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Moduli of curves and moduli of sheaves

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Abstract

Relationships between moduli spaces of curves and sheaves on 3-folds are presented starting with the Gromov-Witten/Donaldson-Thomas correspondence proposed more than 20 years ago with D. Maulik, N. Nekrasov, and A. Okounkov. The descendent and relative correspondences as developed with A. Pixton in the context of stable pairs led to the proof of the correspondence for the Calabi-Yau quintic 3-fold. More recently, the study of correspondences in families has played an important role in connection with other basic moduli problems in algebraic geometry. The full conjectural framework is presented here in the context of families of 3-folds.

Contents

1 Introduction

Let X be a nonsingular complex projective 3-fold. The counting of algebraic curves in X can be approached by stable maps in Gromov-Witten theory and by stable pairs in Donaldson-Thomas theory. In a series of conjectures starting with the original Gromov-Witten/Donaldson-Thomas correspondence [9, 10] for ideal sheaves formulated with D. Maulik, N. Nekrasov, and A. Okounkov, an equivalence is proposed between the different approaches to counting via non-trivial transformations. My goal here is to present the Gromov-Witten/Pairs descendent correspondence in a general form for families and to discuss a few cases in which the correspondence has been proven (including the case of the Calabi-Yau quintic 3-fold established with A. Pixton in [24]).

The full Gromov-Witten/Pairs descendent correspondence for families in the form of Conjectures II and III of Section 5 is more general than in the previous perspective on the subject presented in Conjecture I of Section 3. Sections 2-4 of paper are essentially a survey of the conjectures and results of [23, 24] and related work. Other introductions to the subject can be found in [19, 27].

2 Stable maps and stable pairs

2.1 Families

We will state all the conjectures in the context of families. Let

$$\nu : \mathcal{X} \to \mathcal{Y}$$

be a flat projective family of nonsingular complex 3-folds over an irreducible quasi-projective base scheme $\mathcal{Y}$. Later, in the context of relative and log theories, the morphism ν may be taken to be log smooth instead of smooth. The focus of the paper, however, will be on the smooth case.

2.2 Gromov-Witten theory: stable maps

Let $\overline{\mathcal{M}}_{g,r}(X,\beta)$ denote the moduli space of r-pointed stable maps from connected genus g curves to a nonsingular projective 3-fold X representing a class $\beta \in H_2(X,\mathbb{Z})$. Let

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,r}(X,\beta) \to X\,, \quad \mathbb{L}_i \to \overline{\mathcal{M}}_{g,r}(X,\beta)$$

denote the evaluation maps and the cotangent line bundles associated to the marked points.

We will be interested in the moduli space of stable maps

$$\overline{\mathcal{M}}_{g,r}(\nu,\beta) \quad \text{associated to} \quad \nu : \mathcal{X} \to \mathcal{Y}\,,$$

where β is a fiber class well-defined for the family. The points of $\overline{\mathcal{M}}_{g,r}(\nu,\beta)$ are stable maps to the fibers of ν.

Let $\gamma_1, \ldots, \gamma_r \in H^*(\mathcal{X})$ be cohomology[1] classes, and let

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}(\nu,\beta))\,.$$

The *standard descendent* insertions, denoted by $\tau_k(\gamma)$, correspond to the classes $\psi_i^k \mathrm{ev}_i^*(\gamma)$ on the moduli space of stable maps. Let

$$\Big\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\Big\rangle_{g,\beta}^{\mathsf{GW}} = \epsilon_{\mathsf{GW}*}\left(\prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \,\cap\, [\overline{\mathcal{M}}_{g,r}(\nu,\beta)]^{vir}\right) \in H_*(\mathcal{Y})$$

[1] All homology and cohomology groups will be taken with $\mathbb{Q}$-coefficients unless explicitly denoted otherwise.

denote the descendent Gromov-Witten invariants where

$$\epsilon_{\mathsf{GW}} : \overline{\mathcal{M}}_{g,r}(\nu,\beta) \to \mathcal{Y}$$

is the map to the base of the family. Foundational aspects of deformation theory and the virtual class are treated in [1, 8].

Let C be a possibly disconnected curve with at worst nodal singularities. The genus of C is defined by $1-\chi(\mathcal{O}_C)$. Let $\overline{\mathcal{M}}'_{g,r}(\nu,\beta)$ denote the moduli space of maps mapping to fibers of ν with possibly disconnected domain curves C of genus g with *no* collapsed connected components. In particular, C must represent a nonzero fiber class β.

We define the descendent invariants in the disconnected case by

$$\Big\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\Big\rangle^{\mathsf{GW}'}_{g,\beta} = \epsilon'_{\mathsf{GW}*}\left(\prod_{i=1}^r \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i)\ \cap\ [\overline{\mathcal{M}}'_{g,r}(\nu,\beta)]^{vir}\right) \in H_*(\mathcal{Y})\,,$$

where $\epsilon'_{\mathsf{GW}} : \overline{\mathcal{M}}'_{g,r}(\nu,\beta) \to \mathcal{Y}$. The associated Gromov-Witten descendent partition function is defined by

$$\mathsf{Z}'_{\mathsf{GW}}\Big(\nu;u\ \Big|\ \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta = \sum_{g\in\mathbb{Z}} \Big\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big\rangle^{\mathsf{GW}'}_{g,\beta} u^{2g-2}. \tag{2.1}$$

Since the domain components must map non-trivially, an elementary argument shows the genus g in the sum on the right side of (2.1) is bounded from below. As a result, the partition function (2.1) is a Laurent series in u with coefficients in $H_*(\mathcal{Y})$.

2.3 Product evaluations for stable maps

Let $\nu^r : \mathcal{X}^r \to \mathcal{Y}$ be the fiber product over $\mathcal{Y}$,

$$\mathcal{X}^r = \underbrace{\mathcal{X}\times_{\mathcal{Y}}\cdots\times_{\mathcal{Y}}\mathcal{X}}_{r}\,.$$

We can consider cohomology classes $\delta \in H^*(\mathcal{X}^r)$ and define *general descendent* insertions

$$\tau_{k_1,\ldots,k_r}(\delta) = \prod_{i=1}^r \psi_i^{k_i}\cdot \mathrm{ev}^*_{\mathrm{full}}(\delta)$$

using the evaluation map

$$\mathrm{ev}_{\mathrm{full}} : \overline{\mathcal{M}}'_{g,r}(\nu,\beta) \to \mathcal{X}^r$$

obtained from the full marking set.

We define the associated descendent invariants in the disconnected case by

$$\Big\langle \tau_{k_1,\ldots,k_r}(\delta)\Big\rangle^{\mathsf{GW}'}_{g,\beta} = \epsilon'_{\mathsf{GW}*}\left(\prod_{i=1}^r \psi_i^{k_i}\cdot \mathrm{ev}^*_{\mathrm{full}}(\delta)\ \cap\ [\overline{\mathcal{M}}'_{g,r}(\nu,\beta)]^{vir}\right) \in H_*(\mathcal{Y})\,.$$

The associated Gromov-Witten descendent partition function is defined by

$$\mathsf{Z}'_{\mathsf{GW}}\Big(\nu; u \ \Big|\ \tau_{k_1,\ldots,k_r}(\delta)\Big)_\beta = \sum_{g\in\mathbb{Z}} \Big\langle \tau_{k_1,\ldots,k_r}(\delta)\Big\rangle^{\mathsf{GW}'}_{g,\beta} \ u^{2g-2}.$$

If we have a factorization $\delta = \gamma_1 \otimes \cdots \otimes \gamma_r$ for classes $\gamma_i \in H^*(\mathcal{X})$ pulled-back from the r distinct factor projections

$$\mathcal{X}^r \to \mathcal{X}\,,$$

then $\tau_{k_1,\ldots,k_r}(\delta) = \prod_{i=1}^r \tau_{k_i}(\gamma_i)$, and we have an equality of partition functions

$$\mathsf{Z}'_{\mathsf{GW}}\Big(\nu; u \ \Big|\ \tau_{k_1,\ldots,k_r}(\delta)\Big)_\beta = \mathsf{Z}'_{\mathsf{GW}}\Big(\nu; u \ \Big|\ \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta\,.$$

However, $H^*(\mathcal{X}^r)$ is not in general spanned by such factorizations.[2]

2.4 Donaldson-Thomas theory: stable pairs

Let X be a nonsingular complex projective 3-fold. A *stable pair* (F, s) on X is a coherent sheaf F on X and a section $s \in H^0(X, F)$ satisfying the following stability conditions:

- F is *pure* of dimension 1,
- the section $s : \mathcal{O}_X \to F$ has cokernel of dimensional 0.

To a stable pair, we associate the Euler characteristic and the class of the support C of F,

$$\chi(F) = n \in \mathbb{Z} \quad \text{and} \quad [C] = \beta \in H_2(X, \mathbb{Z})\,.$$

For fixed n and β, there is a projective moduli space of stable pairs $\mathcal{P}_n(X, \beta)$.

The moduli space of stable pairs is also defined in families of 3-folds. We will be interested in the moduli space of stable pairs

$$\mathcal{P}_n(\nu, \beta) \quad \text{associated to} \quad \nu : \mathcal{X} \to \mathcal{Y}\,.$$

A foundational treatment of the moduli space of stable pairs is presented in [25] via the results of Le Potier [7]. For a fixed fiber class β, the moduli space $\mathcal{P}_n(\nu, \beta)$ is empty for all sufficiently negative n.

Denote the universal stable pair over $\mathcal{X} \times_{\mathcal{Y}} \mathcal{P}_n(\nu, \beta)$ by

$$\mathcal{O}_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{P}_n(\nu,\beta)} \xrightarrow{s} \mathbb{F}\,.$$

For $y \in \mathcal{Y}$ and a stable pair $(F, s) \in \mathcal{P}_n(\mathcal{X}_y, \beta)$, the restriction of the universal stable pair to the fiber

$$\mathcal{X}_y \times (F, s) \ \subset\ \mathcal{X}_y \times \mathcal{P}_n(\mathcal{X}_y, \beta)$$

[2] If $\mathcal{Y}$ is a point, then the Künneth decomposition proves that factorizations span.

is canonically isomorphic to $\mathcal{O}_{\mathcal{X}_y} \xrightarrow{s} F$. Let

$$\pi_{\mathcal{X}}\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{P}_n(\nu,\beta) \to \mathcal{X},$$

$$\pi_{\mathcal{P}}\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{P}_n(\nu,\beta) \to \mathcal{P}_n(\nu,\beta)$$

be the projections onto the first and second factors. Since ν is smooth and $\mathbb{F}$ is $\pi_{\mathcal{P}}$-flat, $\mathbb{F}$ has a finite resolution by locally free sheaves. Hence, the Chern character of the universal sheaf $\mathbb{F}$ on $\mathcal{X} \times_{\mathcal{Y}} \mathcal{P}_n(\nu,\beta)$ is well-defined.

For each cohomology class $\gamma \in H^*(\mathcal{X})$ and integer $k \in \mathbb{Z}_{\geq 0}$, the action of the *standard descendent* insertion $\tau_k(\gamma)$ is defined by

$$\tau_k(\gamma) = \pi_{\mathcal{P}*}\big(\pi_{\mathcal{X}}^*(\gamma) \cdot \mathrm{ch}_{2+k}(\mathbb{F}) \cap \pi_{\mathcal{P}}^*(\ \cdot\ )\big)\,.$$

The pull-back $\pi_{\mathcal{P}}^*$ is well-defined in homology since $\pi_{\mathcal{P}}$ is flat [5].

We may view the descendent action as defining a cohomology class

$$\tau_k(\gamma) \in H^*(\mathcal{P}_n(\nu,\beta))$$

or as defining an endomorphism

$$\tau_k(\gamma) : H_*(\mathcal{P}_n(\nu,\beta)) \to H_*(\mathcal{P}_n(\nu,\beta))\,.$$

We define the stable pairs invariant with descendent insertions by

$$\Big\langle \tau_{k_1}(\gamma_1)\ldots\tau_{k_r}(\gamma_r)\Big\rangle_{n,\beta}^{\mathsf{P}} = \epsilon_{\mathsf{P}*}\left(\prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\ \cap\ [\mathcal{P}_n(\nu,\beta)]^{vir}\right) \in H_*(\mathcal{Y})\,,$$

where $\epsilon_{\mathsf{P}} : \mathcal{P}_n(\nu,\beta) \to \mathcal{Y}$. The partition function is

$$\mathsf{Z}_{\mathsf{P}}\Big(\nu; q\ \Big|\ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_{\beta} = \sum_{n\in\mathbb{Z}} \Big\langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big\rangle_{n,\beta}^{\mathsf{P}} q^n.$$

Since $\mathcal{P}_n(\nu,\beta)$ is empty for sufficiently negative n, the partition function is a Laurent series in q with coefficients in $H_*(\mathcal{Y})$.

2.5 Product evaluations for stable pairs

Let $\nu^r : \mathcal{X}^r \to \mathcal{Y}$ be the fiber product over $\mathcal{Y}$ as in Section 2.3. We can consider cohomology classes $\delta \in H^*(\mathcal{X}^r)$ and define *general descendent* insertions for stable pairs by

$$\tau_{k_1,\ldots,k_r}(\delta) = \pi_{\mathcal{P}*}\Big(\pi_{\mathcal{X}^r}^*(\delta)\cdot\prod_{i=1}^{r}\mathrm{ch}_{2+k_i}(\mathbb{F}_i)\cap\pi_{\mathcal{P}}^*(\ \cdot\ )\Big)\ : H_*(\mathcal{P}_n(\nu,\beta)) \to H_*(\mathcal{P}_n(\nu,\beta))$$

using the maps

$$\pi_{\mathcal{X}^r}\colon \mathcal{X}^r \times_{\mathcal{Y}} \mathcal{P}_n(\nu,\beta) \to \mathcal{X}^r,$$

$$\pi_{\mathcal{P}}\colon \mathcal{X}^r \times_{\mathcal{Y}} \mathcal{P}_n(\nu,\beta) \to \mathcal{P}_n(\nu,\beta)\,.$$

Here, $\mathbb{F}_i$ denotes the universal sheaf from the i^{th} factor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{P}_n(\mathcal{X}, \beta)$.

We define the associated descendent invariants by

$$\Big\langle \tau_{k_1,\ldots,k_r}(\delta) \Big\rangle^{\mathsf{P}}_{n,\beta} = \epsilon_{\mathsf{P}*}\left(\tau_{k_1,\ldots,k_r}(\delta) \cap [\mathcal{P}_n(\nu,\beta)]^{vir}\right) \in H_*(\mathcal{Y})\,.$$

The partition function is

$$\mathsf{Z}_{\mathsf{P}}\Big(\nu; q \;\Big|\tau_{k_1,\ldots,k_r}(\delta)\Big)_\beta = \sum_{n\in\mathbb{Z}} \Big\langle \tau_{k_1,\ldots,k_r}(\delta) \Big\rangle^{\mathsf{P}}_{n,\beta} q^n.$$

If we have a factorization $\delta = \gamma_1 \otimes \cdots \otimes \gamma_r$ for classes $\gamma_i \in H^*(\mathcal{X})$ pulled-back from the r distinct factor projections, then

$$\mathsf{Z}_{\mathsf{P}}\Big(\nu; q \;\Big|\tau_{k_1,\ldots,k_r}(\delta)\Big)_\beta = \mathsf{Z}_{\mathsf{P}}\Big(\nu; q \;\Big| \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta\,.$$

2.6 Dimension constraints

The stable maps and stable pairs descendent series

$$\mathsf{Z}'_{\mathsf{GW}}\Big(\nu; u \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta\,, \qquad \mathsf{Z}_{\mathsf{P}}\Big(\nu; q \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta \tag{2.2}$$

for the family $\nu : \mathcal{X} \to \mathcal{Y}$ both satisfy dimension constraints.

For $\gamma_i \in H^{e_i}(\mathcal{X})$, the (real) dimension of the descendent theories are

$$\tau_{k_i}(\gamma_i) \in H^{e_i+2k_i}(\overline{\mathcal{M}}'_{g,r}(\nu,\beta))\,, \qquad \tau_{k_i}(\gamma_i) \in H^{e_i+2k_i-2}(P_n(\nu,\beta))\,.$$

The virtual dimensions are

$$\dim_{\mathbb{C}} [\overline{\mathcal{M}}'_{g,r}(\nu,\beta)]^{vir} = \int_\beta c_1(T_\nu) + r + \dim_{\mathbb{C}} \mathcal{Y}\,, \quad \dim_{\mathbb{C}} [\mathcal{P}_n(\nu,\beta)]^{vir} = \int_\beta c_1(T_\nu) + \dim_{\mathbb{C}} \mathcal{Y}$$

where T_ν is the relative tangent bundle. The coefficients of both series (2.2) therefore take values in the *same* homology group

$$H_{2\int_\beta c_1(T_\nu) + 2\dim_{\mathbb{C}} \mathcal{Y} - \sum_{i=1}^r (e_i+2k_i-2)}(\mathcal{Y})\,.$$

Is there a relationship between the Gromov-Witten and stable pairs descendent series (2.2)? The immediate issues are:

(i) the series involve different moduli spaces and universal structures,
(ii) the variables u and q of the two series are different.

The *descendent correspondence* proposes a precise relationship between the Gromov-Witten and stable pairs descendent series, but only after a change of variables to address (ii).

3 Descendent correspondence

3.1 Descendent notation

A partition $\sigma = (\sigma_1, \ldots, \sigma_{\ell(\sigma)})$ of positive size $|\sigma|$ and length $\ell(\sigma)$ consists of integers σ_k satisfying the properties

$$\sigma_1 \geq \ldots \geq \sigma_{\ell(\sigma)} > 0\,, \qquad \sum_{k=1}^{\ell(\sigma)} \sigma_k = |\sigma|\,.$$

For the family of 3-folds $\nu : \mathcal{X} \to \mathcal{Y}$, let

$$\iota_\Delta : \Delta_{\ell(\sigma)} \to \mathcal{X}^{\ell(\sigma)}$$

denote the inclusion of the fiberwise small diagonal[3] in the product $\mathcal{X}^{\ell(\sigma)}$ over $\mathcal{Y}$. For $\gamma \in H^*(\mathcal{X})$, we write

$$\gamma \cdot \Delta_{\widehat{\ell}} = \iota_{\Delta *}(\gamma) \in H^*(\mathcal{X}^{\ell(\sigma)})\,.$$

The *diagonal descendent* insertion[4]

$$\tau_\sigma(\gamma) = \tau_{\sigma_1 - 1, \ldots, \sigma_{\ell(\sigma)}-1}(\gamma \cdot \Delta_{\ell(\sigma)}) \tag{3.1}$$

will play an important role in the descendent correspondence.

Warning. If $\sigma = (\sigma_1)$ has a single part, we recover the standard descendent

$$\tau_\sigma(\gamma) = \tau_{\sigma_1 - 1}(\gamma)\,.$$

The shift by 1 is a confusing aspect of the definition (but is necessary when using language of partitions to describe descendents). When reading a formula with descendents, care must be taken to determine whether the descendent subscript is a *partition* or an *integer*. If the subscript is a partition, then the descendent is a *diagonal descendent*. If the subscript is an integer (or a list of integers), then the descendent is either a *standard* or a *general descendent*.

3.2 Correspondence matrix

A central result of [23] is the construction of a universal correspondence matrix $\widetilde{\mathsf{K}}$ indexed by partitions α and $\widehat{\alpha}$ of positive size with[5]

$$\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))\,.$$

The elements of $\widetilde{\mathsf{K}}$ are constructed from the capped descendent vertex [23] and satisfy two basic properties:

[3]The small diagonal is the set of points of each fiber $\mathcal{X}_y^{\ell(\sigma)}$ for which the coordinates $(x_1, \ldots, x_{\widehat{\ell}})$ are all equal $x_i = x_j$.
[4]We follow the notation of Sections 2.3 for stable maps and 2.5 for stable pairs.
[5]Here, $i^2 = -1$.

(i) The vanishing $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} = 0$ holds unless $|\alpha| \geq |\widehat{\alpha}|$.

(ii) The coefficients of $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))$ as a series in u are homogeneous[6] in the variables c_i of degree

$$|\alpha| + \ell(\alpha) - |\widehat{\alpha}| - \ell(\widehat{\alpha}) - 3(\ell(\alpha) - 1).$$

Let $\mathcal{T}_\nu \to \mathcal{X}$ be the rank 3 relative tangent bundle of the family $\nu : \mathcal{X} \to \mathcal{Y}$. Via the substitution

$$c_i = c_i(\mathcal{T}_\nu),$$

the matrix elements of $\widetilde{\mathsf{K}}$ act by cup product on the cohomology of $\mathcal{X}$ with $\mathbb{Q}[i]((u))$-coefficients.

The matrix $\widetilde{\mathsf{K}}$ is used to define a correspondence rule

$$\tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell) \quad \mapsto \quad \overline{\tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)} \tag{3.2}$$

with $\gamma_i \in H^*(\mathcal{X})$. The left side is a product of standard descendent insertions.

The definition of the right side of (3.2) requires a sum over all set partitions P of $\{1, \ldots, \ell\}$. For such a set partition P, each element $S \in P$ is a subset of $\{1, \ldots, \ell\}$. Let α_S be the associated subpartition of α, and let

$$\gamma_S = \prod_{j \in S} \gamma_j.$$

In case all cohomology classes γ_i are even[7] we define the right side of the correspondence rule (3.2) by

$$\overline{\tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{P \text{ set partition of } \{1,\ldots,\ell\}} \ \prod_{S \in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \cdot \gamma_S)\ . \tag{3.3}$$

The second sum in (3.3) is over all partitions $\widehat{\alpha}$ of positive size. However, by the vanishing of property (i),

$$\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} = 0 \ \text{ unless } \ |\alpha_S| \geq |\widehat{\alpha}|\,,$$

the summation index may be restricted to partitions $\widehat{\alpha}$ of positive size bounded by $|\alpha_S|$.

The terms on the right side of (3.3) are diagonal descendent insertions (3.1). For example, let $\ell = 5$ and consider the set partition P of $\{1, 2, 3, 4, 5\}$ defined by the data

$$S_1 \cup S_2 = \{1, 2, 3, 4, 5\}\,,\ S_1 = \{1, 2, 3\}\,,\ S_2 = \{4, 5\}$$

Terms on the right side of (3.3) associated to P are of the form

$$\tau_{\widehat{\alpha}^1}(\widetilde{\mathsf{K}}_{\alpha_{S_1},\widehat{\alpha}^1} \cdot \gamma_{S_1}) \cdot \tau_{\widehat{\alpha}^2}(\widetilde{\mathsf{K}}_{\alpha_{S_2},\widehat{\alpha}^2} \cdot \gamma_{S_2})\,.$$

[6] The variable c_k has degree k for the homogeneity (and the complex number i has degree 0).

[7] If classes are odd, a sign must be introduced in the correspondence rule. Signs will be discussed in Section 5.3.

Here, $\widehat{\alpha}^1$ and $\widehat{\alpha}^2$ are two partitions of positive size – the superscript index serves to distinguish them as both are summed over in (3.3).

Suppose $|\alpha_S| = |\widehat{\alpha}|$ in the second sum in (3.3). The homogeneity property (ii) then places a strong constraint. The u coefficients of

$$\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))$$

are homogeneous of degree

$$3 - 2\ell(\alpha_S) - \ell(\widehat{\alpha}) \,. \tag{3.4}$$

For the matrix element $\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}}$ to be nonzero, the degree (3.4) must be non-negative. Since the lengths of α_S and $\widehat{\alpha}$ are at least 1, non-negativity of (3.4) is only possible if

$$\ell(\alpha_S) = \ell(\widehat{\alpha}) = 1 \,.$$

Then, we also have $\alpha_S = \widehat{\alpha}$ since the sizes match.

The above argument shows that the descendents on the right side of (3.3) all correspond to partitions of size *less* than $|\alpha|$ except for the *leading term* obtained from the the maximal set partition

$$\{1\} \cup \{2\} \cup \ldots \cup \{\ell\} = \{1, 2, \ldots, \ell\}$$

in ℓ parts. The leading term of the descendent correspondence, calculated in [23], is a third basic property of $\widetilde{\mathsf{K}}$:

(iii) $\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = (iu)^{\ell(\alpha)-|\alpha|}\,\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) + \ldots .$

In case $\alpha = 1^\ell$ has all parts equal to 1, then α_S also has all parts equal to 1 for every $S \in P$. By property (ii), the u coefficients of $\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}}$ are homogeneous of degree

$$3 - \ell(\alpha_S) - |\widehat{\alpha}| - \ell(\widehat{\alpha}),$$

and hence vanish unless

$$\alpha_S = \widehat{\alpha} = (1) \ .$$

Therefore, if α has all parts equal to 1, the leading term is therefore the entire formula. We obtain a fourth property of the matrix $\widetilde{\mathsf{K}}$:

(iv) $\overline{\tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)} = \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell) \,.$

The geometric construction of $\widetilde{\mathsf{K}}$ in [23] expresses the coefficients explicitly in terms of the 1-legged capped descendent vertex for stable pairs and stable maps. These vertices can be computed (as a rational function in the stable pairs case and term by term in the genus parameter for stable maps). Hence, the coefficient

$$\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))$$

can, in principle, be calculated term by term in u. The calculations in practice are quite difficult, and complete closed formulas are not known[8] for all of the coefficients.

[8]See [14, 16] for formulas is the stationary case for fixed 3-folds X.

3.3 Conjecture I

To state the Gromov-Witten/Pairs correspondence proposed and studied in [23, 24] for the family $\nu : \mathcal{X} \to \mathcal{Y}$, the basic degree

$$d_\beta = \int_\beta c_1(T_\nu) \ \in \mathbb{Z}$$

associated to the fiber class β is required.

Conjecture I [GW/P descendent correspondence]. *For $\gamma_i \in H^*(\mathcal{X})$, we have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\Big(\nu; q \ \Big| \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)_\beta$$
$$= (-iu)^{d_\beta} \mathsf{Z}'_{\mathsf{GW}}\Big(\nu; u \ \Big| \ \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \ \Big)_\beta$$

under the variable change $-q = e^{iu}$.

Since the stable pairs side of the correspondence

$$\mathsf{Z}_{\mathsf{P}}\Big(\nu; q \ \Big| \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)_\beta \in H^*(\mathcal{Y})((q))$$

is defined as a series in q, the change of variable $-q = e^{iu}$ is *not* a priori well-defined. However, the stable pairs descendent series is conjectured to be a rational function in q. The change of variable $-q = e^{iu}$ is well-defined for a rational function in q by substitution. The well-posedness of the descendent correspondence depends upon the rationality conjecture.

4 Results: past and future

4.1 Toric 3-folds

The basic case of the GW/P descendent correspondence for families is for the torus equivariant Gromov-Witten and stable pairs theories of a nonsingular projective toric 3-fold X. The torus equivariant theory is the families theory for the homotopy quotient

$$\nu : \mathcal{X} = \mathsf{X} \times_{\mathsf{T}} \mathsf{ET} \to \mathsf{BT} = \mathcal{Y},$$

where T is the torus and BT has an algebraic approximation by products of projective spaces. In case of primary insertions (where all descendents are of the form $\tau_0(\gamma)$), the correspondence was conjectured in [9, 10] and proven in [11]. The descendent correspondence in the toric case was suggested in [10, 25] and proven in [23]. See [20, 21, 22, 26, 30] for stable pairs calculations in the toric case.

Theorem 1 [P.-Pixton]. *The GW/P descendent correspondence holds for the torus equivariant theories of nonsingular projective toric 3-folds X.*

Since Virasoro constraints are proven for the descendent Gromov-Witten theory of toric 3-folds [6, 31], Theorem 1 implies Virasoro constraints for moduli spaces of sheaves, see [2, 13, 14, 32] for developments in these directions.

4.2 Relative theories

In order to use Theorem 1 to prove the descendent correspondence for more general 3-folds X, a degeneration strategy was pursued in [24]. There are several steps, but the end result is a proof for 3-folds X that admit certain simple degenerations to toric 3-folds. A basic case[9] was the Calabi-Yau quintic 3-fold $X_5 \subset \mathbb{CP}^4$.

Theorem 2 [P.-Pixton]. *The* GW/P *correspondence holds for the* 3*-fold* X_5.

Crucial to the strategy of [24] is the lifting of the entire descendent correspondence to the situation of relative Gromov-Witten and stable pairs theories for families ν over a 1-dimensional base $\mathcal{Y}$ where the fibers are allowed to have normal crossing degenerations into two components. As a consequence, many interesting relative cases of the correspondence were proven in [24]. For example, let $S_4 \subset \mathbb{CP}^3$ be a nonsingular $K3$ quartic surface.

Theorem 3 [P.-Pixton]. *The* GW/P *descendent correspondence holds for the log Calabi-Yau* 3*-fold* $(\mathbb{CP}^3/S_4)$.

In the past few years, Maulik and Ranganathan have promoted the descendent correspondence conjecture to the logarithmic case where the family ν is allowed to have arbitrary normal crossings degenerations [12]. Their method should yield further cases of the GW/P descendent correspondence for more general 3-folds X and log 3-folds (X/Y).

4.3 Non-negative geometries

In recent work [29], Pardon has used new transversality arguments to reduce the case of the correspondence for primary insertions to the correspondence for local curves proven in [4, 18]. The outcome is the following result.

Theorem 4 [Pardon]. *The* GW/P *correspondence holds for primary insertions for families* $\nu : \mathcal{X} \to \mathcal{Y}$ *where the fibers are nonsingular projective Fano or Calabi-Yau 3-folds.*

In particular, Pardon's results yield a new proof of Theorem 2 and a new approach to the GW/P correspondence in the primary case for rich geometries such as $S_4 \times E$ where E is an elliptic curve [15]. The GW/P descendent correspondence[10] for the universal family

$$\nu : \mathcal{X} = \mathbb{C} \times \mathcal{S} \to \mathcal{M}^{K3} = \mathcal{Y}$$

[9] For a fixed Calabi-Yau 3-fold X, the descendent and primary correspondence is equivalent since the moduli spaces are all essentially 0 dimensional.

[10] The 3-folds here are noncompact, so residue theories are required for the definitions. As usual, care must be taken to treat the reduced virtual class properly.

over the moduli of $K3$ surfaces is an interesting open direction.

4.4 Moduli of curves

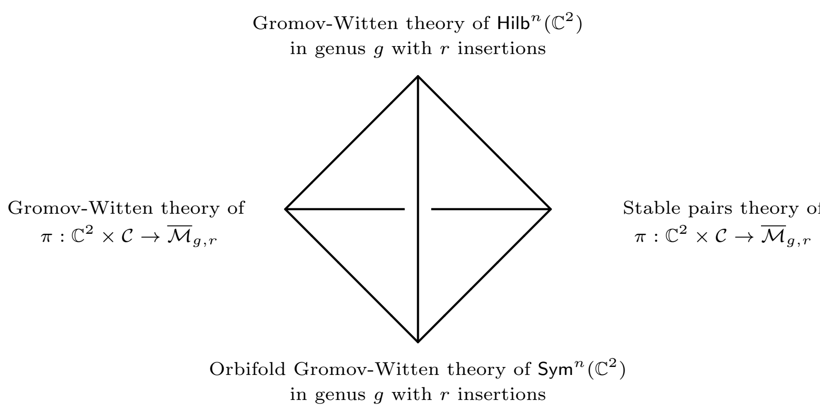
A beautiful family of varieties is the universal curve, $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$, over the moduli space of stable curves. An associated family of 3-folds is

$$\nu : \mathcal{X} = \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,n} = \mathcal{Y}.$$

The definition of Gromov-Witten and stable pairs theories for ν requires residues and relative geometry, see [28].

Theorem 5 [P.- H.-H. Tseng]. *The* $\mathsf{GW/P}$ *correspondence (with no insertions) holds for all relative boundary conditions for the family* $\mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,n}$.

The $\mathsf{GW/P}$ correspondence of Theorem 5 implies (and is equivalent to) the Crepant Resolution Conjecture relating the Gromov-Witten theory in all genera[11] of the Hilbert scheme of points of $\mathbb{C}^2$ to the orbifold Gromov-Witten theory of the symmetric product of $\mathbb{C}^2$.



5 Stronger forms

5.1 Asymmetry for families

The two sides of the $\mathsf{GW/P}$ descendent correspondence of Section 3.3 are *not* symmetric for families

$$\nu : \mathcal{X} \to \mathcal{Y}.$$

The stable pairs side has a fully factored form in terms of standard descendents while the stable maps side has diagonal descendents insertions (3.1). In case the base $\mathcal{Y}$ is a point, the Künneth decomposition of the diagonal can be used to express diagonal classes as sums of fully factorized terms, but we do not have such Künneth decompositions over arbitrary bases $\mathcal{Y}$.

[11] For the genus 0 theory, see [3, 17] and the local curve results of [4, 18].

5.2 Conjecture II

There is a canonical promotion of the GW/P descendent correspondence of Section 3.3 to include diagonal descendent insertions (3.1) on both sides.

Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a partition of positive size $|\alpha|$ and length ℓ. Let D be a set partition of $\{1, \ldots, \ell\}$ in *ordered form*:

$$D_1 \cup \cdots \cup D_d = \{1, \ldots, \ell\}$$

and the elements of D_i come before the elements of D_j if $i < j$. For example,

$$D_1 \cup D_2 = \{1, 2, 3\}\,, \;\; D_1 = \{1, 2\}\,, \;\; D_2 = \{3\}$$

is in ordered form.

We will define a correspondence rule

$$\tau_{\alpha_{D_1}}(\gamma_1) \cdots \tau_{\alpha_{D_d}}(\gamma_d) \quad \mapsto \quad \overline{\tau_{\alpha_{D_1}}(\gamma_1) \cdots \tau_{\alpha_{D_d}}(\gamma_d)}\,,$$

where α_{D_i} is the partition obtained from α by selecting the parts with indices $k_1, \ldots, k_{\ell(D_i)}$ in D_i, and

$$\tau_{\alpha_{D_i}}(\gamma_i) = \tau_{\alpha_{k_1}-1, \ldots, \alpha_{k_{\ell(D_i)}}-1}(\gamma_i \cdot \Delta_{\ell(D_i)})$$

is the diagonal descendent insertion with $\gamma_i \in H^*(\mathcal{X})$ following the notation of (3.1).

Set partitions of $\{1, \ldots, \ell\}$ are partially ordered by refinement. Given another set partition P of $\{1, \ldots, \ell\}$, let $D \wedge P$ denote the finest set partition for which D and P are both refinements. A part $I \in D \wedge P$ is simultaneously[12] a union of parts of D and a union of parts of P (and no proper subset of I has this property),

$$I = D_{i_1} \cup \cdots \cup D_{i_m}\,, \quad I = S_{j_1} \cup \cdots \cup S_{j_n}\,.$$

Let $I \subset \{1, \ldots, \ell\}$ have $|I|$ elements.

In case all cohomology classes γ_i are even[13], the right side of the correspondence rule is:

$$\overline{\tau_{\alpha_{D_1}}(\gamma_1) \cdots \tau_{\alpha_{D_d}}(\gamma_d)} = \sum_{P \text{ set partition of } \{1, \ldots, \ell\}} \; \prod_{I \in D \wedge P} \mathsf{T}_I\,, \tag{5.1}$$

where we have

$$\mathsf{T}_I = \sum_{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n} \tau_{\widehat{\alpha}^1 \cup \cdots \cup \widehat{\alpha}^n} \left(\prod_{k=1}^{n} \widetilde{\mathsf{K}}_{\alpha_{S_{j_k}}, \widehat{\alpha}^k} \cdot \prod_{l=1}^{m} \gamma_{i_l} \cdot c_3(\mathcal{T}_\nu)^{|I|+1-n-m} \right).$$

Here, $\widehat{\alpha}^1 \cup \cdots \cup \widehat{\alpha}^n$ is the concatenation of the partitions $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$.

[12] For the parts of D in I, we always view them in their natural order. Later, the ordering will be relevant for the sign.

[13] If classes are odd, a sign must be introduced in the correspondence rule. Signs will be discussed in Section 5.3.

Conjecture II [GW/P diagonal descendent correspondence].

For $\gamma_i \in H^*(\mathcal{X})$, we have

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(\nu; q \;\Big|\tau_{\alpha_{D_1}}(\gamma_1)\cdots\tau_{\alpha_{D_d}}(\gamma_d)\Big)_\beta$$
$$= (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\Big(\nu; u \;\Big|\; \overline{\tau_{\alpha_{D_1}}(\gamma_1)\cdots\tau_{\alpha_{D_d}}(\gamma_d)}\;\Big)_\beta$$

under the variable change $-q = e^{iu}$.

As before, Conjecture II relies upon a rationality conjecture for the diagonal descendent stable pairs theory of ν. Conjecture II specializes to Conjecture I when D is the set partition of $\{1,\ldots,\ell\}$ with every part of size 1. As a consequence of Conjecture II, the diagonal descendent Gromov-Witten and stable pairs theories of the family ν are equivalent.

5.3 Signs

Signs play an important role both in Conjectures I and II. When the cohomology classes $\gamma_i \in H^*(\mathcal{X})$ are not all even, a natural sign must be included in correspondence rule (5.1).

Given a set partition P of $\{1,\ldots,\ell\}$ indexing the sum on the right side of the rule (5.1), let

$$I_1 \cup \cdots \cup I_{\ell(D\wedge P)} = \{1,\ldots,\ell\}\,.$$

be the parts of $D \wedge P$. The parts I_j of $D \wedge P$ are unordered, but we choose an ordering. We then obtain a permutation of the parts $D_1 \cup \cdots \cup D_d$ by moving each part D_i to the associated ordered part $D_i \subset I_j$ (and respecting the original order of the D_i in each I_j). The permutation, in turn, determines a sign $\mathbf{s}(P)$ by the anti-commutation of the odd classes γ_i associated to the parts D_i. We then write

$$\overline{\tau_{\alpha_{D_1}}(\gamma_1)\cdots\tau_{\alpha_{D_d}}(\gamma_d)} = \sum_{P \text{ set partition of } \{1,\ldots,\ell\}} (-1)^{\mathbf{s}(P)} \prod_{j=1}^{\ell(D\wedge P)} \mathsf{T}_{I_j}\,.$$

The descendent $\overline{\tau_{\alpha_{D_1}}(\gamma_1)\cdots\tau_{\alpha_{D_d}}(\gamma_d)}$ is easily seen to have the same commutation rules with respect to odd cohomology as $\tau_{\alpha_{D_1}}(\gamma_1)\cdots\tau_{\alpha_{D_d}}(\gamma_d)$.

5.4 Conjecture III

What about the general descendent $\tau_{k_1,\ldots,k_r}(\delta)$ discussed in Sections 2.3 and 2.5? Can we write a GW/P descendent correspondence for these descendents in a symmetric form which generalizes Conjectures I and II? The answer is *yes*: there is a generalization of Conjectures I and II for the general descendent $\tau_{k_1,\ldots,k_r}(\delta)$, but the form is more intricate.

Let $\alpha = (\alpha_1,\ldots,\alpha_\ell)$ be a partition of positive size $|\alpha|$ and length ℓ. We will define a correspondence rule

$$\tau_{\alpha_1-1,\ldots,\alpha_\ell-1}(\delta) \quad\mapsto\quad \overline{\tau_{\alpha_1-1,\ldots,\alpha_\ell-1}(\delta)}$$

without any restrictions on the class $\delta \in H^*(\mathcal{X}^\ell)$.

The right side of the correspondence rule is

$$\overline{\tau_{\alpha_1-1,\ldots,\alpha_\ell-1}(\delta)} = \sum_{P \text{ set partition of } \{1,\ldots,\ell\}} \sum_{\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}} \mathsf{U}_P(\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}) .$$

The second sum is over all $\ell(P)$-tuples of partitions of positive size (but only finitely many terms will be nonzero).

To define the contribution $\mathsf{U}_P(\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)})$, we first order the parts of P,

$$S_1 \cup \cdots \cup S_{\ell(P)} = \{1,\ldots,\ell\}$$

We place the elements within each S_k in the canonical increasing order (inherited from $\{1,\ldots,\ell\}$). We thus obtain a canonical permutation $\theta_P \in \Sigma_\ell$. The permutation θ_P yields a canonical automorphism

$$\theta_P : \mathcal{X}^\ell \to \mathcal{X}^\ell$$

over $\mathcal{Y}$ by permuting the product factors (to simplify the notation, we use the same symbol for both the permutation and the automorphism).

For example, let $\ell = 5$ and consider the set partition P of $\{1,2,3,4,5\}$ defined by the data

$$S_1 \cup S_2 = \{1,2,3,4,5\}\,,\ S_1 = \{2,4,5\}\,,\ S_2 = \{1,3\}\,.$$

Then, the permutation $\theta_P \in \Sigma_5$ is

$$1 \to 2\,,\quad 2 \to 4\,,\quad 3 \to 5\,,\quad 4 \to 1\,,\quad 5 \to 3\,.$$

The contribution $\mathsf{U}_P(\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)})$ is a general descendent insertion with $\sum_{k=1}^{\ell(P)} \ell(\widehat{\alpha}^k)$ descendent indices,

$$\mathsf{U}_P(\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}) = \tau_{\widehat{\alpha}^1_1-1,\ldots,\widehat{\alpha}^1_{\ell(\widehat{\alpha}^1)}-1,\ldots,\widehat{\alpha}^{\ell(P)}_1-1,\ldots,\widehat{\alpha}^{\ell(P)}_{\ell(\widehat{\alpha}^{\ell(P)})}-1}(\delta_P(\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}))\,.$$

The last step is to define the class $\delta_P(\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}) \in H^*(\mathcal{X}^{\sum_{k=1}^{\ell(P)} \ell(\widehat{\alpha}^k)})$.

Let $\iota : \Delta_P \hookrightarrow \mathcal{X}^\ell$ be the product of diagonals,

$$\Delta_P = \prod_{k=1}^{\ell(P)} \mathrm{pr}^*_{S_k}(\Delta_{S_k})\,,$$

where $\Delta_{S_k} \subset \mathcal{X}^{S_k}$ is the small diagonal of the factor $\mathrm{pr}_{S_k} : \mathcal{X}^\ell \to \mathcal{X}^{S_k}$ corresponding to $S_k \subset \{1,\ldots,\ell\}$. There is a corresponding product of diagonals

$$\widehat{\iota} : \Delta_{\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}} \hookrightarrow \mathcal{X}^{\sum_{k=1}^{\ell(P)} \ell(\widehat{\alpha}^k)}$$

defined by pulling-back the small diagonals of the product factors,

$$\Delta_{\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}} = \prod_{k=1}^{\ell(P)} \mathrm{pr}^*_{\widehat{\alpha}^k}(\Delta_{\widehat{\alpha}^k})\,,$$

where $\mathrm{pr}_{\widehat{\alpha}^k} : \mathcal{X}^{\sum_{k=1}^{\ell(P)} \ell(\widehat{\alpha}^k)} \to \mathcal{X}^{\ell(\widehat{\alpha}^k)}$ corresponds to the indices of $\widehat{\alpha}^k$. There is a canonical isomorphism

$$\phi_P : \Delta_P \to \Delta_{\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}}$$

which takes the factor $\mathcal{X}$ corresponding to S_k on the domain to the factor $\mathcal{X}$ corresponding to $\widehat{\alpha}^k$ on the target. Then,

$$\delta_P(\widehat{\alpha}^1, \ldots, \widehat{\alpha}^{\ell(P)}) = \widehat{\iota}_* \left(\prod_{k=1}^{\ell(P)} \widetilde{\mathsf{K}}_{\alpha_{S_k},\widehat{\alpha}^k} \cdot \phi_{P*}\big(\iota^* \theta_P^*(\delta)\big) \right),$$

where each term $\widetilde{\mathsf{K}}_{\alpha_{S_k},\widehat{\alpha}^k}$ acts on the factor $\mathcal{X}$ corresponding to $\widehat{\alpha}^k$ of $\Delta_{\widehat{\alpha}^1,\ldots,\widehat{\alpha}^{\ell(P)}}$.

Conjecture III [GW/P families descendent correspondence].

For $\delta \in H^*(\mathcal{X}^\ell)$, we have

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\Big(\nu; q \;\Big|\, \tau_{\alpha_1-1,\ldots,\alpha_\ell-1}(\delta)\Big)_\beta = (-iu)^{d_\beta} \mathsf{Z}'_{\mathsf{GW}}\Big(\nu; u \;\Big|\; \overline{\tau_{\alpha_1-1,\ldots,\alpha_\ell-1}(\delta)}\;\Big)_\beta$$

under the variable change $-q = e^{iu}$.

Conjecture III relies upon a rationality conjecture for the general descendent stable pairs theory of ν. A nice exercise is to specialize Conjecture III to the cases of Conjectures I and II and to derive the sign rules there from Conjecture III.

At the moment, Conjectures II and III are known for families only in cases where Künneth decompositions are available and Conjecture I is proven. Perhaps the most interesting known case is that of the T-equivariant theory of a nonsingular projective toric 3-fold X.

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