(\mathbb{P}^4)))}{e(T_{p_i}(\mathbb{P}^4))} \cdot \frac{(5\lambda_i)}{e(\mathbb{E}_g^* \otimes (5\lambda_i))}\,, \tag{26}$$

obtained from the Hodge bundle $\mathbb{E}_g \to \overline{M}_{g,n+k}$.

Similarly, the application of the virtual localization formula to the moduli of stable quotients yields classes

$$F_{p_i,k} \in H^*(\overline{M}_{g,n|k}) \otimes_{\mathbb{C}} \Lambda_i$$

for which the contribution of $\overline{Q}_{g,n}(\mathbb{P}^4,d)^{T,p_i}$ is given by

$$\sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_{g,n}(\mathbb{P}^4,d)^{\mathsf{T},p_i}]^{\mathrm{vir}}} \frac{e(\mathrm{Obs})}{e(\mathrm{Nor})} \psi_1^{a_1}\cdots\psi_n^{a_n} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n|k}} \mathsf{H}_g^{p_i}\, \psi_1^{a_1}\cdots\psi_n^{a_n}\, F_{p_i,k}.$$

Here $\overline{M}_{g,n|k}$ is the moduli space of genus g curves with markings

$$\{p_1,\cdots,p_n\} \cup \{\hat{p}_1\cdots\hat{p}_k\} \in C^{\mathrm{ns}} \subset C$$

satisfying the conditions

(i) the points p_i are distinct,
(ii) the points $\hat{p}_j$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C(\sum_{i=1}^{m} p_i + \epsilon \sum_{j=1}^{k} \hat{p}_j)$$

for every strictly positive $\epsilon \in \mathbb{Q}$.

The Hodge class $\mathsf{H}_g^{p_i}$ is given again by formula (26) using the Hodge bundle

$$\mathbb{E}_g \to \overline{M}_{g,n|k}\,.$$

Definition 15. *For $\gamma \in H^*(\overline{M}_{g,n})$, let*

$$\langle\langle \psi_1^{a_1},\ldots,\psi_n^{a_n} \,|\, \gamma \rangle\rangle_{g,n}^{p_i,\infty} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma\, \psi_1^{a_1}\cdots\psi_n^{a_n} \prod_{j=1}^{k} S_{p_i}(\psi_{n+j})\,,$$

$$\langle\langle \psi_1^{a_1},\ldots,\psi_n^{a_n} \,|\, \gamma \rangle\rangle_{g,n}^{p_i,0+} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n|k}} \gamma\, \psi_1^{a_1}\cdots\psi_n^{a_n}\, F_{p_i,k}\,.$$

Proposition 16 (Ciocan-Fontanine, Kim [8])**.** *For $2g-2+n>0$, we have the wall crossing relation*

$$\langle\langle \psi_1^{a_1},\ldots,\psi_n^{a_n} \,|\, \gamma \rangle\rangle_{g,n}^{p_i,\infty}(Q(q)) = (I_0^{\mathsf{Q}})^{2g-2+n} \langle\langle \psi_1^{a_1},\ldots,\psi_n^{a_n} \,|\, \gamma \rangle\rangle_{g,n}^{p_i,0+}(q)$$

where $Q(q)$ is the mirror map

$$Q(q) = \exp\left(\frac{I_1^{\mathsf{Q}}(q)}{I_0^{\mathsf{Q}}(q)}\right)\,.$$

Proposition 16 is a consequence of [8, Lemma 5.5.1]. The mirror map here is the mirror map for quintic discussed in Section 0.1. Propositions 12 and 16 together yield

$$\begin{aligned}\langle\langle 1,\ldots,1\,|\,\gamma\rangle\rangle^{p_i,\infty}_{g,n} &= \mathsf{P}^{0,\ldots,0,\gamma}_{g,n}\big(\langle\langle 1,1,1\rangle\rangle^{p_i,\infty}_{0,3}, \langle\langle 1,1,1,1\rangle\rangle^{p_i,\infty}_{0,4},\ldots\big)\,,\\ \langle\langle 1,\ldots,1\,|\,\gamma\rangle\rangle^{p_i,0+}_{g,n} &= \mathsf{P}^{0,\ldots,0,\gamma}_{g,n}\big(\langle\langle 1,1,1\rangle\rangle^{p_i,0+}_{0,3}, \langle\langle 1,1,1,1\rangle\rangle^{p_i,0+}_{0,4},\ldots\big)\,.\end{aligned}$$

Similarly, using Propositions 13 and 16, we obtain

$$\left\langle\!\!\left\langle \frac{1}{x_1-\psi},\ldots,\frac{1}{x_n-\psi}\,\Big|\,\gamma\right\rangle\!\!\right\rangle^{p_i,\infty}_{g,n} =$$

$$e^{\langle\langle 1,1\rangle\rangle^{p_i,\infty}_{0,2}\left(\sum_i \frac{1}{x_i}\right)} \sum_{a_1,\ldots,a_n} \frac{\mathsf{P}^{a_1,\ldots,a_n,\gamma}_{g,n}\big(\langle\langle 1,1,1\rangle\rangle^{p_i,\infty}_{0,3}, \langle\langle 1,1,1,1\rangle\rangle^{p_i,\infty}_{0,4},\ldots\big)}{x_1^{a_1+1}\cdots x_n^{a_n+1}}\,,$$

$$\left\langle\!\!\left\langle \frac{1}{x_1-\psi},\ldots,\frac{1}{x_n-\psi}\,\Big|\,\gamma\right\rangle\!\!\right\rangle^{p_i,0+}_{g,n} = \tag{27}$$

$$e^{\langle\langle 1,1\rangle\rangle^{p_i,0+}_{0,2}\left(\sum_i \frac{1}{x_i}\right)} \sum_{a_1,\ldots,a_n} \frac{\mathsf{P}^{a_1,\ldots,a_n,\gamma}_{g,n}\big(\langle\langle 1,1,1\rangle\rangle^{p_i,0+}_{0,3}, \langle\langle 1,1,1,1\rangle\rangle^{p_i,0+}_{0,4},\ldots\big)}{x_1^{a_1+1}\cdots x_n^{a_n+1}}\,.$$

4. Higher genus series on the formal quintic

4.1. **Overview.** We apply Givental's the localization strategy [15, 16, 20] for Gromov-Witten theory to the stable quotient invariants of formal quintic. The contribution $\mathrm{Cont}_\Gamma(q)$ discussed in Section 1 of a graph $\Gamma \in \mathsf{G}_g(\mathbb{P}^4)$ can be separated into vertex and edge contributions. We express the vertex and edge contributions in terms of the series $\mathbb{S}_i$ and $\mathbb{V}_{ij}$ of Section 2.3. Our treatment here follows our study of $K\mathbb{P}^2$ in [22, Section 5]

4.2. **Edge terms.** Recall the definition[15]of $\mathbb{V}_{ij}$ given in Section 2.3,

$$\mathbb{V}_{ij} = \left\langle\!\!\left\langle \frac{\phi_i}{x-\psi},\frac{\phi_j}{y-\psi}\right\rangle\!\!\right\rangle^{0+,0+}_{0,2}\,. \tag{28}$$

Let $\overline{\mathbb{V}}_{ij}$ denote the restriction of $\mathbb{V}_{ij}$ to $t=0$. Via formula (9), $\overline{\mathbb{V}}_{ij}$ is a summation of contributions of fixed loci indexed by a graph Γ consisting of two vertices connected by a unique edge. Let w_1 and w_2 be T-weights. Denote by

$$\overline{\mathbb{V}}^{w_1,w_2}_{ij}$$

[15]We use the variables x_1 and x_2 here instead of x and y.

the summation of contributions of T-fixed loci with tangent weights precisely w_1 and w_2 on the first rational components which exit the vertex components over p_i and p_j.

The series $\overline{\mathbb{V}}_{ij}^{w_1,w_2}$ includes *both* vertex and edge contributions. By definition (28) and the virtual localization formula, we find the following relationship between $\overline{\mathbb{V}}_{ij}^{w_1,w_2}$ and the corresponding pure edge contribution $\mathsf{E}_{ij}^{w_1,w_2}$,

$$e_i\overline{\mathbb{V}}_{ij}^{w_1,w_2}e_j = \left\langle\left\langle \frac{1}{w_1-\psi}, \frac{1}{x_1-\psi}\right\rangle\right\rangle_{0,2}^{p_i,0+} \mathsf{E}_{ij}^{w_1,w_2} \left\langle\left\langle \frac{1}{w_2-\psi}, \frac{1}{x_2-\psi}\right\rangle\right\rangle_{0,2}^{p_j,0+}$$

$$= \frac{e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{w_1}+\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{x_1}}}{w_1+x_1} \mathsf{E}_{ij}^{w_1,w_2} \frac{e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{w_2}+\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{x_2}}}{w_2+x_2}$$

$$= \sum_{a_1,a_2} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{x_1}+\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{w_1}} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{x_2}+\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{w_2}} (-1)^{a_1+a_2}\frac{\mathsf{E}_{ij}^{w_1,w_2}}{w_1^{a_1}w_2^{a_2}} x_1^{a_1-1}x_2^{a_2-1}\,.$$

After summing over all possible weights, we obtain

$$e_i\left(\overline{\mathbb{V}}_{ij}-\frac{\delta_{ij}}{e_i(x_1+x_2)}\right)e_j = \sum_{w_1,w_2} e_i\overline{\mathbb{V}}_{ij}^{w_1,w_2}e_j\,.$$

The above calculations immediately yield the following result.

Lemma 17. *We have*

$$\left[e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{x_1}} e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{x_2}} e_i\left(\overline{\mathbb{V}}_{ij}-\frac{\delta_{ij}}{e_i(x_1+x_2)}\right)e_j\right]_{x_1^{a_1-1}x_2^{a_2-1}} =$$

$$\sum_{w_1,w_2} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{w_1}} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{w_2}} (-1)^{a_1+a_2}\frac{\mathsf{E}_{ij}^{w_1,w_2}}{w_1^{a_1}w_2^{a_2}}\,.$$

The notation $[\ldots]_{x_1^{a_1-1}x_2^{a_2-1}}$ in Lemma 17 denotes the coefficient of $x_1^{a_1-1}x_2^{a_2-1}$ in the series expansion of the argument.

4.3. **A simple graph.** Before treating the general case, we present the localization formula for a simple graph[16]. Let $\Gamma \in \mathsf{G}_g(\mathbb{P}^4)$ consist of two vertices and one edge,

$$v_1, v_2 \in \Gamma(V)\,, \qquad e \in \Gamma(E)$$

[16]We follow here the notation of Section 1.

with genus and T-fixed point assignments

$$\mathsf{g}(v_i) = g_i\,, \qquad \mathsf{p}(v_i) = p_i\,.$$

Let w_1 and w_2 be tangent weights at the vertices p_1 and p_2 respectively. Denote by $\mathrm{Cont}_{\Gamma,w_1,w_2}$ the summation of contributions to

$$\sum_{d>0} q^d \left(e(\mathrm{Obs}) \cap \left[\overline{Q}_g(\mathbb{P}^4,d)\right]^{\mathrm{vir}}\right) \tag{29}$$

of T-fixed loci with tangent weights precisely w_1 and w_2 on the first rational components which exit the vertex components over p_1 and p_2. We can express the localization formula for (29) as

$$\left\langle\left\langle \frac{1}{w_1-\psi} \,\middle|\, \mathsf{H}^{p_1}_{g_1} \right\rangle\right\rangle^{p_1,0+}_{g_1,1} \mathsf{E}^{w_1,w_2}_{12} \left\langle\left\langle \frac{1}{w_2-\psi} \,\middle|\, \mathsf{H}^{p_2}_{g_2} \right\rangle\right\rangle^{p_2,0+}_{g_2,1}$$

which equals

$$\sum_{a_1,a_2} e^{\frac{\langle\langle 1,1\rangle\rangle^{p_1,0+}_{0,2}}{w_1}} \frac{\mathsf{P}\left[\psi^{a_1-1} \,\middle|\, \mathsf{H}^{p_1}_{g_1}\right]^{p_1,0+}_{g_1,1}}{w_1^{a_1}} \mathsf{E}^{w_1,w_2}_{12} e^{\frac{\langle\langle 1,1\rangle\rangle^{p_2,0+}_{0,2}}{w_2}} \frac{\mathsf{P}\left[\psi^{a_2-1} \,\middle|\, \mathsf{H}^{p_2}_{g_2}\right]^{p_2,0+}_{g_2,1}}{w_2^{a_2}}$$

where $\mathsf{H}^{p_i}_{g_i}$ is the Hodge class (26). We have used here the notation

$$\mathsf{P}\left[\psi_1^{k_1},\ldots,\psi_n^{k_n} \,\middle|\, \mathsf{H}^{p_i}_h\right]^{p_i,0+}_{h,n} = \\ \mathsf{P}^{k_1,\ldots,k_n,\mathsf{H}^{p_i}_h}_{h,1}\left(\langle\langle 1,1,1\rangle\rangle^{p_i,0+}_{0,3}, \langle\langle 1,1,1,1\rangle\rangle^{p_i,0+}_{0,4}, \ldots\right)$$

and applied (27).

After summing over all possible weights w_1, w_2 and applying Lemma 17, we obtain the following result for the full contribution

$$\mathrm{Cont}_\Gamma = \sum_{w_1,w_2} \mathrm{Cont}_{\Gamma,w_1,w_2}$$

of Γ to $\sum_{d\geq 0} q^d (e(\mathrm{Obs}) \cap \left[\overline{Q}_g(\mathbb{P}^4,d)\right]^{\mathrm{vir}})$.

Proposition 18. *We have*

$$\mathrm{Cont}_\Gamma = \sum_{a_1,a_2>0} \mathsf{P}\left[\psi^{a_1-1} \,\middle|\, \mathsf{H}^{p_i}_{g_1}\right]^{p_i,0+}_{g_1,1} \mathsf{P}\left[\psi^{a_2-1} \,\middle|\, \mathsf{H}^{p_j}_{g_2}\right]^{p_j,0+}_{g_2,1} \\ \cdot(-1)^{a_1+a_2} \left[e^{-\frac{\langle\langle 1,1\rangle\rangle^{p_i,0+}_{0,2}}{x_1}} e^{-\frac{\langle\langle 1,1\rangle\rangle^{p_j,0+}_{0,2}}{x_2}} e_i \left(\mathbb{V}_{ij} - \frac{\delta_{ij}}{e_i(x_1+x_2)}\right) e_j \right]_{x_1^{a_1-1}x_2^{a_2-1}}.$$

4.4. **A general graph.** We apply the argument of Section 4.3 to obtain a contribution formula for a general graph Γ.

Let $\Gamma \in \mathsf{G}_{g,0}(\mathbb{P}^4)$ be a decorated graph as defined in Section 1. The *flags* of Γ are the half-edges[17]. Let F be the set of flags. Let

$$\mathsf{w} : \mathsf{F} \to \mathrm{Hom}(\mathsf{T}, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be a fixed assignment of T-weights to each flag.

We first consider the contribution $\mathrm{Cont}_{\Gamma,\mathsf{w}}$ to

$$\sum_{d\geq 0} q^d (e(\mathrm{Obs}) \cap \left[\overline{Q}_g(\mathbb{P}^4, d)\right]^{\mathrm{vir}})$$

of the T-fixed loci associated Γ satisfying the following property: the tangent weight on the first rational component corresponding to each $f \in \mathsf{F}$ is exactly given by $\mathsf{w}(f)$. We have

$$\mathrm{Cont}_{\Gamma,\mathsf{w}} = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A}\in\mathbb{Z}^{\mathsf{F}}_{>0}} \prod_{v\in\mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_{\Gamma,\mathsf{w}}(v) \prod_{e\in\mathsf{E}} \mathrm{Cont}^{\mathsf{A}}_{\Gamma,\mathsf{w}}(e)\,. \tag{30}$$

The terms on the right side of (30) require definition:

- The sum on the right is over the set $\mathbb{Z}^{\mathsf{F}}_{>0}$ of all maps
$$\mathsf{A} : \mathsf{F} \to \mathbb{Z}_{>0}$$
corresponding to the sum over a_1, a_2 in Proposition 18.
- For $v \in \mathsf{V}$ with n incident flags with w-values $(w_1, \ldots, w_n)$ and A-values $(a_1, a_2, ..., a_n)$,
$$\mathrm{Cont}^{\mathsf{A}}_{\Gamma,\mathsf{w}}(v) = \frac{\mathsf{P}\left[\psi_1^{a_1-1}, \ldots, \psi_n^{a_n-1} \,\middle|\, \mathsf{H}^{\mathsf{p}(v)}_{\mathsf{g}(v)}\right]^{\mathsf{p}(v),0+}_{\mathsf{g}(v),n}}{w_1^{a_1}\cdots w_n^{a_n}}\,.$$
- For $e \in \mathsf{E}$ with assignments $(\mathsf{p}(v_1), \mathsf{p}(v_2))$ for the two associated vertices[18] and w-values (w_1, w_2) for the two associated flags,
$$\mathrm{Cont}_{\Gamma,\mathsf{w}}(e) = e^{\frac{\langle\langle 1,1\rangle\rangle^{\mathsf{p}(v_1),0+}_{0,2}}{w_1}} e^{\frac{\langle\langle 1,1\rangle\rangle^{\mathsf{p}(v_2),0+}_{0,2}}{w_2}} \mathsf{E}^{w_1,w_2}_{\mathsf{p}(v_1),\mathsf{p}(v_2)}\,.$$

The localization formula then yields (30) just as in the simple case of Section 4.3.

By summing the contribution (30) of Γ over all the weight functions w and applying Lemma 17, we obtain the following result which generalizes Proposition 18.

[17]Flags are either half-edges or markings.

[18]In case e is self-edge, $v_1 = v_2$.

Proposition 19. *We have*

$$\mathrm{Cont}_\Gamma = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}}_{>0}} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \prod_{e \in \mathsf{E}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e)\,,$$

where the vertex and edge contributions with incident flag A*-values* $(a_1, \ldots, a_n)$ *and* (b_1, b_2) *respectively are*

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(v) = \mathsf{P}\left[\psi_1^{a_1-1}, \ldots, \psi_n^{a_n-1} \,\middle|\, \mathsf{H}^{\mathsf{p}(v)}_{\mathsf{g}(v)}\right]^{\mathsf{p}(v),0+}_{\mathsf{g}(v),n}\,,$$

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(e) = (-1)^{b_1+b_2}\left[e^{-\frac{\langle\langle 1,1\rangle\rangle^{\mathsf{p}(v_1),0+}_{0,2}}{x_1}} e^{-\frac{\langle\langle 1,1\rangle\rangle^{\mathsf{p}(v_2),0+}_{0,2}}{x_2}} e_i\left(\overline{\mathbb{V}}_{ij} - \frac{1}{e_i(x_1+x_2)}\right)e_j\right]_{x_1^{b_1-1}x_2^{b_2-1}}\,,$$

where $\mathsf{p}(v_1) = p_i$ *and* $\mathsf{p}(v_2) = p_j$ *in the second equation.*

4.5. **Legs.** Let $\Gamma \in \mathsf{G}_{g,n}(\mathbb{P}^4)$ be a decorated graph with markings. While no markings are needed to define the stable quotient invariants of formal quintic, the contributions of decorated graphs with markings will appear in the proof of the holomorphic anomaly equation. The formula for the contribution $\mathrm{Cont}_\Gamma(H^{k_1}, \ldots, H^{k_n})$ of Γ to

$$\sum_{d \geq 0} q^d \prod_{j=0}^{n} \mathrm{ev}^*(H^{k_j}) \cdot e(\mathrm{Obs}) \cap \left[\overline{Q}_g(\mathbb{P}^4, d)\right]^{\mathrm{vir}}$$

is given by the following result.

Proposition 20. *We have*

$$\mathrm{Cont}_\Gamma(H^{k_1}, \ldots, H^{k_n}) = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}}_{>0}} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \prod_{e \in \mathsf{E}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e) \prod_{l \in \mathsf{L}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(l)\,,$$

where the leg contribution is

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(l) = (-1)^{\mathsf{A}(l)-1}\left[e^{-\frac{\langle\langle 1,1\rangle\rangle^{\mathsf{p}(l),0+}_{0,2}}{z}} \overline{\mathbb{S}}_{\mathsf{p}(l)}(H^{k_l})\right]_{z^{\mathsf{A}(l)-1}}\,.$$

The vertex and edge contributions are same as before.

The proof of Proposition 20 follows the vertex and edge analysis. We leave the details as an exercise for the reader. The parallel statement for Gromov-Witten theory can be found in [15, 16, 20].

5. Vertices, edges, and legs

5.1. **Overview.** Following the analysis of $K\mathbb{P}^2$ in [22, Section 6] which uses results of Givental [15, 16, 20] and the wall-crossing of [8], we calculate here the vertex and edge contributions in terms of the function R_k of Section 2.4.4.

5.2. **Calculations in genus 0.** We follow the notation introduced in Section 3.1. Recall the series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \dots .$$

Proposition 21. (Givental [15, 16, 20]) *For $n \geq 3$, we have*

$$\langle\langle 1, \dots, 1 \rangle\rangle_{0,n}^{p_i,\infty} =$$

$$(\sqrt{\Delta_i})^{2g-2+n} \left(\sum_{k\geq 0} \frac{1}{k!} \int_{\overline{M}_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \Big|_{t_0=0, t_1=0, t_{j\geq 2}=(-1)^j \frac{Q_{j-1}}{\lambda_i^{j-1}}}$$

where the functions $\sqrt{\Delta_i}$, Q_l are defined by

$$\overline{\mathbb{S}}_i^{\infty}(1) = e_i \Big\langle\Big\langle \frac{\phi_i}{z-\psi}, 1 \Big\rangle\Big\rangle_{0,2}^{p_i,\infty} = \frac{e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,\infty}}{z}}}{\sqrt{\Delta_i}} \left(1 + \sum_{l=1}^{\infty} Q_l \left(\frac{z}{\lambda_i} \right)^l \right) .$$

From (21) and Proposition 16, we have

$$\langle\langle 1, 1 \rangle\rangle_{0,2}^{p_i,\infty} = \mu\lambda_i ,$$

$$\sqrt{\Delta_i} = \frac{C_0}{R_0} ,$$

$$Q_k = \frac{R_k}{R_0} .$$

Using Proposition 16 again, we have proven the following result.

Proposition 22. *For $n \geq 3$, we have*

$$\langle\langle 1, \dots, 1 \rangle\rangle_{0,n}^{p_i,0+} =$$

$$R_0^{2-n} \left(\sum_{k\geq 0} \frac{1}{k!} \int_{\overline{M}_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \Big|_{t_0=0, t_1=0, t_{j\geq 2}=(-1)^j \frac{R_{j-1}}{\lambda_i^{j-1} R_0}} .$$

Proposition 22 immediately implies the evaluation

$$\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^{p_i,0+} = \frac{1}{R_0} . \tag{31}$$

Another simple consequence of Proposition 22 is the following basic property.

Corollary 23. *For $n \geq 3$, we have $\langle\langle 1, \dots, 1 \rangle\rangle_{0,n}^{p_i,0+} \in \mathbb{C}[R_0^{\pm 1}, R_1, R_2, ...][\lambda_i^{-1}]$.*

5.3. **Vertex and edge analysis.** By Proposition 19, we have decomposition of the contribution to $\Gamma \in \mathsf{G}_g(\mathbb{P}^4)$ to the stable quotient theory of formal quintic into vertex terms and edge terms

$$\mathrm{Cont}_\Gamma = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}}_{>0}} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \prod_{e \in \mathsf{E}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e)\,.$$

Lemma 24. *We have* $\mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \in \mathbb{C}(\lambda_0, \ldots, \lambda_4)[L^{\pm 1}]$.

Proof. By Proposition 19,

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(v) = \mathsf{P}\left[\psi_1^{a_1-1}, \ldots, \psi_n^{a_n-1} \,\middle|\, \mathsf{H}^{\mathsf{p}(v)}_{\mathsf{g}(v)}\right]^{\mathsf{p}(v),0+}_{\mathsf{g}(v),n}\,.$$

The right side of the above formula is a polynomial in the variables

$$\frac{1}{\langle\langle 1,1,1\rangle\rangle^{\mathsf{p}(v),0+}_{0,3}} \quad \text{and} \quad \left\{ \langle\langle 1, \ldots, 1\rangle\rangle^{\mathsf{p}(v),0+}_{0,n}|_{t_0=0} \right\}_{n\geq 3}$$

with coefficients in $\mathbb{C}(\lambda_0, \ldots, \lambda_4)$. The Lemma then follows from the evaluation (31), Corollary 23, and Proposition 5.

Both the positive and the negative powers of $\langle\langle 1,1,1\rangle\rangle^{\mathsf{p}(v),0+}_{0,3}$ are required here, since $R_0^{\pm 1}$ occurs in Corollary 23. □

Let $e \in \mathsf{E}$ be an edge connecting the T-fixed points $p_i, p_j \in \mathbb{P}^4$. Let the A-values of the respective half-edges be (k,l).

Lemma 25. *We have* $\mathrm{Cont}^{\mathsf{A}}_\Gamma(e) \in \mathbb{C}(\lambda_0, \ldots, \lambda_4)[L^{\pm 1}, \mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}]$ *and*

- *the degree of* $\mathrm{Cont}^{\mathsf{A}}_\Gamma(e)$ *with respect to* $\mathcal{Y}$ *is* 1,
- *the coefficient of* $\mathcal{Y}$ *in* $\mathrm{Cont}^{\mathsf{A}}_\Gamma(e)$ *is*

$$(-1)^{k+l+1} \frac{R_{1\,k-1} R_{1\,l-1}}{5L^3 \lambda_i^{k-2} \lambda_j^{l-2}}\,.$$

Proof. By Proposition 19,

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(e) = (-1)^{k+l} \left[e^{-\frac{\mu\lambda_i}{x} - \frac{\mu\lambda_j}{y}} e_i \left(\overline{\mathbb{V}}_{ij} - \frac{\delta_{ij}}{e_i(x+y)} \right) e_j \right]_{x^{k-1}y^{l-1}}\,.$$

Using also the equation

$$e_i \overline{\mathbb{V}}_{ij}(x,y) e_j = \frac{\sum_{r=0}^4 \overline{\mathbb{S}}_i(\phi_r)|_{z=x}\, \overline{\mathbb{S}}_j(\phi^r)|_{z=y}}{x+y}\,,$$

we write $\mathrm{Cont}^{\mathsf{A}}_\Gamma(e)$ as

$$\left[(-1)^{k+l} e^{-\frac{\mu\lambda_i}{x} - \frac{\mu\lambda_j}{y}} \sum_{r=0}^4 \overline{\mathbb{S}}_i(\phi_r)|_{z=x}\, \overline{\mathbb{S}}_j(\phi^r)|_{z=y} \right]_{x^k y^{l-1} - x^{k+1}y^{l-2} + \ldots + (-1)^{k-1} x^{k+l-1}}$$

where the subscript signifies a (signed) sum of the respective coefficients. If we substitute the asymptotic expansions (21) for

$$\overline{\mathbb{S}}_i(1)\,,\quad \overline{\mathbb{S}}_i(H)\,,\quad \overline{\mathbb{S}}_i(H^2)\,,\quad \overline{\mathbb{S}}_i(H^3)\,,\quad \overline{\mathbb{S}}_i(H^4)$$

in the above expression, the Lemma follows from Proposition 7. □

Similarly, we obtain the following result using Proposition 8.

Lemma 26. *We have* $\mathrm{Cont}^{\mathsf{A}}_\Gamma(e) \in \mathbb{C}(\lambda_0,\ldots,\lambda_4)[L^{\pm1},\mathcal{X},A_2,A_4,A_6]$ *and*

- *the degree of* $\mathrm{Cont}^{\mathsf{A}}_\Gamma$ *with respect to* $\mathcal{X}$ *is 1,*
- *the coefficient of* $\mathcal{X}$ *in* $\mathrm{Cont}^{\mathsf{A}}_\Gamma$ *is*

$$(-1)^{k+l+1}\left(\frac{R_{0\,k-1}R_{2\,l-1}}{5L^3\lambda_i^{k-1}\lambda_j^{l-3}}+\frac{R_{2\,k-1}R_{0\,l-1}}{5L^3\lambda_i^{k-3}\lambda_j^{l-1}}\right).$$

5.4. **Legs.** Using the contribution formula of Proposition 20,

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(l) \;=\; (-1)^{\mathsf{A}(l)-1}\left[e^{-\frac{\langle\langle 1,1\rangle\rangle^{\mathsf{p}(l),0+}_{0,2}}{z}}\,\overline{\mathbb{S}}_{\mathsf{p}(l)}(H^{k_l})\right]_{z^{\mathsf{A}(l)-1}},$$

we easily conclude

- when the insertion at the marking l is H^0,

$$C_0\cdot \mathrm{Cont}^{\mathsf{A}}_\Gamma(l)\in\mathbb{C}(\lambda_0,\ldots,\lambda_4)[L^{\pm1}]\,,$$

- when the insertion at the marking l is H^1,

$$C_0C_1\cdot \mathrm{Cont}^{\mathsf{A}}_\Gamma(l)\in\mathbb{C}(\lambda_0,\ldots,\lambda_4)[L^{\pm1},\mathcal{X}]\,,$$

- when the insertion at the marking l is H^2,

$$C_0C_1C_2\cdot \mathrm{Cont}^{\mathsf{A}}_\Gamma(l)\in\mathbb{C}(\lambda_0,\ldots,\lambda_4)[L^{\pm1},\mathcal{X},\mathcal{X}_1,\mathcal{Y}]\,,$$

- when the insertion at the marking l is H^3,

$$C_0C_1C_2C_3\cdot \mathrm{Cont}^{\mathsf{A}}_\Gamma(l)\in\mathbb{C}(\lambda_0,\ldots,\lambda_4)[L^{\pm1},\mathcal{X},\mathcal{X}_1,\mathcal{X}_2]\,,$$

- when the insertion at the marking l is H^4,

$$C_0C_1C_2C_3C_4\cdot \mathrm{Cont}^{\mathsf{A}}_\Gamma(l)\in\mathbb{C}(\lambda_0,\ldots,\lambda_4)[L^{\pm1},\mathcal{X},\mathcal{X}_1,\mathcal{X}_2]\,.$$

6. Holomorphic anomaly for the formal quintic

6.1. **Proof of Theorem 1.** By definition, we have

$$
\begin{aligned}
K_2(q) =& -\frac{1}{L^5}\mathcal{X}\,,\\
A_2(q) =& \frac{1}{L^5}\left(-\frac{1}{5}\mathcal{Y}-\frac{2}{5}\mathcal{X}-\frac{3}{25}\right),\\
A_4(q) =& \frac{1}{L^{10}}\left(-\frac{1}{25}\mathcal{X}^2-\frac{1}{25}\mathcal{X}\mathcal{Y}+\frac{1}{25}\mathcal{X}_1+\frac{2}{25^2}\right),\\
A_6(q) =& \frac{1}{10\cdot 5^5 L^{15}}\Big(4+125\mathcal{X}_1+50\mathcal{X}(1+10\mathcal{X}_1)\\
& -5L^5(1+10\mathcal{X}+25\mathcal{X}^2+25\mathcal{X}_1)+125\mathcal{X}_2-125\mathcal{X}^2(\mathcal{Y}-1)\Big)\,.
\end{aligned}
$$

Hence, statement (i),

$$\widetilde{\mathcal{F}}^{\mathsf{B}}_g(q)\in\mathbb{C}[L^{\pm1}][C_0^{\pm1},K_2,A_2,A_4,A_6]\,,$$

follows from Proposition 7, 8, 19 and Lemmas 24 - 26.

Since

$$\frac{\partial}{\partial T}=\frac{q}{C_1}\frac{\partial}{\partial q}\,,$$

statement (ii),

$$\frac{\partial^k\widetilde{\mathcal{F}}^{\mathsf{B}}_g}{\partial T^k}(q)\in\mathbb{C}[L^{\pm1}][C_0^{\pm1},C_1^{-1},K_2,A_2,A_4,A_6]\,, \tag{32}$$

follows since the ring

$$\mathbb{C}[L^{\pm1}][C_0^{\pm1},C_1^{-1},K_2,A_2,A_4,A_6]=\mathbb{C}[L^{\pm1}][C_0^{\pm1},C_1^{-1},\mathcal{X},\mathcal{X}_1,\mathcal{X}_2,\mathcal{Y}]$$

is closed under the action of the differential operator

$$\mathsf{D}=q\frac{\partial}{\partial q}$$

by (23). The degree of C_1^{-1} in (32) is 1 which yields statement (iii). ☐

Remark 27. The proof of Theorem 1 actually yields:

$$C_0^{2-2g}\cdot\widetilde{\mathcal{F}}^{\mathsf{B}}_g\,,\quad C_0^{2-2g}C_1^k\cdot\frac{\partial^k\widetilde{\mathcal{F}}^{\mathsf{B}}_g}{\partial T^k}\;\in\;\mathbb{C}[L^{\pm1}][A_2,A_4,A_6,K_2]\,.$$

6.2. **Proof of Theorem 2: first equation.** Let $\Gamma\in\mathsf{G}_g(\mathbb{P}^4)$ be a decorated graph. Let us fix an edge $f\in\mathsf{E}(\Gamma)$:

- if Γ is connected after deleting f, denote the resulting graph by

$$\Gamma^0_f\in\mathsf{G}_{g-1,2}(\mathbb{P}^4)\,,$$

- •• if Γ is disconnected after deleting f, denote the resulting two graphs by

$$\Gamma^1_f \in \mathsf{G}_{g_1,1}(\mathbb{P}^4) \quad \text{and} \quad \Gamma^2_f \in \mathsf{G}_{g_2,1}(\mathbb{P}^4)$$

where $g = g_1 + g_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph Γ^0_f in case • should be viewed as sum of 2 graphs

$$\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)}\,.$$

Similarly, in case ••, we will sum over the ordering of g_1 and g_2. As usually, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.

By Proposition 19, we have the following formula for the contribution of the graph Γ to the theory of the formal quintic,

$$\mathrm{Cont}_\Gamma = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}}_{\geq 0}} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \prod_{e \in \mathsf{E}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e)\,.$$

Let f connect the T-fixed points $p_i, p_j \in \mathbb{P}^4$. Let the A-values of the respective half-edges be (k, l). By Lemma 25, we have

$$\frac{\partial \mathrm{Cont}^{\mathsf{A}}_\Gamma(f)}{\partial \mathcal{Y}} = (-1)^{k+l+1} \frac{R_{1\,k-1} R_{1\,l-1}}{5L^3 \lambda_i^{k-2} \lambda_j^{l-2}}\,. \tag{33}$$

• If Γ is connected after deleting f, we have

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}}_{\geq 0}} \left(-\frac{5L^5}{C_0^2 C_1^2}\right) \frac{\partial \mathrm{Cont}^{\mathsf{A}}_\Gamma(f)}{\partial \mathcal{Y}} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \prod_{e \in \mathsf{E},\, e \neq f} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e)$$
$$= \mathrm{Cont}_{\Gamma^0_f}(H, H)\,.$$

The derivation is simply by using (33) on the left and Proposition 20 on the right.

•• If Γ is disconnected after deleting f, we obtain

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}}_{\geq 0}} \left(-\frac{5L^5}{C_0^2 C_1^2}\right) \frac{\partial \mathrm{Cont}^{\mathsf{A}}_\Gamma(f)}{\partial \mathcal{Y}} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \prod_{e \in \mathsf{E},\, e \neq f} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e)$$
$$= \mathrm{Cont}_{\Gamma^1_f}(H)\, \mathrm{Cont}_{\Gamma^2_f}(H)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_g(\mathbb{P}^4)$ and using the vanishing

$$\frac{\partial \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(v)}{\partial \mathcal{Y}} = 0$$

of Lemma 24, we obtain

$$\left(-\frac{L^5}{5C_0^2C_1^2}\right)\frac{\partial}{\partial \mathcal{Y}}\langle\rangle^{\mathsf{SQ}}_{g,0} \;=\; \frac{1}{2}\sum_{i=1}^{g-1}\langle H\rangle^{\mathsf{SQ}}_{g-i,1}\langle H\rangle^{\mathsf{SQ}}_{i,1} + \frac{1}{2}\langle H,H\rangle^{\mathsf{SQ}}_{g-1,2}\,. \tag{34}$$

We have followed here the notation of Section 0.1.

By definition of A_2, A_4, A_6, we have following equations.

$$\left(\frac{1}{C_1^2}\frac{\partial}{\partial A_2} - \frac{K_2}{5C_1^2}\frac{\partial}{\partial A_4} + \frac{K_2^2}{50C_1^2}\frac{\partial}{\partial A_6}\right)\mathcal{Y} = -5L^5\,,$$

$$\left(\frac{1}{C_1^2}\frac{\partial}{\partial A_2} - \frac{K_2}{5C_1^2}\frac{\partial}{\partial A_4} + \frac{K_2^2}{50C_1^2}\frac{\partial}{\partial A_6}\right)\mathcal{X}_1 = 0\,,$$

$$\left(\frac{1}{C_1^2}\frac{\partial}{\partial A_2} - \frac{K_2}{5C_1^2}\frac{\partial}{\partial A_4} + \frac{K_2^2}{50C_1^2}\frac{\partial}{\partial A_6}\right)\mathcal{X}_2 = 0\,.$$

Since $I_0^{2g-2}\langle\rangle^{\mathsf{SQ}}_g = \widetilde{\mathcal{F}}^{\mathsf{B}}_g$, the left side of (34) after multiplication by I_0^{2g-2} is, by the chain rule,

$$\frac{1}{C_0^2C_1^2}\frac{\partial\widetilde{\mathcal{F}}^{\mathsf{B}}_g}{\partial A_2} - \frac{K_2}{5C_0^2C_1^2}\frac{\partial\widetilde{\mathcal{F}}^{\mathsf{B}}_g}{\partial A_4} + \frac{K_2^2}{50C_0^2C_1^2}\frac{\partial\widetilde{\mathcal{F}}^{\mathsf{B}}_g}{\partial A_6} \in \mathbb{C}[L^{\pm 1}][C_0^{\pm 1}, C_1^{-1}, K_2, A_2, A_4, A_6]\,.$$

On the right side of (34), we have

$$I_0^{2(g-i)-2+1}\langle H\rangle^{\mathsf{SQ}}_{g-i,1} \;=\; \widetilde{\mathcal{F}}^{\mathsf{B}}_{g-i,1}(q) \;=\; \widetilde{\mathcal{F}}^{\mathsf{GW}}_{g-i,1}(Q(q))\,,$$

where the first equality is by definition and the second is by wall-crossing (6). Then,

$$\widetilde{\mathcal{F}}^{\mathsf{GW}}_{g-i,1}(Q(q)) \;=\; \frac{\partial\widetilde{\mathcal{F}}^{\mathsf{GW}}_{g-i}}{\partial T}(Q(q)) \;=\; \frac{\partial\widetilde{\mathcal{F}}^{\mathsf{B}}_{g-i}}{\partial T}(q)$$

where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing (6). So we conclude

$$I_0^{2(g-i)-2+1}\langle H\rangle^{\mathsf{SQ}}_{g-i,1} = \frac{\partial\widetilde{\mathcal{F}}^{\mathsf{B}}_{g-i}}{\partial T}(q) \;\in \mathbb{C}[[q]]\,. \tag{35}$$

Similarly, we obtain

$$I_0^{2(g-i)-2+1}\langle H\rangle^{\mathsf{SQ}}_{i,1} \;=\; \frac{\partial\widetilde{\mathcal{F}}^{\mathsf{B}}_{i}}{\partial T}(q) \;\in \mathbb{C}[[q]]\,, \tag{36}$$

$$I_0^{2(g-1)-2+2}\langle H,H\rangle^{\mathsf{SQ}}_{g-1,2} \;=\; \frac{\partial^2\widetilde{\mathcal{F}}^{\mathsf{B}}_{g-1}}{\partial T^2}(q) \;\in \mathbb{C}[[q]]\,. \tag{37}$$

The above equations transform (34), after multiplication by I_0^{2g-2}, to exactly the first holomorphic anomaly equation of Theorem 2,

$$\frac{1}{C_0^2C_1^2}\frac{\partial\widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_2}-\frac{1}{5C_0^2C_1^2}\frac{\partial\widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_4}K_2+\frac{1}{50C_0^2C_1^2}\frac{\partial\widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_6}K_2^2=\frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial\widetilde{\mathcal{F}}_{g-i}^{\mathsf{B}}}{\partial T}\frac{\partial\widetilde{\mathcal{F}}_i^{\mathsf{B}}}{\partial T}+\frac{1}{2}\frac{\partial^2\widetilde{\mathcal{F}}_{g-1}^{\mathsf{B}}}{\partial T^2}$$

as an equality in $\mathbb{C}[[q]]$.

In order to lift the first holomorphic anomaly equation to the ring

$$\mathbb{C}[L^{\pm 1}][A_2,A_4,A_6,C_0^{\pm 1},C_1^{-1},K_2]\,,$$

we must lift the equalities (35)-(37). The proof is identical to the parallel lifting for $K\mathbb{P}^2$ given in [22, Section 7.3].

6.3. **Proof of Theorem 2: second equation.** By Proposition 19, we have the following formula for the contribution of the graph Γ to the stable quotient theory of formal quintic,

$$\mathrm{Cont}_\Gamma=\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^{\mathsf{F}}}\prod_{v\in\mathsf{V}}\mathrm{Cont}_\Gamma^{\mathsf{A}}(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)\,.$$

Let f connect the T-fixed points $p_i,p_j\in\mathbb{P}^4$. Let the A-values of the respective half-edges be (k,l). By Lemma 26, we have

$$\frac{\partial\mathrm{Cont}_\Gamma^{\mathsf{A}}(f)}{\partial\mathcal{X}}=(-1)^{k+l+1}\left(\frac{R_{0\,k-1}R_{2\,l-1}}{5L^3\lambda_i^{k-1}\lambda_j^{l-3}}+\frac{R_{2\,k-1}R_{0\,l-1}}{5L^3\lambda_i^{k-3}\lambda_j^{l-1}}\right)\,. \tag{38}$$

• If Γ is connected after deleting f, we have

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^{\mathsf{F}}}\left(-\frac{5L^5}{C_1^2}\right)\frac{\partial\mathrm{Cont}_\Gamma^{\mathsf{A}}(f)}{\partial\mathcal{X}}\prod_{v\in\mathsf{V}}\mathrm{Cont}_\Gamma^{\mathsf{A}}(v)\prod_{e\in\mathsf{E},\,e\neq f}\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)$$
$$=\mathrm{Cont}_{\Gamma_f^0}(1,H^2)\,.$$

The derivation is simply by using (38) on the left and Proposition 20 on the right.

•• If Γ is disconnected after deleting f, we obtain

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^{\mathsf{F}}}\left(-\frac{5L^5}{C_1^2}\right)\frac{\partial\mathrm{Cont}_\Gamma^{\mathsf{A}}(f)}{\partial\mathcal{X}}\prod_{v\in\mathsf{V}}\mathrm{Cont}_\Gamma^{\mathsf{A}}(v)\prod_{e\in\mathsf{E},\,e\neq f}\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)$$
$$=\mathrm{Cont}_{\Gamma_f^1}(1)\,\mathrm{Cont}_{\Gamma_f^2}(H^2)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_g(\mathbb{P}^4)$ and using the vanishing

$$\frac{\partial \mathrm{Cont}^{\mathsf{A}}_\Gamma(v)}{\partial \mathcal{X}} = 0$$

of Lemma 24, we obtain

$$\left(-\frac{5L^5}{C_1^2}\right)\frac{\partial}{\partial \mathcal{X}}\langle\rangle^{\mathsf{SQ}}_{g,0} \;=\; \sum_{i=1}^{g-1}\langle 1\rangle^{\mathsf{SQ}}_{g-i,1}\langle H^2\rangle^{\mathsf{SQ}}_{i,1} + \langle 1, H^2\rangle^{\mathsf{SQ}}_{g-1,2} \;=\; 0\,. \tag{39}$$

The second equality in the above equations follow from the string equation for formal stable quotient invariants. Since

$$K_2 = -\frac{1}{L^5}\mathcal{X}\,,$$

the equation (39) is equivalent to the second holomorphic anomaly equation of Theorem 2 as an equality in $\mathbb{C}[[q]]$.

In order to lift the second holomorphic anomaly equation to the ring

$$\mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2]\,,$$

we must lift the equalities

$$\begin{aligned} \langle 1\rangle^{\mathsf{SQ}}_{g-i,1} &= 0\,, \\ \langle 1, H^2\rangle^{\mathsf{SQ}}_{g-1,2} &= 0\,. \end{aligned}$$

The proof follows from the properties of the unit 1 in a CohFT. Specifically, the method of the proof of [25, Proposition 2.12] is used. We leave the details to the reader.

6.4. **Genus one invariants.** We do not study the genus 1 unpointed series $\widetilde{\mathcal{F}}^{\mathsf{B}}_1(q)$ in the paper, so we take

$$\begin{aligned} I_0 \cdot \langle H\rangle^{\mathsf{SQ}}_{1,1} &= \frac{\partial \widetilde{\mathcal{F}}^{\mathsf{B}}_1}{\partial T}\,, \\ I_0^2 \cdot \langle H, H\rangle^{\mathsf{SQ}}_{1,2} &= \frac{\partial^2 \widetilde{\mathcal{F}}^{\mathsf{B}}_1}{\partial T^2}\,. \end{aligned}$$

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using Proposition 20,

$$\begin{aligned} \frac{\partial \widetilde{\mathcal{F}}^{\mathsf{B}}_1}{\partial T} &= \frac{L^5}{C_1}\left(\frac{1}{2}A_2 + \frac{19}{24}K_2 + \frac{1}{12} - \frac{19}{120}\frac{1}{L^5}\right), \\ \frac{\partial^2 \widetilde{\mathcal{F}}^{\mathsf{B}}_1}{\partial T^2} &= \frac{1}{C_1}\mathsf{D}\left(\frac{L^5}{C_1}\left(\frac{1}{2}A_2 + \frac{19}{24}K_2 + \frac{1}{12} - \frac{19}{120}\frac{1}{L^5}\right)\right). \end{aligned}$$

6.5. **Bounding the degree.** For the holomorphic anomaly equation for $K\mathbb{P}^2$, the integration constants can be bounded [22, Section 7.5]. A parallel result hold for the formal quintic.

The degrees in L of the term of

$$\widetilde{\mathcal{F}}_g^{\mathsf{SQ}} \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, K_2]$$

for formal quintic always fall in the range

$$[15 - 15g, 10g - 10] \tag{40}$$

In particular, the constant (in A_2, A_4, A_6, K_2) term of $\widetilde{\mathcal{F}}_g^{\mathsf{SQ}}$ missed by the holomorphic anomaly equation for formal quintic is a Laurent polynomial in L with degree in the range (40). The bound (40) is a consequence of Proposition 19, the vertex and edge analysis of Section 5, and the following result.

Lemma 28. *The degrees in L of R_{ip} fall in the range*

$$[-i, 4p+1]\,.$$

Proof. The proof for the functions R_{0p} follows from the arguments of [27]. The proof for the other R_{ip} follows from Lemma 17. ☐

References

[1] M. Alim, E. Scheidegger, S.-T. Yau, J. Zhou, *Special polynomial rings, quasi modular forms and duality of topological strings*, Adv. Theor. Math. Phys. **18** (2014), 401–467.

[2] K. Behrend, B. Fantechi, *The intrinsic normal cone,* Invent. Math. **128** (1997), 45–88.

[3] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic anomalies in topological field theories*, Nucl. Phys. B**405** (1993), 279–304.

[4] I. Ciocan-Fontanine and B. Kim, *Moduli stacks of stable toric quasimaps,* Adv. in Math. **225** (2010), 3022–3051.

[5] I. Ciocan-Fontanine and B. Kim, *Wall-crossing in genus zero quasimap theory and mirror maps,* Algebr. Geom. **1** (2014), 400–448.

[6] I. Ciocan-Fontanine and B. Kim, *Big I-functions* in *Development of moduli theory Kyoto 2013*, 323–347, Adv. Stud. Pure Math. **69**, Math. Soc. Japan, 2016.

[7] I. Ciocan-Fontanine and B. Kim, *Quasimap wallcrossings and mirror symmetry*, arXiv:1611.05023.

[8] I. Ciocan-Fontanine and B. Kim, *Higher genus quasimap wall-crossing for semi-positive targets*, JEMS **19** (2017), 2051-2102.

[9] I. Ciocan-Fontanine, B. Kim, and D. Maulik, *Stable quasimaps to GIT quotients,* J. Geom. Phys. **75** (2014), 17–47.

[10] E. Clader, F. Janda, and Y. Ruan, *Higher genus quasimap wall-crossing via localization*, arXiv:1702.03427.

[11] Y. Cooper and A. Zinger, *Mirror symmetry for stable quotients invariants*, Michigan Math. J. **63** (2014), 571–621.

[12] D. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**: Amer. Math. Soc., Providence, RI, 1999.

[13] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry – Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math. **62**, Part 2: Amer. Math. Soc., Providence, RI, 1997.

[14] A. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **13** (1996), 613–663.

[15] A. Givental, *Elliptic Gromov-Witten invariants and the generalized mirror conjecture,* Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 107–155, World Sci. Publ., River Edge, NJ, 1998.

[16] A. Givental, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices **23** (2001), 613–663.

[17] T. Graber, R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.

[18] B. Kim and H. Lho, *Mirror theorem for elliptic quasimap invariants*, Geom. and Top. **22** (2018), 1459–1481.

[19] M. Kontsevich, *Enumeration of rational curves via torus actions* in *The moduli space of curves (Texel Island, 1994)*, 335–368, Progr. Math. **129**: Birkhäuser Boston, Boston, MA, 1995.

[20] Y.-P. Lee and R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory and Virasoro constraints,* https://people.math.ethz.ch/~rahul/, 2004.

[21] H. Lho, *Equivariant holomorphic anomaly equation for the formal quintic*, in preparation.

[22] H. Lho and R. Pandharipande, *Stable quotients and the holomorphic anomaly equation*, Adv. Math. **332** (2018), 349–402.

[23] H. Lho and R. Pandharipande, *Crepant resolution and the holomorphic anomaly equation for* $\mathbb{C}^3/\mathbb{Z}_3$, arXiv:1804.03168.

[24] A. Marian, D. Oprea, Dragos, R. Pandharipande, *The moduli space of stable quotients,* Geom. Topol. **15** (2011), 1651–1706.

[25] R. Pandharipande, A. Pixton, and D. Zvonkine, *Relations on* $\overline{M}_{g,n}$ *via 3-spin structures*, JAMS **28** (2015), 279–309.

[26] S. Yamaguchi and S. T. Yau, *Topological string partition functions as polynomials*, JHEP **047** (2004), arXiv:hep-th/0406078.

[27] D. Zagier and A. Zinger, *Some properties of hypergeometric series associated with mirror symmetry* in *Modular Forms and String Duality*, 163-177, Fields Inst. Commun. **54**, AMS 2008.

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