

Cohomological field theory calculations

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Abstract

Cohomological field theories (CohFTs) were defined in the mid 1990s by Kontsevich and Manin to capture the formal properties of the virtual fundamental class in Gromov-Witten theory. A beautiful classification result for semisimple CohFTs (via the action of the Givental group) was proven by Teleman in 2012. The Givental-Teleman classification can be used to explicitly calculate the full CohFT in many interesting cases not approachable by earlier methods.

My goal here is to present an introduction to these ideas together with a survey of the calculations of the CohFTs obtained from

- Witten's classes on the moduli spaces of r -spin curves,
- Chern characters of the Verlinde bundles on the moduli of curves,
- Gromov-Witten classes of Hilbert schemes of points of \mathbb{C}^2 .

The subject is full of basic open questions.

0 Introduction

0.1 Moduli of curves

The moduli space \mathcal{M}_g of complete, nonsingular, irreducible, algebraic curves over \mathbb{C} of genus g has been a central object in mathematics since Riemann's work in the middle of the 19th century. The Deligne-Mumford compactification

$$\mathcal{M}_g \subset \overline{\mathcal{M}}_g$$

by nodal curves was defined almost 50 years ago [10].

We will be concerned here with the moduli space of curves with marked points,

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n},$$

in the stable range $2g - 2 + n > 0$. As a Deligne-Mumford stack (or orbifold), $\overline{\mathcal{M}}_{g,n}$ is nonsingular, irreducible, and of (complex) dimension $3g - 3 + n$. There are natural forgetful morphisms

$$p : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$$

dropping the last marking.

The *boundary*¹ of the Deligne-Mumford compactification is the closed locus parameterizing curves with a least one node,

$$\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}.$$

By identifying the last two markings of a single $(n+2)$ -pointed curve of genus $g-1$, we obtain a morphism

$$q : \overline{\mathcal{M}}_{g-1,n+2} \rightarrow \overline{\mathcal{M}}_{g,n}.$$

Similarly, by identifying the last markings of separate pointed curves, we obtain

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \rightarrow \overline{\mathcal{M}}_{g,n},$$

where $n = n_1 + n_2$ and $g = g_1 + g_2$. The images of both q and r lie in the boundary $\partial\overline{\mathcal{M}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$.

The cohomology and Chow groups of the moduli space of curves are

$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \quad \text{and} \quad A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

While there has been considerable progress in recent years, many basic questions about the cohomology and algebraic cycle theory remain open.²

0.2 Gromov-Witten classes

Let X be a nonsingular projective variety over \mathbb{C} , and let

$$\overline{\mathcal{M}}_{g,n}(X, \beta)$$

be the moduli space of genus g , n -pointed stable maps to X representing the class $\beta \in H_2(X, \mathbb{Z})$. The basic structures carried by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ are forgetful maps,

$$\pi : \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{g,n},$$

to the moduli space of curves via the domain (in case $2g - 2 + n > 0$) and evaluation maps,

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow X,$$

for each marking $1 \leq i \leq n$.

Given cohomology classes $v_1, \dots, v_n \in H^*(X, \mathbb{Q})$, the associated *Gromov-Witten class* is defined by

$$\Omega_{g,n,\beta}^X(v_1, \dots, v_n) = \pi_* \left(\prod_{i=1}^n \text{ev}_i^*(v_i) \cap [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \right) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

¹Since $\overline{\mathcal{M}}_{g,n}$ is a closed nonsingular orbifold, the boundary here is *not* in the sense of orbifold with boundary. If $g = 0$, there is no boundary map q .

²See [31] for a survey of results and open questions.

Central to the construction is the *virtual fundamental class* of the moduli space of stable maps,

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \in H_{2 \cdot \text{vir dim}}(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}),$$

of *virtual dimension*

$$\text{vir dim} = \int_{\beta} c_1(X) + (1 - g) \cdot (\dim_{\mathbb{C}}(X) - 3) + n.$$

Gromov-Witten classes contain much more information than the *Gromov-Witten invariants* defined by integration,

$$\langle v_1, \dots, v_n \rangle_{g,n,\beta}^X = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n,\beta}^X(v_1, \dots, v_n).$$

We refer the reader to [3, 4, 9, 15] for a detailed treatment of stable maps, virtual fundamental classes, and Gromov-Witten invariants in algebraic geometry.

The Gromov-Witten classes satisfy formal properties with respect to the natural forgetful and boundary maps p , q , and r discussed in Section 0.1. The idea of a cohomological field theory was introduced by Kontsevich and Manin [21] to fully capture these formal properties.

0.3 Cohomological field theories

The starting point for defining a cohomological field theory is a triple of data $(V, \eta, \mathbf{1})$ where

- V is a finite dimensional \mathbb{Q} -vector space,
- η is a non-degenerate symmetric 2-form on V ,
- $\mathbf{1} \in V$ is a distinguished element.

Given a \mathbb{Q} -basis $\{e_i\}$ of V , the symmetric form η can be written as a matrix

$$\eta_{jk} = \eta(e_j, e_k).$$

The inverse matrix is denoted, as usual, by η^{jk} .

A *cohomological field theory* consists of a system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of tensors

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}.$$

The tensor $\Omega_{g,n}$ associates a cohomology class in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ to vectors

$$v_1, \dots, v_n \in V$$

assigned to the n markings. We will use both

$$\Omega_{g,n}(v_1 \otimes \dots \otimes v_n) \quad \text{and} \quad \Omega_{g,n}(v_1, \dots, v_n)$$

to denote the associated cohomology class in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

In order to define a cohomological field theory, the system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ must satisfy the CohFT axioms:

- (i) Each tensor $\Omega_{g,n}$ is Σ_n -invariant for the natural action of the symmetric group Σ_n on

$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

obtained by simultaneously permuting the n marked points of $\overline{\mathcal{M}}_{g,n}$ and the n factors of V^* .

- (ii) The tensor $q^*(\Omega_{g,n}) \in H^*(\overline{\mathcal{M}}_{g-1,n+2}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$, obtained via pull-back by the boundary morphism

$$q : \overline{\mathcal{M}}_{g-1,n+2} \rightarrow \overline{\mathcal{M}}_{g,n} ,$$

is required to equal the contraction of $\Omega_{g-1,n+2}$ by the bi-vector

$$\sum_{j,k} \eta^{jk} e_j \otimes e_k$$

inserted at the two identified points:

$$q^*(\Omega_{g,n}(v_1, \dots, v_n)) = \sum_{j,k} \eta^{jk} \Omega_{g-1,n+2}(v_1, \dots, v_n, e_j, e_k)$$

in $H^*(\overline{\mathcal{M}}_{g-1,n+2}, \mathbb{Q})$ for all $v_i \in V$.

The tensor $r^*(\Omega_{g,n})$, obtained via pull-back by the boundary morphism

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \rightarrow \overline{\mathcal{M}}_{g,n} ,$$

is similarly required to equal the contraction of $\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1}$ by the same bi-vector:

$$r^*(\Omega_{g,n}(v_1, \dots, v_n)) = \sum_{j,k} \eta^{jk} \Omega_{g_1,n_1+1}(v_1, \dots, v_{n_1}, e_j) \otimes \Omega_{g_2,n_2+1}(v_{n_1+1}, \dots, v_n, e_k)$$

in $H^*(\overline{\mathcal{M}}_{g_1,n_1+1}, \mathbb{Q}) \otimes H^*(\overline{\mathcal{M}}_{g_2,n_2+1}, \mathbb{Q})$ for all $v_i \in V$.

- (iii) The tensor $p^*(\Omega_{g,n})$, obtained via pull-back by the forgetful map

$$p : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n} ,$$

is required to satisfy

$$\Omega_{g,n+1}(v_1, \dots, v_n, \mathbf{1}) = p^*\Omega_{g,n}(v_1, \dots, v_n)$$

for all $v_i \in V$. In addition, the equality

$$\Omega_{0,3}(v_1, v_2, \mathbf{1}) = \eta(v_1, v_2)$$

is required for all $v_i \in V$.

Definition 1 A system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of tensors

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

satisfying (i) and (ii) is a cohomological field theory or a CohFT. If (iii) is also satisfied, Ω is a CohFT with unit.

The simplest example of a cohomological field theory with unit is given by the *trivial CohFT*,

$$V = \mathbb{Q}, \quad \eta(1,1) = 1, \quad \mathbf{1} = 1, \quad \Omega_{g,n}(1, \dots, 1) = 1 \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

A more interesting example is given by the total Chern class

$$c(\mathbb{E}) = 1 + \lambda_1 + \dots + \lambda_g \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

of the rank g Hodge bundle $\mathbb{E} \rightarrow \overline{\mathcal{M}}_{g,n}$,

$$V = \mathbb{Q}, \quad \eta(1,1) = 1, \quad \mathbf{1} = 1, \quad \Omega_{g,n}(1, \dots, 1) = c(\mathbb{E}) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Definition 2 For a CohFT $\Omega = (\Omega_{g,n})_{2g-2+n>0}$, the topological part ω of Ω is defined by

$$\omega_{g,n} = [\Omega_{g,n}]^0 \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}.$$

The degree 0 part $[\]^0$ of Ω is simply obtained from the canonical summand projection

$$[\]^0 : H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \rightarrow H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

If Ω is a CohFT with unit, then ω is also a CohFT with unit. The topological part of the CohFT obtained from the total Chern class of the Hodge bundle is the trivial CohFT.

The motivating example of a CohFT with unit is obtained from the Gromov-Witten theory of a nonsingular projective variety X . Here,

$$V = H^*(X, \mathbb{Q}), \quad \eta(v_1, v_2) = \int_X v_1 \cup v_2, \quad \mathbf{1} = 1.$$

Of course, the Poincaré pairing on $H^*(X, \mathbb{Q})$ is symmetric only if X has no odd cohomology.³ The tensor $\Omega_{g,n}$ is defined using the Gromov-Witten classes $\Omega_{g,n,\beta}^X$ of Section 0.2 (together with a Novikov⁴ parameter q),

$$\Omega_{g,n}(v_1, \dots, v_n) = \sum_{\beta \in H_2(X, \mathbb{Q})} \Omega_{g,n,\beta}^X q^\beta.$$

The CohFT axioms here coincide exactly with the axioms⁵ of Gromov-Witten theory related to the morphisms p , q , and r . For example, axiom (ii) of a CohFT here is the *splitting axiom* of Gromov-Witten theory, see [21].

³To accommodate the case of arbitrary X , the definition of a CohFT can be formulated with signs and $\mathbb{Z}/2\mathbb{Z}$ -gradings. We do not take the super vector space path here.

⁴Formally, we must extend scalars in the definition of a CohFT from \mathbb{Q} to the Novikov ring to capture the Gromov-Witten theory of X .

⁵The divisor axiom of Gromov-Witten theory (which concerns divisor and curve classes on X) is not part of the CohFT axioms.

0.4 Semisimplicity

A CohFT with unit Ω defines a *quantum product* \bullet on V by⁶

$$\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3).$$

The quantum product \bullet is commutative by CohFT axiom (i). The associativity of \bullet follows from CohFT axiom (ii). The element $\mathbf{1} \in V$ is the identity for \bullet by the second clause of CohFT axiom (iii). Hence,

$$(V, \bullet, \mathbf{1})$$

is a commutative \mathbb{Q} -algebra.

Lemma 3 *The topological part ω of Ω is uniquely and effectively determined by the coefficients*

$$\Omega_{0,3}(v_1, v_2, v_3) \in H^*(\overline{\mathcal{M}}_{0,3}, \mathbb{Q})$$

of the quantum product \bullet .

Proof. Let the moduli point $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ correspond to a maximally degenerate curve (with every component isomorphic to \mathbb{P}^1 with exactly 3 special points). Since

$$\omega_{g,n}(v_1, \dots, v_n) \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

is a multiple of the identity class, $\omega_{g,n}(v_1, \dots, v_n)$ is determined by the pull-back to the point $[C, p_1, \dots, p_n]$. The equality

$$\omega_{g,n}(v_1, \dots, v_n)|_{[C, p_1, \dots, p_n]} = \Omega_{g,n}(v_1, \dots, v_n)|_{[C, p_1, \dots, p_n]}$$

holds, and the latter restriction is determined by 3-point values $\Omega_{0,3}(w_1, w_2, w_3)$ from repeated application of CohFT axiom (ii). \diamond

A finite dimensional \mathbb{Q} -algebra is *semisimple* if there exists a basis $\{e_i\}$ of idempotents,

$$e_i e_j = \delta_{ij} e_i,$$

after an extension of scalars to \mathbb{C} .

Definition 4 *A CohFT with unit $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ is semisimple if $(V, \bullet, \mathbf{1})$ is a semisimple algebra.*

⁶Since $\overline{\mathcal{M}}_{0,3}$ is a point, we canonically identify $H^*(\overline{\mathcal{M}}_{0,3}, \mathbb{Q}) \cong \mathbb{Q}$, so $\Omega_{0,3}(v_1, v_2, v_3) \in \mathbb{Q}$.

0.5 Classification and calculation

The Givental-Teleman classification concerns semisimple CohFTs with unit.⁷ The form of the classification result is as follows: *a semisimple CohFT with unit Ω is uniquely determined by the following two structures:*

- the topological part ω of Ω ,
- an R -matrix

$$R(z) = \text{Id} + R_1 z + R_2 z^2 + R_3 z^3 + \dots, \quad R_k \in \text{End}(V)$$

satisfying the symplectic property

$$R(z) \cdot R^*(-z) = \text{Id},$$

where \star denotes the adjoint with respect to the metric η .

The precise statement of the Givental-Teleman classification will be discussed in Section 1.

Via the Givental-Teleman classification, a semisimple CohFT with unit Ω can be calculated in three steps:

- (i) determine the ring $(V, \bullet, \mathbf{1})$ as explicitly as possible,
- (ii) find a closed formula for the topological part ω of Ω via Lemma 3,
- (iii) calculate the R -matrix of the theory.

In the language of Gromov-Witten theory, step (i) is the determination of the *small quantum cohomology* ring $QH^*(X, \mathbb{Q})$ via the 3-pointed genus 0 Gromov-Witten invariants. Step (ii) is then to calculate the Gromov-Witten invariants where the domain has a *fixed* complex structure of higher genus. New ideas are often required for the leap to higher genus moduli in step (iii). Finding a closed formula for the R -matrix requires a certain amount of luck.

Explaining how the above path to calculation plays out in three important CohFTs is my goal here. The three theories are:

- Witten's class on the moduli of r -spin curves,
- the Chern character of the Verlinde bundle on the moduli of curves,
- the Gromov-Witten theory of the Hilbert scheme of points of \mathbb{C}^2 .

While each theory has geometric interest and the calculations have consequences in several directions, the focus of the paper will be on the CohFT determination. The paths to calculation pursued here are applicable in many other cases.

⁷Semisimple CohFTs without unit are also covered, but we are interested here in the unital case. Semisimplicity is an essential condition.

0.6 Past and future directions

The roots of the classification of semisimple CohFTs can be found in Givental's analysis [16, 17, 22] of the torus localization formula [18] for the higher genus Gromov-Witten theory of toric varieties. The three CohFTs treated here are not directly accessible via the older torus localization methods. Givental's approach to the R -matrix via oscillating integrals (used often in the study of toric geometries) is not covered in the paper.

Many interesting CohFTs are *not* semisimple. For example, the Gromov-Witten theory of the famous Calabi-Yau quintic 3-fold,

$$X_5 \subset \mathbb{P}^4,$$

does *not* define a semisimple CohFT. However, in the past year, an approach to the quintic via the semisimple *formal quintic* theory [20, 23] appears possible. These developments are not surveyed here.

0.7 Acknowledgments

Much of what I know about the Givental-Teleman classification was learned through writing [22] with Y.-P. Lee and [32] with A. Pixton and D. Zvonkine. For the study of the three CohFTs discussed in the paper, my collaborators have been J. Bryan, F. Janda, A. Marian, A. Okounkov, D. Oprea, A. Pixton, H.-H. Tseng, and D. Zvonkine. More specifically,

- Sections 1-2 are based on the papers [32, 33] and the Appendix of [33],
- Section 3 is based on the paper [24],
- Section 4 is based on the papers [6, 29] and especially [34].

Discussions with A. Givental, T. Graber, H. Lho, and Y. Ruan have played an important role in my view of the subject. I was partially supported by SNF grant 200021-143274, ERC grant AdG-320368-MCSK, SwissMAP, and the Einstein Stiftung.

1 Givental-Teleman classification

1.1 Stable graphs

The boundary strata of the moduli space of curves correspond to *stable graphs*

$$\Gamma = (V, H, L, g : V \rightarrow \mathbb{Z}_{\geq 0}, v : H \rightarrow V, i : H \rightarrow H)$$

satisfying the following properties:

- (i) V is a vertex set with a genus function $g : V \rightarrow \mathbb{Z}_{\geq 0}$,
- (ii) H is a half-edge set equipped with a vertex assignment $v : H \rightarrow V$ and an involution i ,

- (iii) E , the edge set, is defined by the 2-cycles of i in H (self-edges at vertices are permitted),
- (iv) L , the set of legs, is defined by the fixed points of i and endowed with a bijective correspondence with a set of markings,
- (v) the pair (V, E) defines a *connected* graph,
- (vi) for each vertex v , the stability condition holds:

$$2g(v) - 2 + n(v) > 0,$$

where $n(v)$ is the valence of Γ at v including both half-edges and legs.

An automorphism of Γ consists of automorphisms of the sets V and H which leave invariant the structures g , v , and i (and hence respect E and L). Let $\text{Aut}(\Gamma)$ denote the automorphism group of Γ .

The genus of a stable graph Γ is defined by:

$$g(\Gamma) = \sum_{v \in V} g(v) + h^1(\Gamma).$$

Let $\mathbf{G}_{g,n}$ denote the set of all stable graphs (up to isomorphism) of genus g with n legs. The strata⁸ of the moduli space $\overline{\mathcal{M}}_{g,n}$ of Deligne-Mumford stable curves are in bijective correspondence to $\mathbf{G}_{g,n}$ by considering the dual graph of a generic pointed curve parameterized by the stratum.

To each stable graph Γ , we associate the moduli space

$$\overline{\mathcal{M}}_{\Gamma} = \prod_{v \in V} \overline{\mathcal{M}}_{g(v), n(v)}.$$

Let π_v denote the projection from $\overline{\mathcal{M}}_{\Gamma}$ to $\overline{\mathcal{M}}_{g(v), n(v)}$ associated to the vertex v . There is a canonical morphism

$$\iota_{\Gamma} : \overline{\mathcal{M}}_{\Gamma} \rightarrow \overline{\mathcal{M}}_{g,n} \tag{1}$$

with image⁹ equal to the boundary stratum associated to the graph Γ .

1.2 R -matrix action

1.2.1 First action

Let $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ be a CohFT¹⁰ on the vector space (V, η) . Let R be a matrix series

$$R(z) = \sum_{k=0}^{\infty} R_k z^k \in \text{Id} + z \cdot \text{End}(V)[[z]]$$

⁸We consider here the standard stratification by topological type of the pointed curve

⁹The degree of ι_{Γ} is $|\text{Aut}(\Gamma)|$.

¹⁰ Ω is not assumed here to be unital – only CohFT axioms (i) and (ii) are imposed.

which satisfies the symplectic condition

$$R(z) \cdot R^*(-z) = \text{Id}.$$

We define a new CohFT $R\Omega$ on the vector space (V, η) by summing over stable graphs Γ with summands given by a product of vertex, edge, and leg contributions,

$$(R\Omega)_{g,n} = \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \iota_{\Gamma^*} \left(\prod_{v \in V} \text{Cont}(v) \prod_{e \in E} \text{Cont}(e) \prod_{l \in L} \text{Cont}(l) \right), \quad (2)$$

where

- (i) the vertex contribution is

$$\text{Cont}(v) = \Omega_{g(v), n(v)},$$

where $g(v)$ and $n(v)$ denote the genus and number of half-edges and legs of the vertex,

- (ii) the leg contribution is the $\text{End}(V)$ -valued cohomology class

$$\text{Cont}(l) = R(\psi_l),$$

where $\psi_l \in H^2(\overline{\mathcal{M}}_{g(v), n(v)}, \mathbb{Q})$ is the cotangent class at the marking corresponding to the leg,

- (iii) the edge contribution is

$$\text{Cont}(e) = \frac{\eta^{-1} - R(\psi'_e)\eta^{-1}R(\psi''_e)^\top}{\psi'_e + \psi''_e},$$

where ψ'_e and ψ''_e are the cotangent classes at the node which represents the edge e . The symplectic condition guarantees that the edge contribution is well-defined.

We clarify the meaning of the edge contribution (iii),

$$\text{Cont}(e) \in V^{\otimes 2} \otimes H^*(\overline{\mathcal{M}}_{g', n'}) \otimes H^*(\overline{\mathcal{M}}_{g'', n''}),$$

where (g', n') and (g'', n'') are the labels of the vertices adjacent to the edge e by writing the formula explicitly in coordinates.

Let $\{e_\mu\}$ be a \mathbb{Q} -basis of V . The components of the R -matrix in the basis are $R_\mu^\nu(z)$,

$$R(z)(e_\mu) = \sum_\nu R_\mu^\nu(z) \cdot e_\nu.$$

The components of $\text{Cont}(e)$ are

$$\text{Cont}(e)^{\mu\nu} = \frac{\eta^{\mu\nu} - \sum_{\rho, \sigma} R_\rho^\mu(\psi'_e) \cdot \eta^{\rho\sigma} \cdot R_\sigma^\nu(\psi''_e)}{\psi'_e + \psi''_e} \in H^*(\overline{\mathcal{M}}_{g', n'}) \otimes H^*(\overline{\mathcal{M}}_{g'', n''}).$$

The fraction

$$\frac{\eta^{\mu\nu} - \sum_{\rho,\sigma} R_\rho^\mu(z) \cdot \eta^{\rho\sigma} \cdot R_\sigma^\nu(w)}{z + w}$$

is a power series in z and w since the numerator vanishes when $z = -w$ as a consequence of the symplectic condition which, in coordinates, takes the form

$$\sum_{\rho,\sigma} R_\rho^\mu(z) \cdot \eta^{\rho\sigma} \cdot R_\sigma^\nu(-z) = \eta^{\mu\nu}.$$

The substitution $z = \psi'_e$ and $w = \psi''_e$ is therefore unambiguously defined.

Definition 5 Let $R\Omega$ be the CohFT obtained from Ω by the R -action (2).

The above R -action was first defined¹¹ on Gromov-Witten potentials by Givental [16]. An abbreviated treatment of the lift to CohFTs appears in papers by Teleman [38] and Shadrin [37]. A careful proof that $R\Omega$ satisfies CohFT axioms (i) and (ii) can be found in [32, Section 2].

If Ω is a CohFT with unit on $(V, \eta, \mathbf{1})$, then $R\Omega$ may not respect the unit $\mathbf{1}$. To handle the unit, a second action is required.

1.2.2 Second action

A second action on the CohFT¹² Ω on (V, η) is given by translations. Let $T \in V[[z]]$ be a series with *no terms of degree 0 or 1*,

$$T(z) = T_2 z^2 + T_3 z^3 + \dots, \quad T_k \in V.$$

Definition 6 Let $T\Omega$ be the CohFT obtained from Ω by the formula

$$(T\Omega)_{g,n}(v_1, \dots, v_n) = \sum_{m=0}^{\infty} \frac{1}{m!} p_{m\star} \left(\Omega_{g,n+m}(v_1, \dots, v_n, T(\psi_{n+1}), \dots, T(\psi_{n+m})) \right),$$

where $p_m : \overline{\mathcal{M}}_{g,n+m} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the morphism forgetting the last m markings.

The right side of the formula in Definition 6 is a formal expansion by distributing the powers of the ψ classes as follows:

$$\Omega_{g,n+m}(\dots, T(\psi_\bullet), \dots) = \sum_{k=2}^{\infty} \psi_\bullet^k \cdot \Omega_{g,n+m}(\dots, T_k, \dots).$$

The summation is finite because T has no terms of degree 0 or 1.

¹¹To simplify our formulas, we have changed Givental's and Teleman's conventions by replacing R with R^{-1} . Equation (2) above then determines a right group action on CohFTs rather than a left group action as in Givental's and Teleman's papers.

¹²To define the translation action, Ω is required only to be CohFT and not necessarily a CohFT with unit.

1.3 Reconstruction

We can now state the Givental-Teleman classification result [38]. Let Ω be a semisimple CohFT with unit on $(V, \eta, \mathbf{1})$, and let ω be the topological part of Ω . For a symplectic matrix R , define

$$R.\omega = R(T(\omega)) \quad \text{with} \quad T(z) = z((\text{Id} - R(z)) \cdot \mathbf{1}) \in V[[z]].$$

By [32, Proposition 2.12], $R.\omega$ is a CohFT with unit on $(V, \eta, \mathbf{1})$. The Givental-Teleman classification asserts the *existence* of a unique R -matrix which exactly recovers Ω .

Theorem 7 *There exists a unique symplectic matrix*

$$R \in \text{Id} + z \cdot \text{End}(V)[[z]]$$

which reconstructs Ω from ω ,

$$\Omega = R.\omega,$$

as a CohFT with unit.

The first example concerns the total Chern class CohFT of Section 0.3,

$$V = \mathbb{Q}, \quad \eta(1,1) = 1, \quad \mathbf{1} = 1, \quad \Omega_{g,n}(1, \dots, 1) = c(\mathbb{E}) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

The topological part is the trivial CohFT, and the R -matrix is

$$R(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} z^{2k-1}\right).$$

That the above R -matrix reconstructs the total Chern class CohFT is a consequence of Mumford's calculation [27] of the Chern character of the Hodge bundle by Grothendieck-Riemann-Roch.

1.4 Chow field theories

Let $(V, \eta, \mathbf{1})$ be a \mathbb{Q} -vector space with a non-degenerate symmetric 2-form and a distinguished element. Let $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ be a system of tensors

$$\Omega_{g,n} \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

where A^* is the Chow group of algebraic cycles modulo rational equivalence. In order to define a Chow field theory, the system Ω must satisfy the CohFT axioms of Section 0.3 with cohomology H^* replaced everywhere by Chow A^* .

Definition 8 *A system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of elements*

$$\Omega_{g,n} \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

satisfying (i) and (ii) is a Chow field theory or a ChowFT. If (iii) is also satisfied, Ω is a ChowFT with unit.

For ChowFTs, the quantum product $(V, \bullet, \mathbf{1})$ and semisimplicity are defined just as for CohFTs. The R - and T -actions of Sections 1.2 also lift immediately to ChowFTs. However, the classification of semisimple ChowFTs is an open question.

Question 9 *Does the Givental-Teleman classification of Theorem 7 hold for a semisimple Chow field theory Ω with unit?*

2 Witten's r -spin class

2.1 r -spin CohFT

Let $r \geq 2$ be an integer. Let $(V_r, \eta, \mathbf{1})$ be the following triple:

- V_r is an $(r-1)$ -dimensional \mathbb{Q} -vector space with basis e_0, \dots, e_{r-2} ,
- η is the non-degenerate symmetric 2-form

$$\eta_{ab} = \langle e_a, e_b \rangle = \delta_{a+b, r-2},$$

- $\mathbf{1} = e_0$.

Witten's r -spin theory provides a family of classes

$$\mathcal{W}_{g,n}^r(a_1, \dots, a_n) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

for $a_1, \dots, a_n \in \{0, \dots, r-2\}$ which define a CohFT $\mathcal{W}^r = (\mathcal{W}_{g,n}^r)_{2g-2+n>0}$ by

$$\mathcal{W}_{g,n}^r : V_r^{\otimes n} \rightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}), \quad \mathcal{W}_{g,n}^r(e_{a_1} \otimes \dots \otimes e_{a_n}) = \mathcal{W}_{g,n}^r(a_1, \dots, a_n).$$

The class $\mathcal{W}_{g,n}^r(a_1, \dots, a_n)$ has (complex) degree¹³

$$\begin{aligned} \deg_{\mathbb{C}} \mathcal{W}_{g,n}^r(a_1, \dots, a_n) &= D_{g,n}^r(a_1, \dots, a_n) \\ &= \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}. \end{aligned} \quad (3)$$

If $D_{g,n}^r(a_1, \dots, a_n)$ is not an integer, the corresponding Witten's class vanishes.

The construction of $\mathcal{W}_{0,n}^r(a_1, \dots, a_n)$ in genus 0 was carried out by Witten [42] using r -spin structures. Let $\overline{\mathcal{M}}_{0,n}^r(a_1, \dots, a_n)$ be the Deligne-Mumford moduli space parameterizing r^{th} roots,

$$\mathcal{L}^{\otimes r} \cong \omega_C \left(- \sum_{i=1}^n a_i p_i \right) \quad \text{where} \quad [C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{0,n}.$$

The class $\frac{1}{r} \mathcal{W}_{0,n}^r(a_1, \dots, a_n)$ is defined to be the push-forward to $\overline{\mathcal{M}}_{0,n}$ of the top Chern class of the bundle on $\overline{\mathcal{M}}_{0,n}^r(a_1, \dots, a_n)$ with fiber $H^1(C, \mathcal{L})^*$.

¹³So $\mathcal{W}_{g,n}^r(a_1, \dots, a_n) \in H^{2 \cdot D_{g,n}^r(a_1, \dots, a_n)}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

The existence of Witten's class in higher genus is both remarkable and highly non-trivial. Polishchuk and Vaintrob [35, 36] constructed

$$\mathcal{W}_{g,n}^r(a_1, \dots, a_n) \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

as an algebraic cycle class and proved the CohFT axioms (i-iii) for a Chow field theory hold. The algebraic approach was later simplified in [7, 8]. Analytic constructions appear in [13, 26].

2.2 Genus 0

2.2.1 3 and 4 markings

Witten [42] determined the following initial conditions in genus 0 with $n = 3, 4$:

$$\int_{\overline{\mathcal{M}}_{0,3}} \mathcal{W}_{0,3}^r(a_1, a_2, a_3) = \begin{cases} 1 & \text{if } a_1 + a_2 + a_3 = r - 2, \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

$$\int_{\overline{\mathcal{M}}_{0,4}} \mathcal{W}_{0,4}^r(1, 1, r - 2, r - 2) = \frac{1}{r}.$$

Uniqueness of the r -spin CohFT in genus 0 follows easily from the initial conditions (4) and the axioms of a CohFT with unit.

The genus 0 sector of the CohFT W^r defines a quantum product¹⁴ \bullet on V_r . The resulting algebra $(V_r, \bullet, \mathbf{1})$, even after extension to \mathbb{C} , is *not* semisimple. Therefore, the Givental-Teleman classification can not be directly applied.

2.2.2 Witten's r -spin class and representations of $\mathfrak{sl}_2(\mathbb{C})$.

Consider the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$. Denote by ρ_k the k^{th} symmetric power of the standard 2-dimensional representation of \mathfrak{sl}_2 ,

$$\rho_k = \text{Sym}^k(\rho_1), \quad \dim \rho_k = k + 1.$$

The following complete solution of the genus 0 part of the CohFT W^r (after integration) was found by Pixton, see [33] for a proof.

Theorem 10 *Let $\mathbf{a} = (a_1, \dots, a_{n \geq 3})$ with $a_i \in \{0, \dots, r - 2\}$ satisfy the degree constraint $D_{0,n}^r(\mathbf{a}) = n - 3$. Then,*

$$\int_{\overline{\mathcal{M}}_{0,n}} \mathcal{W}_{0,n}^r(\mathbf{a}) = \frac{(n - 3)!}{r^{n-3}} \dim \left[\rho_{r-2-a_1} \otimes \dots \otimes \rho_{r-2-a_n} \right]^{\mathfrak{sl}_2},$$

where the superscript \mathfrak{sl}_2 denotes the \mathfrak{sl}_2 -invariant subspace.

¹⁴See Section 0.4.

2.2.3 Shifted Witten class

Definition 11 For $\gamma \in V_r$, the shifted r -spin CohFT $W^{r,\gamma}$ is defined by

$$W_{g,n}^{r,\gamma}(v_1 \otimes \cdots \otimes v_n) = \sum_{m \geq 0} \frac{1}{m!} p_{m*} W_{g,n+m}^r(v_1 \otimes \cdots \otimes v_n \otimes \gamma^{\otimes m}),$$

where $p_m: \overline{\mathcal{M}}_{g,n+m} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the map forgetting the last m markings.

Using degree formula $D_{g,n}^r$, the summation in the definition of the shift is easily seen to be finite. The shifted Witten class $W^{r,\gamma}$ determines a CohFT with unit, see [32, Section 1.1].

Definition 12 Define the CohFT \widehat{W}^r with unit on $(V_r, \eta, \mathbf{1})$ by the shift

$$\widehat{W}^r = W^{r,(0,\dots,0,r)}$$

along the special vector $re_{r-2} \in V_r$. Let $(V_r, \widehat{\bullet}, \mathbf{1})$ be the \mathbb{Q} -algebra determined by the quantum product defined by \widehat{W}^r .

The Verlinde algebra of level r for \mathfrak{sl}_2 is spanned by the weights of \mathfrak{sl}_2 from 0 to $r-2$. The coefficient of c in the product $a \bullet b$ is equal to the dimension of the \mathfrak{sl}_2 -invariant subspace of the representation $\rho_a \otimes \rho_b \otimes \rho_c$ provided the inequality

$$a + b + c \leq 2r - 4$$

is satisfied. Using Theorem 10 for the integral r -spin theory in genus 0, the following basic result is proven in [33].

Theorem 13 The algebra $(V_r, \widehat{\bullet}, \mathbf{1})$ is isomorphic to the Verlinde algebra of level r for \mathfrak{sl}_2 .

Since the Verlinde algebra is well-known to be semisimple¹⁵, the Givental-Teleman classification of Theorem 7 can be applied to the CohFT \widehat{W}^r . Using the degree formula (3) and Definition 12, we see the (complex) degree $D_{g,n}^r(a_1, \dots, a_n)$ part of \widehat{W}^r equals W^r ,

$$\left[\widehat{W}_{g,n}^r(a_1, \dots, a_n) \right]^{D_{g,n}^r(a_1, \dots, a_n)} = W_{g,n}^r(a_1, \dots, a_n).$$

Hence, a complete computation of \widehat{W}^r also provides a computation of W^r .

2.3 The topological field theory

After the studying genus 0 theory, we turn our attention to the topological part $\widehat{\omega}^r$ of \widehat{W}^r . The following two results of [33] provide a complete calculation.

¹⁵An explicit normalized idempotent basis is given in Proposition 14 below.

Proposition 14 *The basis of normalized idempotents of $(V_r, \widehat{\bullet}, \mathbf{1})$ is given by*

$$v_k = \sqrt{\frac{2}{r}} \sum_{a=0}^{r-2} \sin\left(\frac{(a+1)k\pi}{r}\right) e_a, \quad k \in \{1, \dots, r-1\}.$$

More precisely, we have

$$\eta(v_k, v_l) = (-1)^{k-1} \delta_{k,l}, \quad v_k \widehat{\bullet} v_l = \frac{\sqrt{r/2}}{\sin(\frac{k\pi}{r})} v_k \delta_{k,l}.$$

Once the normalized idempotents are found, the computation of $\widehat{\omega}^r$ is straightforward by Lemma 3 and elementary trigonometric identities.

Proposition 15 *For $a_1, \dots, a_n \in \{0, \dots, r-2\}$, we have*

$$\widehat{\omega}_{g,n}^r(e_{a_1}, \dots, e_{a_n}) = \left(\frac{r}{2}\right)^{g-1} \sum_{k=1}^{r-1} \frac{(-1)^{(k-1)(g-1)} \prod_{i=1}^n \sin\left(\frac{(a_i+1)k\pi}{r}\right)}{\left(\sin\left(\frac{k\pi}{r}\right)\right)^{2g-2+n}}.$$

In Proposition 15, the CohFT $\widehat{\omega}^r$ is viewed as taking values in \mathbb{Q} via the canonical identification

$$H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \cong \mathbb{Q}.$$

2.4 The R -matrix

The last (and often hardest) step in the computation of a semisimple CohFT via the Givental-Telean classification is to find the unique R -matrix. Remarkably, there exists a closed formula in hypergeometric series for the R -matrix of the CohFT \widehat{W}^r . The precise shift in Definition 12 of the CohFT \widehat{W}^r is crucial: the shift re_{r-2} is (up to scale) the *only* shift of W^r for which closed formulas for the R -matrix are known.

The method of finding the unique R -matrix for W^r uses the *Euler field* e_{r-2} at the shift re_{r-2} . The operator of quantum multiplication $\widehat{\bullet}$ by the Euler field in the basis e_0, \dots, e_{r-2} is

$$\xi = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 2 \\ 0 & & & 0 & 2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \\ 0 & 2 & 0 & & 0 & \\ 2 & 0 & \cdots & \cdots & 0 & \end{pmatrix}.$$

In the same frame, the *shifted degree operator* is

$$\mu = \frac{1}{2r} \begin{pmatrix} -(r-2) & 0 & \cdots & \cdots & 0 \\ 0 & -(r-4) & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & r-4 & 0 \\ 0 & \cdots & \cdots & 0 & r-2 \end{pmatrix}.$$

Since \widehat{W}^r has an Euler field with an associated degree operator, the unique R -matrix for the classification is given by the solution of

$$[R_{m+1}, \xi] = (m - \mu)R_m \quad (5)$$

with the initial condition $R_0 = \text{Id}$, see [38].

Definition 16 For each integer $a \in \{0, \dots, r-2\}$, define the hypergeometric series

$$\mathbf{B}_{r,a}(z) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^m \frac{((2i-1)r - 2(a+1))((2i-1)r + 2(a+1))}{i} \right] \left(-\frac{z}{16r^2} \right)^m.$$

Let $\mathbf{B}_{r,a}^{\text{even}}$ and $\mathbf{B}_{r,a}^{\text{odd}}$ the even and odd summands¹⁶ of the series $\mathbf{B}_{r,a}$.

The unique solution to (5) is computed in [33]. The R -matrix of \widehat{W}^r has a surprisingly simple form.

Theorem 17 The unique R -matrix classifying \widehat{W}^r has coefficients

$$R_a^a = \mathbf{B}_{r,a}^{\text{even}}(z), \quad a \in \{0, \dots, r-2\}$$

on the main diagonal, and

$$R_a^{r-2-a} = \mathbf{B}_{r,a}^{\text{odd}}(z), \quad a \in \{0, \dots, r-2\}$$

on the antidiagonal (if r is even, the coefficient at the intersection of both diagonals is 1), and 0 everywhere else.

In case $r = 2$, the matrix is trivial $R(z) = \text{Id}$. For $r = 3$ and 4 respectively, the R -matrices¹⁷ are

$$R(z) = \begin{pmatrix} \mathbf{B}_{3,0}^{\text{even}}(z) & \mathbf{B}_{3,1}^{\text{odd}}(z) \\ \mathbf{B}_{3,0}^{\text{odd}}(z) & \mathbf{B}_{3,1}^{\text{even}}(z) \end{pmatrix},$$

$$R(z) = \begin{pmatrix} \mathbf{B}_{4,0}^{\text{even}}(z) & 0 & \mathbf{B}_{4,2}^{\text{odd}}(z) \\ 0 & 1 & 0 \\ \mathbf{B}_{4,0}^{\text{odd}}(z) & 0 & \mathbf{B}_{4,2}^{\text{even}}(z) \end{pmatrix}.$$

¹⁶The even summand consists of all the even powers of z (and likewise for the odd summand).

¹⁷ R here is R^{-1} in [32, 33] because of a change of conventions.

2.5 Calculation of W^r

The analysis of Sections 2.2-2.4 together complete the calculation of \widehat{W}^r ,

$$\widehat{W}^r = R.\widehat{\omega}^r,$$

in exactly the steps (i)-(iii) proposed in Section 0.5 of the Introduction. Then, as we have seen,

$$W_{g,n}^r(a_1, \dots, a_n) = \left[\widehat{W}_{g,n}^r(a_1, \dots, a_n) \right]_{g,n}^{\mathbb{D}^r(a_1, \dots, a_n)}.$$

The calculation has an immediate consequence [33].

Corollary 18 *Witten's r -spin class on $\overline{\mathcal{M}}_{g,n}$ lies in the tautological ring (in cohomology),*

$$\mathcal{W}_{g,n}^r(a_1, \dots, a_n) \in RH^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

We refer the reader to [31] for a discussion of tautological classes on the moduli space of curves. In fact, the first proof of Pixton's relations in $RH^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ was obtained via the calculation of \widehat{W}^3 in [32].

2.6 Questions

Whether Corollary 18 also holds in Chow is an interesting question: is

$$\mathcal{W}_{g,n}^r(a_1, \dots, a_n) \in R^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})? \quad (6)$$

A positive answer to Question 9 about the classification of Chow field theories would imply a positive answer here. The following question may be viewed as a refinement of (6).

Question 19 *Find a formula in algebraic cycles for Witten's r -spin class on $\overline{\mathcal{M}}_{g,n}^r(a_1, \dots, a_n)$ before push-forward to $\overline{\mathcal{M}}_{g,n}$.*

Another open direction concerns the moduli spaces of holomorphic differentials [1, 14]. Let (a_1, \dots, a_n) be a partition of $2g - 2$ with non-negative parts. Let

$$\overline{\mathcal{H}}_g(a_1, \dots, a_n) \subset \overline{\mathcal{M}}_{g,n}$$

be the closure of the locus of moduli points

$$[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n} \quad \text{where} \quad \omega_C \cong \mathcal{O}_C \left(\sum_{i=1}^n a_i p_i \right).$$

For $r - 2 \geq \text{Max}\{a_1, \dots, a_n\}$, Witten's r -spin class $\mathcal{W}_{g,n}^r(a_1, \dots, a_n)$ is well-defined and of degree *independent* of r ,

$$\mathbb{D}_{g,n}^r(a_1, \dots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r} = g-1.$$

By [33, Theorem 7], after scaling by r^{g-1} ,

$$\mathcal{W}_{g,n}(a_1, \dots, a_n)[r] = r^{g-1} \cdot \mathcal{W}_{g,n}^r(a_1, \dots, a_n) \in RH^{g-1}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

is a *polynomial* in r for all sufficiently large r .

Question 20 Prove the following conjecture of [33, Appendix]:

$$(-1)^g \mathcal{W}_{g,n}(a_1, \dots, a_n)[0] = [\overline{\mathcal{H}}_g(a_1, \dots, a_n)] \in H^{2(g-1)}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

3 Chern character of the Verlinde bundle

3.1 Verlinde CohFT

Let G be a complex, simple, simply connected Lie group with Lie algebra \mathfrak{g} . Fix an integer *level* $\ell > 0$. Let $(V_\ell, \eta, \mathbf{1})$ be the following triple:

- V_ℓ is the \mathbb{Q} -vector space with basis indexed by the irreducible representations of \mathfrak{g} at level ℓ ,
- η is the non-degenerate symmetric 2-form

$$\eta(\mu, \nu) = \delta_{\mu, \nu^*}$$

where ν^* denotes the dual representation,

- $\mathbf{1}$ is the basis element corresponding to the trivial representation.

Let μ_1, \dots, μ_n be n irreducible representations of \mathfrak{g} at level ℓ . A vector bundle

$$\mathbb{V}_g(\mu_1, \dots, \mu_n) \rightarrow \overline{\mathcal{M}}_{g,n}$$

is constructed in [41]. Over nonsingular curves, the fibers of $\mathbb{V}_g(\mu_1, \dots, \mu_n)$ are the spaces of non-abelian theta functions – spaces of global sections of the determinant line bundles over the moduli of parabolic G -bundles. To extend $\mathbb{V}_g(\mu_1, \dots, \mu_n)$ over the boundary

$$\partial \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n},$$

the theory of *conformal blocks* is required [41]. The vector bundle $\mathbb{V}_g(\mu_1, \dots, \mu_n)$ has various names in the literature: the Verlinde bundle, the bundle of conformal blocks, and the bundle of vacua. A study in genus 0 and 1 can be found in [12].

A CohFT Ω^ℓ is defined via the Chern character¹⁸ of the Verlinde bundle:

$$\Omega_{g,n}^\ell(\mu_1, \dots, \mu_n) = \text{ch}_t(\mathbb{V}_g(\mu_1, \dots, \mu_n)) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

CohFT axiom (i) for Ω^ℓ is trivial. Axiom (ii) follows from the fusion rules [41]. Axiom (iii) for the unit $\mathbf{1}$ is the *propagation of vacua* [12, Proposition 2.4(i)].

¹⁸For a vector bundle \mathbb{V} with Chern roots r_1, \dots, r_k ,

$$\text{ch}_t(\mathbb{V}) = \sum_{j=1}^k e^{tr_j}.$$

The parameter t may be treated either as a formal variable, in which case the CohFT is defined over the ring $\mathbb{Q}[[t]]$ instead of \mathbb{Q} , or as a rational number $t \in \mathbb{Q}$.

3.2 Genus 0 and the topological part

Since the variable t carries the degree grading, the topological part ω^ℓ of Ω^ℓ is obtained by setting $t = 0$,

$$\omega_{g,n}^\ell = \Omega_{g,n}^\ell \Big|_{t=0}.$$

The result is the just the rank of the Verlinde bundle,

$$\omega_{g,n}^\ell(\mu_1, \dots, \mu_n) = \text{rk } \mathbb{V}_g(\mu_1, \dots, \mu_n) = d_g(\mu_1, \dots, \mu_n).$$

With the quantum product obtained¹⁹ from ω^ℓ , $(V_\ell, \bullet, \mathbf{1})$ is the *fusion algebra*.

Since the fusion algebra is well-known²⁰ to be semisimple, the CohFT with unit Ω^ℓ is also semisimple. The subject has a history starting in the mid 80s with the discovery and in 90s with several proofs of the *Verlinde formula* for the rank $d_g(\mu_1, \dots, \mu_n)$, see [2] for an overview.

Hence, steps (i) and (ii) of the computational strategy of Section 0.5 for Ω^ℓ are complete (and have been for many years). Step (iii) is the jump to moduli.

3.3 Path to the R -matrix

The shifted r -spin CohFT \widehat{W}^r has an Euler field obtained from the pure dimensionality of Witten's r -spin class which was used to find the unique R -matrix in Section 2. The CohFT Ω^ℓ is not of pure dimension and has no Euler field. A different path to the R -matrix is required here.

The restriction of the tensor $\Omega_{g,n}^\ell$ to the open set of nonsingular curves

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$$

forgets a lot of the data of the CohFT. However, by [24, Lemma 2.2], the restriction is enough to uniquely determine the R -matrix of Ω^ℓ . Fortunately, the restriction is calculable in closed form:

- the first Chern class of the Verlinde bundle over $\mathcal{M}_{g,n}$ is found in [40],
- the existence of a projectively flat connection²¹ [41] on the Verlinde bundle over $\mathcal{M}_{g,n}$ then determines the full Chern character over $\mathcal{M}_{g,n}$.

As should be expected, the computation of Ω^ℓ relies significantly upon the past study of the Verlinde bundles.

3.4 The R -matrix

For a simple Lie algebra \mathfrak{g} and a level ℓ , the conformal anomaly is

$$c = c(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{\hbar + \ell},$$

¹⁹Since the quantum product depends only upon the tensors of genus 0 with 3 markings, the quantum products of Ω^ℓ and ω^ℓ are equal.

²⁰See, for example, [2, Proposition 6.1].

²¹Often called the Hitchin connection.

where \check{h} is the dual Coxeter number. For each representation with highest weight μ of level ℓ , define

$$w(\mu) = \frac{(\mu, \mu + 2\rho)}{2(\check{h} + \ell)}.$$

Here, ρ is half of the sum of the positive roots, and the Cartan-Killing form (\cdot, \cdot) is normalized so that the longest root θ satisfies

$$(\theta, \theta) = 2.$$

Example 21 For $\mathfrak{g} = \mathfrak{sl}(r, \mathbb{C})$, the highest weight of a representation of level ℓ is given by an r -tuple of integers

$$\mu = (\mu^1, \dots, \mu^r), \quad \ell \geq \mu^1 \geq \dots \geq \mu^r \geq 0,$$

defined up to shifting the vector components by the same integer. Furthermore, we have

$$c(\mathfrak{g}, \ell) = \frac{\ell(r^2 - 1)}{\ell + r},$$

$$w(\mu) = \frac{1}{2(\ell + r)} \left(\sum_{i=1}^r (\mu^i)^2 - \frac{1}{r} \left(\sum_{i=1}^r \mu^i \right)^2 + \sum_{i=1}^r (r - 2i + 1) \mu^i \right).$$

Via the path to the R -matrix discussed in Section 3.3, a simple closed form for the R -matrix of Ω^ℓ is found in [24] using the constants $c(\mathfrak{g}, \ell)$ and $w(\mu)$ from representation theory.

Theorem 22 The CohFT Ω^ℓ is reconstructed from the topological part ω^ℓ by the diagonal R -matrix

$$R(z)_\mu^\mu = \exp \left(tz \cdot \left(-w(\mu) + \frac{c(\mathfrak{g}, \ell)}{24} \right) \right).$$

For the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ at level $\ell = 1$, there are only two representations $\{\emptyset, \square\}$ to consider²²,

$$c(\mathfrak{sl}_2, 1) = 1, \quad w(\emptyset) = 0, \quad w(\square) = \frac{1}{4}.$$

As an example of the reconstruction result of Theorem 22,

$$\Omega^\ell = R \cdot \omega^\ell,$$

the total Chern character $\text{ch } \mathbb{V}_g(\square, \dots, \square)$ at $t = 1$ is

$$\exp \left(-\frac{\lambda_1}{2} \right) \cdot \sum_{\Gamma \in \mathbf{G}_{g,n}^{\text{even}}} \frac{2^{g-h^1(\Gamma)}}{|\text{Aut}(\Gamma)|} \cdot {}_{t\Gamma^*} \left(\prod_{e \in E} \frac{1 - \exp \left(-\frac{1}{4}(\psi'_e + \psi''_e) \right)}{\psi'_e + \psi''_e} \cdot \prod_{l \in L} e^{-\psi_l/4} \right)$$

in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. A few remarks about the above formula are required:

²²Here, \emptyset is the trivial representation (corresponding to $\mathbf{1}$) and \square is the standard representation.

- The classes λ_1 and ψ are the first Chern classes of the Hodge bundle and the cotangent line bundle respectively.
- The sum is over the set of *even* stable graphs,

$$\mathbf{G}_{g,n}^{\text{even}} \subset \mathbf{G}_{g,n},$$

defined by requiring the valence $n(v)$ to be even for every vertex v of the graph.

- The Verlinde rank $d_{\mathbf{g}(v)}(\square, \dots, \square)$ with $n(v)$ insertions equals 2^g in the even case, see [2]. The product of $2^{\mathbf{g}(v)}$ over the vertices of Γ yields $2^{g-h^1(\Gamma)}$, where h^1 denotes the first Betti number.

3.5 Questions

A different approach to the calculation of the Chern character of the Verlinde bundle in the \mathfrak{sl}_2 case (for every level) was pursued in [11] using the geometry introduced by Thaddeus [39] to prove the Verlinde formula. The outcome of [11] is a more difficult calculation (with a much more complex answer), but with one advantage: the projective flatness of the Hitchin connection over $\mathcal{M}_{g,n}$ is *not* used. When the flatness is introduced, the method of [11] yields tautological relations. Unfortunately, no such relations are obtained by the above R -matrix calculation of Ω^ℓ since the projective flatness is an input.

Question 23 *Is there an alternative computation of Ω^ℓ which does not use the projective flatness of the Hitchin connection and which systematically produces tautological relations in $RH^*(\mathcal{M}_{g,n}, \mathbb{Q})$?*

Of course, if the answer to Question 23 is yes, then the next question is whether *all* tautological relations are produced.²³

4 Gromov-Witten theory of $\text{Hilb}^m(\mathbb{C}^2)$

4.1 T-equivariant cohomology of $\text{Hilb}^m(\mathbb{C}^2)$

The *Hilbert scheme* $\text{Hilb}^m(\mathbb{C}^2)$ of m points in the plane \mathbb{C}^2 parameterizes ideals $\mathcal{I} \subset \mathbb{C}[x, y]$ of colength m ,

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/\mathcal{I} = m.$$

The Hilbert scheme $\text{Hilb}^m(\mathbb{C}^2)$ is a nonsingular, irreducible, quasi-projective variety of dimension $2m$, see [28] for an introduction. An open dense set of $\text{Hilb}^m(\mathbb{C}^2)$ parameterizes ideals associated to configurations of m distinct points.

The symmetries of \mathbb{C}^2 lift to the Hilbert scheme. The algebraic torus

$$\mathbf{T} = (\mathbb{C}^*)^2$$

²³I first heard an early version of this question from R. Bott at Harvard in the 90s.

acts diagonally on \mathbb{C}^2 by scaling coordinates,

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y).$$

We review the Fock space description of the \mathbb{T} -equivariant cohomology of the Hilbert scheme of points of \mathbb{C}^2 following the notation of [29, Section 2.1].

By definition, the *Fock space* \mathcal{F} is freely generated over \mathbb{Q} by commuting creation operators α_{-k} , $k \in \mathbb{Z}_{>0}$, acting on the vacuum vector v_\emptyset . The annihilation operators α_k , $k \in \mathbb{Z}_{>0}$, kill the vacuum

$$\alpha_k \cdot v_\emptyset = 0, \quad k > 0,$$

and satisfy the commutation relations $[\alpha_k, \alpha_l] = k \delta_{k+l}$.

A natural basis of \mathcal{F} is given by the vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod_i \alpha_{-\mu_i} v_\emptyset \quad (7)$$

indexed by partitions μ . Here, $\mathfrak{z}(\mu) = |\text{Aut}(\mu)| \prod_i \mu_i$ is the usual normalization factor. Let the length $\ell(\mu)$ denote the number of parts of the partition μ .

The *Nakajima basis* defines a canonical isomorphism,

$$\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2] \cong \bigoplus_{n \geq 0} H_{\mathbb{T}}^*(\text{Hilb}^n(\mathbb{C}^2), \mathbb{Q}). \quad (8)$$

The Nakajima basis element corresponding to $|\mu\rangle$ is

$$\frac{1}{\prod_i \mu_i} [\mathcal{V}_\mu]$$

where $[\mathcal{V}_\mu]$ is (the cohomological dual of) the class of the subvariety of $\text{Hilb}^{|\mu|}(\mathbb{C}^2)$ with generic element given by a union of schemes of lengths

$$\mu_1, \dots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points²⁴ of \mathbb{C}^2 . The vacuum vector v_\emptyset corresponds to the unit in

$$1 \in H_{\mathbb{T}}^*(\text{Hilb}^0(\mathbb{C}^2), \mathbb{Q}).$$

The variables t_1 and t_2 are the equivariant parameters corresponding to the weights of the \mathbb{T} -action on the tangent space $\text{Tan}_0(\mathbb{C}^2)$ at the origin of \mathbb{C}^2 .

The subspace $\mathcal{F}_m \subset \mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2]$ corresponding to $H_{\mathbb{T}}^*(\text{Hilb}^m(\mathbb{C}^2), \mathbb{Q})$ is spanned by the vectors (7) with $|\mu| = m$. The subspace can also be described as the m -eigenspace of the *energy operator*:

$$|\cdot| = \sum_{k > 0} \alpha_{-k} \alpha_k.$$

²⁴The points and parts of μ are considered here to be unordered.

The vector $|1^n\rangle$ corresponds to the unit

$$\mathbf{1} \in H_{\mathbb{T}}^*(\mathrm{Hilb}^m(\mathbb{C}^2), \mathbb{Q}).$$

The standard inner product on the \mathbb{T} -equivariant cohomology of $\mathrm{Hilb}^m(\mathbb{C}^2)$ induces the following *nonstandard* inner product on Fock space after an extension of scalars:

$$\langle \mu | \nu \rangle = \frac{(-1)^{|\mu| - \ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}. \quad (9)$$

4.2 Gromov-Witten CohFT of $\mathrm{Hilb}^m(\mathbb{C}^2)$

Let $m > 0$ be a colength. Let $(V_m, \eta, \mathbf{1})$ be the following triple:

- V_m is the free $\mathbb{Q}(t_1, t_2)[[q]]$ -module $\mathcal{F}_m \otimes_{\mathbb{Q}[t_1, t_2]} \mathbb{Q}(t_1, t_2)[[q]]$,
- η is the non-degenerate symmetric 2-form (9),
- $\mathbf{1}$ is the basis element $|1^n\rangle$.

Since $\mathrm{Hilb}^m(\mathbb{C}^2)$ is *not* proper, the Gromov-Witten theory is only defined after localization by \mathbb{T} . The CohFT with unit

$$\Omega^{\mathrm{Hilb}^m(\mathbb{C}^2)} = (\Omega_{g,n}^{\mathrm{Hilb}^m(\mathbb{C}^2)})_{2g-2+n>0}$$

is defined via the localized \mathbb{T} -equivariant Gromov-Witten classes of $\mathrm{Hilb}^m(\mathbb{C}^2)$,

$$\Omega_{g,n}^{\mathrm{Hilb}^m(\mathbb{C}^2)} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}(t_1, t_2)[[q]]) \otimes (V_m^*)^n.$$

Here, q is the Novikov parameter. Curves of degree d are counted with weight q^d , where the curve degree is defined by the pairing with the divisor

$$D = -|2, 1^{m-2}\rangle, \quad d = \int_{\beta} D.$$

Formally, $\Omega^{\mathrm{Hilb}^m(\mathbb{C}^2)}$ is a CohFT *not* over the field \mathbb{Q} as in the r -spin and Verlinde cases, but over the ring $\mathbb{Q}(t_1, t_2)[[q]]$. To simplify notation, let

$$\Omega^m = \Omega^{\mathrm{Hilb}^m(\mathbb{C}^2)}.$$

4.3 Genus 0

Since the \mathbb{T} -action on $\mathrm{Hilb}^m(\mathbb{C}^2)$ has finitely many \mathbb{T} -fixed points, the localized \mathbb{T} -equivariant cohomology

$$H_{\mathbb{T}}^*(\mathrm{Hilb}^m(\mathbb{C}^2), \mathbb{Q}) \otimes_{\mathbb{Q}[t_1, t_2]} \mathbb{Q}(t_1, t_2)$$

is semisimple. At $q = 0$, the quantum cohomology ring,

$$(V_m, \bullet, \mathbf{1}), \quad (10)$$

defined by Ω^m specializes to the localized \mathbb{T} -equivariant cohomology of $\mathbf{Hilb}^m(\mathbb{C}^2)$. Hence, the quantum cohomology (10) is semisimple over the ring $\mathbb{Q}(t_1, t_2)[[q]]$, see [22].

Let M_D denote the operator of \mathbb{T} -equivariant quantum multiplication by the divisor D . A central result of [29] is the following explicit formula for M_D as an operator on Fock space:

$$M_D(q, t_1, t_2) = (t_1 + t_2) \sum_{k>0} \frac{k(-q)^k + 1}{2(-q)^k - 1} \alpha_{-k} \alpha_k - \frac{t_1 + t_2}{2} \frac{(-q) + 1}{(-q) - 1} | \cdot | \\ + \frac{1}{2} \sum_{k, l > 0} \left[t_1 t_2 \alpha_{k+l} \alpha_{-k} \alpha_{-l} - \alpha_{-k-l} \alpha_k \alpha_l \right].$$

The q -dependence of M_D occurs only in the first two terms (which act diagonally in the basis (7)).

Let μ^1 and μ^2 be partitions of m . The \mathbb{T} -equivariant Gromov-Witten invariants of $\mathbf{Hilb}^m(\mathbb{C}^2)$ in genus 0 with 3 cohomology insertions given (in the Nakajima basis) by μ^1 , D , and μ^2 are determined by M_D :

$$\sum_{d=0}^{\infty} \Omega_{0,3,d}^m(\mu^1, D, \mu^2) q^d = \langle \mu^1 | M_D | \mu^2 \rangle.$$

The following result is proven in [29].

Theorem 24 *The restriction of Ω^m to genus 0 is uniquely and effectively determined from the calculation of M_D .*

While Theorem 24 in principle completes the genus 0 study of Ω^m , the result is not as strong as the genus 0 determinations in the r -spin and Verlinde cases. The proof of Theorem 24 provides an effective linear algebraic procedure, but not a formula, for calculating the genus 0 part of Ω^m from M_D .

4.4 The topological part

Let ω^m be the topological part of the CohFT with unit Ω^m . A closed formula for ω^m can not be expected since closed formulas are already missing in the genus 0 study.

The CohFT with unit ω^m has been considered earlier from another perspective. Using fundamental correspondences [25], ω^m is equivalent to the *local GW/DT theory* of 3-folds of the form

$$\mathbb{C}^2 \times C, \tag{11}$$

where C is a curve or arbitrary genus. Such local theories have been studied extensively [6] in the investigation of the GW/DT theory of 3-folds²⁵.

²⁵A natural generalization of the geometry (11) is to consider the 3-fold total space,

$$L_1 \oplus L_2 \rightarrow C,$$

4.5 The R -matrix

Since Ω^m is not of pure dimension (and does not carry an Euler field), the R -matrix is *not* determined by the T -equivariant genus 0 theory alone. As in the Verlinde case, a different method is required. Fortunately, together with the divisor equation, an evaluation of the T -equivariant higher genus theory in degree 0 is enough to uniquely determine the R -matrix.

Let $\text{Part}(m)$ be the set of partitions of m corresponding to the T -fixed points of $\text{Hilb}^m(\mathbb{C}^2)$. For each $\eta \in \text{Part}(m)$, let $\text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2))$ be the T -representation on the tangent space at the T -fixed point corresponding to η . As before, let

$$\mathbb{E} \rightarrow \overline{\mathcal{M}}_{g,n}$$

be the Hodge bundle. The follow result is proven in [34].

Theorem 25 *The R -matrix of Ω^m is uniquely determined by the divisor equation and the degree 0 invariants*

$$\begin{aligned} \langle \mu \rangle_{1,0}^{\text{Hilb}^m(\mathbb{C}^2)} &= \sum_{\eta \in \text{Part}(m)} \mu|_\eta \int_{\overline{\mathcal{M}}_{1,1}} \frac{e(\mathbb{E}^* \otimes \text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2)))}{e(\text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2)))}, \\ \langle \rangle_{g \geq 2,0}^{\text{Hilb}^m(\mathbb{C}^2)} &= \sum_{\eta \in \text{Part}(m)} \int_{\overline{\mathcal{M}}_g} \frac{e(\mathbb{E}^* \otimes \text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2)))}{e(\text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2)))}. \end{aligned}$$

While Theorem 25 is weaker than the explicit R -matrix solutions in the r -spin and Verlinde cases, the result nevertheless has several consequences. The first is a rationality result [34].

Theorem 26 *For all genera $g \geq 0$ and $\mu^1, \dots, \mu^n \in \text{Part}(m)$, the series²⁶*

$$\int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}^m(\mu^1, \dots, \mu^n) \in \mathbb{Q}(t_1, t_2)[[q]]$$

is the Taylor expansion in q of a rational function in $\mathbb{Q}(t_1, t_1, q)$.

The statement of Theorem 26 can be strengthened (with an R -matrix argument using Theorem 25) to prove that the CohFT with unit Ω^m can be defined over the field $\mathbb{Q}(t_1, t_2, q)$.

4.6 Crepant resolution

The Hilbert scheme of points of \mathbb{C}^2 is well-known to be a crepant resolution of the symmetric product,

$$\epsilon : \text{Hilb}^m(\mathbb{C}^2) \rightarrow \text{Sym}^m(\mathbb{C}^2) = (\mathbb{C}^2)^m / S_m.$$

of a sum of line bundles $L_1, L_2 \rightarrow C$. For particular pairs L_1 and L_2 , simple closed form solutions were found [6] and have later played a role in the study of the structure of the Gromov-Witten theory of Calabi-Yau 3-folds by Ionel and Parker [19].

²⁶As always, g and n are required to be in the stable range $2g - 2 + n > 0$.

Viewed as an *orbifold*, the symmetric product $\mathrm{Sym}^m(\mathbb{C}^2)$ has a \mathbb{T} -equivariant Gromov-Witten theory with insertions indexed by partitions of m and an associated CohFT with unit $\Omega^{\mathrm{Sym}^m(\mathbb{C}^2)}$ determined by the Gromov-Witten classes. The CohFT $\Omega^{\mathrm{Sym}^m(\mathbb{C}^2)}$ is defined over the ring $\mathbb{Q}(t_1, t_2)[[u]]$, where u is variable associated to the free ramification points, see [34] for a detailed treatment.

In genus 0, the equivalence of the \mathbb{T} -equivariant Gromov-Witten theories of $\mathrm{Hilb}^m(\mathbb{C}^2)$ and the orbifold $\mathrm{Sym}^m(\mathbb{C}^2)$ was proven²⁷ in [5]. Another consequence of the R -matrix study of $\Omega^{\mathrm{Hilb}^m(\mathbb{C}^2)}$ is the proof in [34] of the *crepant resolution conjecture* here.

Theorem 27 *For all genera $g \geq 0$ and $\mu^1, \dots, \mu^n \in \mathrm{Part}(m)$, we have*

$$\Omega_{g,n}^{\mathrm{Hilb}^m(\mathbb{C}^2)}(\mu^1, \dots, \mu^n) = (-i)^{\sum_{i=1}^n \ell(\mu^i) - |\mu^i|} \Omega_{g,n}^{\mathrm{Sym}^m(\mathbb{C}^2)}(\mu^1, \dots, \mu^n)$$

after the variable change $-q = e^{iu}$.

The variable change of Theorem 27 is well-defined by the rationality of Theorem 26. The analysis [30] of the quantum differential equation of $\mathrm{Hilb}^m(\mathbb{C}^2)$ plays an important role in the proof. Theorem 27 is closely related to the GW/DT correspondence for local curves (11) in families, see [34].

4.7 Questions

The most basic open question is to find an expression for the R -matrix of Ω^m in terms of natural operators on Fock space.

Question 28 *Is there a representation theoretic formula for the R -matrix of the CohFT with unit Ω^m ?*

The difficulty in attacking Question 28 starts with the lack of higher genus calculations in closed form. The first nontrivial example [34] occurs in genus 1 for the Hilbert scheme of 2 points:

$$\int_{\overline{\mathcal{M}}_{1,1}} \Omega_{1,1}^{\mathrm{Hilb}^2(\mathbb{C}^2)}((2)) = -\frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \frac{1+q}{1-q}. \quad (12)$$

While there are numerous calculations to do, the higher m analogue of (12) surely has a simple answer.

Question 29 *Calculate the series*

$$\int_{\overline{\mathcal{M}}_{1,1}} \Omega_{1,1}^{\mathrm{Hilb}^m(\mathbb{C}^2)}((2, 1^{n-2})) \in \mathbb{Q}(t_1, t_2, q),$$

in closed form for all m .

²⁷The prefactor $(-i)^{\sum_{i=1}^n \ell(\mu^i) - |\mu^i|}$ was treated incorrectly in [5] because of an arithmetical error. The prefactor here is correct.

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