Cycles on moduli spaces of abelian varieties

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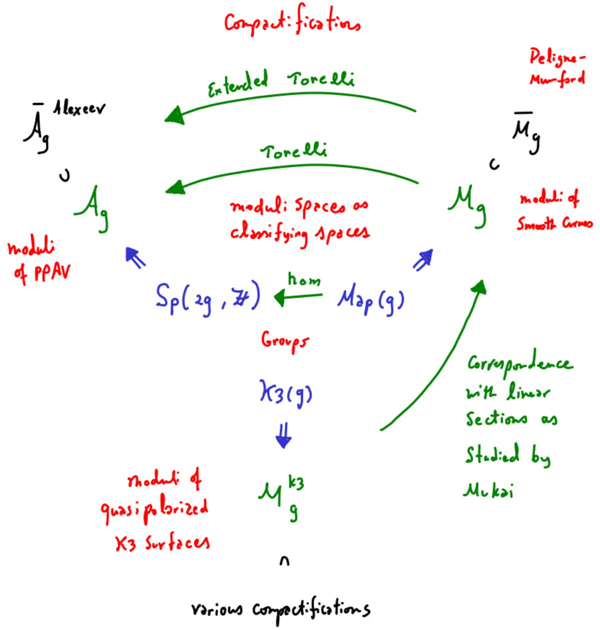

Compactifications
Deligne-Mumford
Extended Torelli
Alexeev
Torelli
moduli spaces as classifying spaces
moduli of smooth curves
moduli of PPAV
hom
Sp(2g, Z)
Map(g)
Groups
K3(g)
Correspondence with linear sections as studied by Mukai
moduli of quasipolarized K3 surfaces
various compactifications

§I. **Abelian varieties**

Abelian varieties with principal polarizations are of the form

$$X = \mathbb{C}^g/\Lambda\,,$$

where $\Lambda \subset \mathbb{C}^g$ is generated by the g basis vectors

$$(1,0,\ldots,0)\,,\ (0,1,0,\ldots,0)\,,\ldots\,,\ (0,\ldots,0,1)$$

together with the columns of a $g \times g$ symmetric matrix τ with positive definite imaginary part

$$\mathrm{Im}(\tau) > 0\,.$$

The upper half plane for τ in dimension 1 generalizes to the Siegel upper half space for τ in higher dimensions:

$$\mathcal{H}_g = \{ \tau \in \mathrm{SymMat}_{g\times g}(\mathbb{C}) \,|\, \mathrm{Im}(\tau) > 0 \}\,.$$

The moduli space of principally polarized abelian varieties

$$\mathcal{A}_g = \mathcal{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z})\,, \quad \dim_{\mathbb{C}} \mathcal{A}_g = \binom{g+1}{2}\,,$$

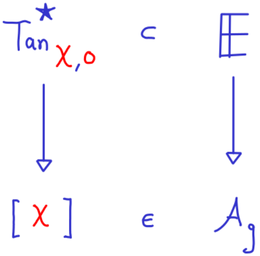
is a quotient of the Siegel upper half space by the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ by a sort of linear fractional transformation:

For $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\tau \in \mathcal{H}_g$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = \frac{A\tau + B}{C\tau + D} \in \mathcal{H}_g\,.$$

§II. **Tautological classes on $\mathcal{A}_g$**

The Hodge bundle $\mathbb{E}$ on $\mathcal{A}_g$ is a $\mathbb{C}$-vector bundle of rank g:



The Chern classes of $\mathbb{E}$ are

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\mathcal{A}_g, \mathbb{Q}) .$$

A result parallel to the Madsen-Weiss Theorem for the moduli space of curves holds:

Theorem (Borel 1974):

$$\lim_{g\to\infty} H^*(\mathcal{A}_g, \mathbb{Q}) = \mathbb{Q}[\lambda_1, \lambda_3, \lambda_5, \ldots] .$$

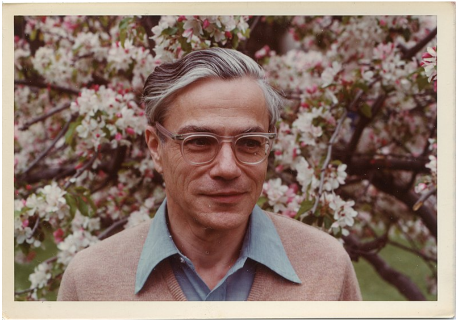
Question: Why are no λ classes of even degree needed?

Answer: Because of Mumford's relation

$$c(\mathbb{E} \oplus \mathbb{E}^*) = 1 \in H^*(\mathcal{A}_g, \mathbb{Q})$$

which expands fully as

$$(1 + \lambda_1 + \lambda_2 + \ldots + \lambda_g) \cdot (1 - \lambda_1 + \lambda_2 + \ldots + (-1)^g \lambda_g) = 1 .$$



For fixed dimension g, we take Borel's result as motivation to restrict our attention to the tautological algebra

$$R^*(\mathcal{A}_g) \subset \mathrm{CH}^*(\mathcal{A}_g, \mathbb{Q})$$

defined (by van der Geer (1996)) to be generated by the λ classes.

Question: What is the structure of the algebra $R^*(\mathcal{A}_g)$?

Question: What is the ideal of relations

$$0 \to \mathcal{J}_g \to \mathbb{Q}[\lambda_1, \lambda_2, \ldots, \lambda_g] \to R^*(\mathcal{A}_g) \to 0 \ ?$$

Theorem (van der Geer 1996):

$$R^*(\mathcal{A}_g) = \frac{\mathbb{Q}[\lambda_1, \lambda_2, \ldots, \lambda_g]}{\langle\, \lambda_g = 0\, ,\, c(\mathbb{E} \oplus \mathbb{E}^*) = 1\, \rangle} \ .$$

The beautiful proof depends upon the algebra satisfying Poincaré duality with socle in degree $\binom{g}{2}$.

§III. **Cycle questions**

Question: Are there any classes of algebraic cycles in $\mathrm{CH}^*(\mathcal{A}_g)$ which are not tautological?

- Are the classes of products

$$\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_{g_1+g_2}$$

tautological in $\mathrm{CH}^*(\mathcal{A}_{g_1+g_2})$?

The product loci are the simplest Nother-Lefschetz loci: loci of abelian varieties with extra line bundles.

- Are the classes of more general Noether-Lefschetz loci tautological?

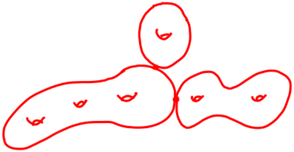
The moduli of curves and abelian varieties are related via the Torelli map:

$$\mathrm{Tor} : \mathcal{M}_g^c \to \mathcal{A}_g$$

defined by the Jacobian of stable curves of compact type,

$$\mathrm{Tor}([C]) = [\mathrm{Jac}(C)]\,.$$

A stable curve $[C] \in \mathcal{M}_g^c$ of compact type is a connected nodal curve with only separating nodes:



The Jacobian of multidegree 0 line bundles on C is a principally polarized abelian variety of dimension g, $[\mathrm{Jac}(C)] \in \mathcal{A}_g$.

For a nonsingular curve C of genus g,

$$\mathrm{Jac}(C) = H^0(C, \Omega^1_C)^* / H_1(C, \mathbb{Z}) .$$

Question: Is $\mathrm{Tor}_*[\mathcal{M}_g^c] \in \mathrm{CH}^*(\mathcal{A}_g)$ tautological?

Question: Does the pull-back

$$\mathrm{Tor}^* : \mathrm{CH}^*(\mathcal{A}_g) \to \mathrm{CH}^*(\mathcal{M}_g^c)$$

yield information about tautological cycles?

To say more, we return to cycles on the moduli space of curves.

§IV. Pixton's **relations on** $\overline{\mathcal{M}}_{g,n}$

We define tautological classes $\mathcal{R}^d_{g,A}$ associated to the data:

- $g, n \in \mathbb{Z}_{\geq 0}$ satisfying $2g - 2 + n > 0$,
- $A = (a_1, \ldots, a_n), \quad a_i \in \{0, 1\}$,
- $d \in \mathbb{Z}_{\geq 0}$ satisfying $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$.

Pixton's relations then take the form

$$\mathcal{R}^d_{g,A} = 0 \quad \in \mathrm{CH}^d(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

The formula for $\mathcal{R}^d_{g,A}$ requires more detail than can be given here, but the shape can be easily shown.

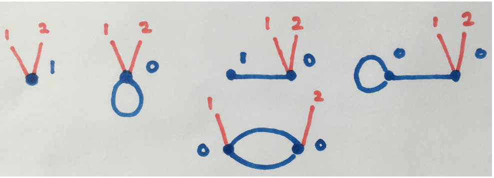
We start with the following two series:

$$B_0(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}(-T)^i = 1 - 60T + 27720T^2 \cdots ,$$

$$B_1(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}\frac{1+6i}{1-6i}(-T)^i = 1 + 84T - 32760T^2 \cdots .$$

- These series control the original set of Faber-Zagier relations.
- These series control Pixton's relations.

Let $\mathsf{G}_{g,n}$ be the finite set of stable graphs of genus g with n legs. For example, $\mathsf{G}_{1,2}$ has 5 elements:



The formula for $\mathcal{R}^d_{g,A}$ is a sum over stable graphs,

$$\mathcal{R}^d_{g,A} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e\right]_d$$

where $\overline{\mathcal{M}}_\Gamma$ is the moduli space associated to Γ,

$$\mathcal{K}_v\,,\ \Psi_\ell\,,\ \Delta_e \in H^*(\overline{\mathcal{M}}_\Gamma)\,,$$

$[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e]$ is the push-forward to $\overline{\mathcal{M}}_{g,n}$ of

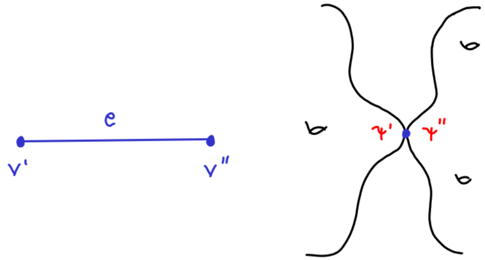
$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{v \in \mathrm{Vertex}(\Gamma)} \mathcal{K}_v \prod_{\ell \in \mathrm{Leg}(\Gamma)} \Psi_\ell \prod_{e \in \mathrm{Edge}(\Gamma)} \Delta_e \cap [\overline{\mathcal{M}}_\Gamma]$$

and $[...]_d$ extracts the part in $\mathrm{CH}^d(\overline{\mathcal{M}}_{g,n})$.

$$\mathcal{R}^d_{g,A} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e \right]_d$$

- Vertex $\mathcal{K}_v$, leg Ψ_v, and edge Δ_e factors have explicit formulas in terms of the κ and ψ classes and the series B_0 and B_1.

- Edge factor is the most interesting:



For $e \in \mathrm{Edge}(\Gamma)$, the formula for the edge factor is:

$$\Delta_e = \frac{2 - B_0(\psi')B_1(\psi'') - B_1(\psi')B_0(\psi'')}{\psi' + \psi''}$$

$$= -24 + 5040(\psi' + \psi'') + \dots .$$

The numerator of Δ_e is divisible by the denominator by the identity

$$B_0(T)B_1(-T) + B_1(T)B_0(-T) = 2 .$$

Warning: A parity factor has been omitted for simplicity.

Theorem (P-Pixton-Zvonkine 2013): For $2g - 2 + n > 0$, $a_i \in \{0, 1\}$, and $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$, the Pixton relation holds

$$\mathcal{R}^d_{g,A} = 0 \ \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

• By Janda's results, Pixton's relations hold in the Chow theory of algebraic cycles:

$$\mathcal{R}^d_{g,A} = 0 \ \in \mathrm{CH}^d(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

• Mumford, in his foundational paper (1983)

Towards an enumerative geometry of the moduli space of curves,

opened the study of the algebra of tautological classes.

Pixton's relations provide the first proposal for their calculus

parallel to the Schubert calculus.

Conjecture (Pixton 2012): These relations are the complete set of relations among tautological classes on $\overline{\mathcal{M}}_{g,n}$.

Pixton's relations can be restricted to the moduli space $\mathcal{M}_g^c$ of curves of compact type (by setting to 0 all terms associated to graphs Γ with non-separating edges).

Conjecture (Pixton 2012): Restriction to $\mathcal{M}_{g,n}^c$ yields a complete set of relations among tautological classes on $\mathcal{M}_{g,n}^c$.

§V. **Pull-back via Torelli**

The Hodge bundle $\mathbb{E}$ on $\mathcal{M}_g^c$ is defined by

$$\begin{array}{ccc} H^0(C, \omega_C) & \subset & \mathbb{E} \\ \downarrow & & \downarrow \\ [C] & \in & \mathcal{M}_g^c \end{array}$$

The Torelli map $\mathrm{Tor} : \mathcal{M}_g^c \to \mathcal{A}_g$ respects the Hodge bundles

$$\mathrm{Tor}^*(\mathbb{E}) = \mathbb{E}\,.$$

The Chern classes of $\mathbb{E} \to \mathcal{M}_g^c$ lie in the tautological algebra by Mumford's calculations:

$$\lambda_i = c_i(\mathbb{E}) \in R^i(\mathcal{M}_g^c)\,.$$

Let $\Lambda^*(\mathcal{M}_g^c) \subset R^*(\mathcal{M}_g^c)$ be generated by $\lambda_1, \ldots, \lambda_g$, then

$$\mathrm{Tor}^* : R^*(\mathcal{A}_g) \to \Lambda^*(\mathcal{M}_g^c)\,.$$

In genus $g = 5$, we have

$$\dim_{\mathbb{Q}} \Lambda^*(\mathcal{M}_5^c) = 11\,, \qquad \dim_{\mathbb{Q}} R^*(\mathcal{M}_5^c) = 66\,,$$

so $\Lambda^*(\mathcal{M}_g^c)$ is a small subspace of $R^*(\mathcal{M}_g^c)$.

We return to the simplest question about cycles on $\mathcal{A}_g$:

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \overset{?}{\in} R^{g-1}(\mathcal{A}_g)\,.$$

The idea is to compute the Torelli pull-back and ask

$$\mathrm{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \overset{?}{\in} \Lambda^{g-1}(\mathcal{M}_g^c)\,.$$

A refined statement is possible:

Proposition (Canning-Oprea-P 2022): If $[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in R^{g-1}(\mathcal{A}_g)$, then we must have

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \frac{(-1)^g g}{6B_{2g}}\,\lambda_{g-1} \;\in\; R^{g-1}(\mathcal{A}_g).$$

Motivated by the Proposition, define

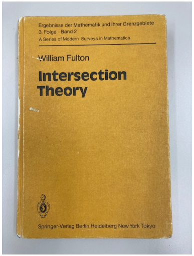
$$\Delta_g = [\mathcal{A}_1 \times \mathcal{A}_{g-1}] - \frac{(-1)^g g}{6B_{2g}} \lambda_{g-1} \in \mathsf{CH}^{g-1}(\mathcal{A}_g)\,.$$

The outcome is an obstruction:

$$\boxed{[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in R^{g-1}(\mathcal{A}_g) \ \Rightarrow\ \mathsf{Tor}^*\Delta_g = 0 \in \mathsf{CH}^{g-1}(\mathcal{M}_g^c)}$$

Can we calculate $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$?

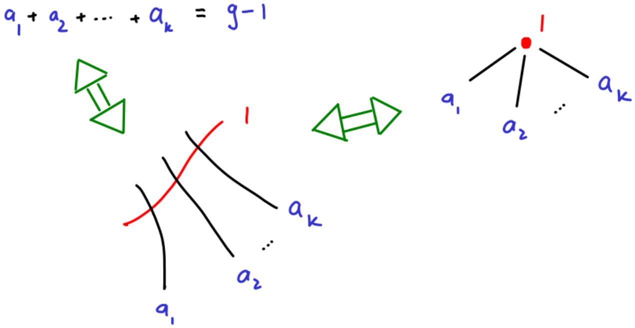
Yes, using Fulton's excess intersection theory.



We must study the subscheme

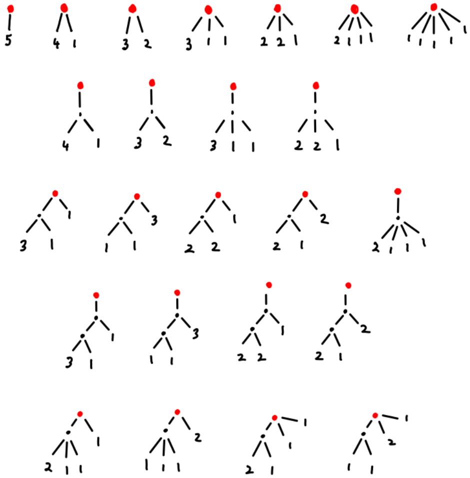
$$\text{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \subset \mathcal{M}_g^c\,.$$

• Irreducible components of $\text{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ are in bijective correspondence with $\text{Part}(g-1)$:



• Irreducible components are usually excess dimensional and intersect in a complicated configuration of strata in $\mathcal{M}_g^c$.

- In genus $g = 6$, a complete list of strata (indexing intersections of irreducible components) is:



Excess intersection theory $\Rightarrow$

$$\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \sum_{\text{All strata } \Gamma} \mathsf{Cont}(\Gamma)\,.$$

- Sum is over all strata of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$.
- $\mathsf{Cont}(\Gamma)$ is a tautological class on $\overline{\mathcal{M}}_\Gamma$.

Example: Cont() = $-3\lambda_2 + 4\lambda_1\psi_1 - 5\psi_1^2$

↑ all on the $M^c_{4,1}$ factor

X 

Y 

Z  = X ∩ Y

$$6\,c_1(E)\,c_1(N_{Z,Y}) - 10\,c_1(N_{Z,Y})^2 + 4\,c_1(E)\,c_1(N_{Z,X}) - 10\,c_1(N_{Z,X})\,c_1(N_{Z,Y}) - 5\,c_1(N_{Z,X})^2 - 3\,c_2(E) + 5\,c_2(N_{Z,X})$$

N denotes normal bundle

E is the pull back of $N_{A_1 \times A_5,\, A_6}$

We are now in a position to check

$$\mathrm{Tor}^*\Delta_g \stackrel{?}{=} 0 \in R^{g-1}(\mathcal{M}_g^c)$$

using Admcycles (a SAGE package which calculates in the tautological algebra of the moduli of curves using Pixton's relations).

Admcycles calculations show

$$\mathrm{Tor}^*\Delta_g = 0 \quad \text{for } g = 1, 2, 3, 4, 5\,.$$

We know Pixton's relations are complete for $\mathcal{M}_{g\leq 5}^c$.

The most interesting case is $g = 6$.

§VI. **Genus** $g = 6$

The first result provides full knowledge of $R^*(\mathcal{M}_6^c)$.

Theorem (Canning-Larson-Schmitt 2023): Pixton's relations are complete for $\mathcal{M}_6^c$.

- For all g, by Faber-P (2003),

$$R^{2g-3}(\mathcal{M}_g^c) \cong \mathbb{Q}\,, \quad R^{>2g-3}(\mathcal{M}_g^c) = 0\,.$$

- For Pixton's conjecture, non-vanishing must be proven after his relations are imposed. The ranks of the pairings

$$R^k(\mathcal{M}_6^c) \times R^{9-k}(\mathcal{M}_6^c) \to R^9(\mathcal{M}_6^c) \cong \mathbb{Q}$$

can be computed by Admcycles and show Pixton's relations are complete in all cases with the possible exception of $R^5(\mathcal{M}_6^c)$.

• Pixton predicts $\dim_{\mathbb{Q}} R^5(\mathcal{M}_6^c) = 72$, but the corresponding pairing rank has dimension 71.

• The proof is completed by establishing the exact sequence

$$R^4(\overline{\mathcal{M}}_{5,2}) \xrightarrow{\alpha} R^5(\overline{\mathcal{M}}_6) \longrightarrow R^5(\mathcal{M}_6^c) \longrightarrow 0$$

and computing with Admcycles:

$$\dim_{\mathbb{Q}} \mathrm{Im}(\alpha) = 916\,, \quad \dim_{\mathbb{Q}} R^5(\overline{\mathcal{M}}_6) = 988\,.$$

• The result is the first case where Pixton's conjecture is proven without relying only upon the non-vanishings obtained from the ranks of the pairings.

We can now use Admcycles to calculate $\mathrm{Tor}^*\Delta_6$:

Theorem (Canning-Oprea-P 2023): $\mathrm{Tor}^*\Delta_6 \neq 0 \in R^5(\mathcal{M}_6^c)$, so

$$[\mathcal{A}_1 \times \mathcal{A}_5] \notin R^5(\mathcal{A}_6)\,.$$

- The relevant pairing is

$$R^4(\mathcal{M}_6^c) \times R^5(\mathcal{M}_6^c) \to R^9(\mathcal{M}_6^c) \cong \mathbb{Q}$$

is of rank 71. By Canning-Larson-Schmitt,

$$\dim_{\mathbb{Q}} R^4(\mathcal{M}_6^c) = 71\,, \qquad \dim_{\mathbb{Q}} R^5(\mathcal{M}_6^c) = 72\,.$$

Hence, there is a 1 dimensional kernel of the paring in $R^5(\mathcal{M}_6^c)$.

- The calculation shows that $\mathrm{Tor}^*\Delta_6 \neq 0$ is the generator of the kernel of the pairing!

§VII. **Projection**

Tautological classes determine a $\mathbb{Q}$-linear subspace

$$R^*(\mathcal{A}_g) \subset \mathrm{CH}^*(\mathcal{A}_g)\,.$$

The cycle theory of $\mathcal{A}_g$ is special (compared to the other moduli spaces that we study).

Theorem (Canning-Molcho-Oprea-P 2024): There is a canonical projection,

$$\mathrm{taut} : \mathrm{CH}^*(\mathcal{A}_g) \to R^*(\mathcal{A}_g)\,,$$

$\mathrm{taut}|_{R^*(\mathcal{A}_g)} = \mathrm{Id}_{R^*(\mathcal{A}_g)}$.

• Projection is defined via an integration map (which requires a new vanishing result).

- Projection yields a canonical direct sum decomposition:

$$\mathrm{CH}^*(\mathcal{A}_g) \cong R^*(\mathcal{A}_g) \oplus NT^*(\mathcal{A}_g)\,,$$

where $NT^*(\mathcal{A}_g) \subset \mathrm{CH}^*(\mathcal{A}_g)$ is the $\mathbb{Q}$-linear subspace of purely non-tautological classes: classes with trivial projection.

- For any cycle class $\alpha \in \mathrm{CH}^*(\mathcal{A}_g)$, we can ask:

Question (i) What is $\mathrm{taut}(\alpha) \in R^*(\mathcal{A}_g)$?

Question (ii) Is $\alpha - \mathrm{taut}(\alpha) \neq 0$?

Consider the classes of products

$$\mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_\ell} \to \mathcal{A}_g \,.$$

The following result by Canning-Molcho-Oprea-P (2024) answers

Question (i) for all products:

Theorem 6. *For $g_1 + \ldots + g_\ell = g$, the tautological projection of the product locus $\mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_\ell}$ in $\mathcal{A}_g$ is given by a $(g-\ell)\times(g-\ell)$ determinant,*

$$\mathsf{taut}([\mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_\ell}]) = \frac{\gamma_{g_1}\cdots\gamma_{g_\ell}}{\gamma_g} \cdot \lambda_{g-1}\cdots\lambda_{g-\ell+1} \cdot \begin{vmatrix} \lambda_{\beta_1} & \lambda_{\beta_1+1} & \cdots & \lambda_{\beta_1+g^*-1} \\ \lambda_{\beta_2-1} & \lambda_{\beta_2} & \cdots & \lambda_{\beta_2+g^*-2} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{\beta_{g^*}-g^*+1} & \lambda_{\beta_{g^*}-g^*+2} & \cdots & \lambda_{\beta_{g^*}} \end{vmatrix},$$

for the partition

$$\beta = (\underbrace{g^* - g_1^*, \ldots, g^* - g_1^*}_{g_1^*}, \underbrace{g^* - g_1^* - g_2^*, \ldots, g^* - g_1^* - g_2^*}_{g_2^*}, \ldots, \underbrace{g^* - g_1^* - \ldots - g_\ell^*}_{g_\ell^*}),$$

where $g^* = g - \ell$ and $g_i^* = g_i - 1$.

The prefactors are defined by $\gamma_g = \prod_{i=1}^{g} \frac{|B_{2i}|}{4i}$.

Some examples:

$$\mathsf{taut}\left([\mathcal{A}_1 \times \mathcal{A}_{g-1}]\right) = \frac{g}{6|B_{2g}|}\lambda_{g-1}\,,$$

$$\mathsf{taut}\left([\mathcal{A}_2 \times \mathcal{A}_{g-2}]\right) = \frac{1}{360} \cdot \frac{g(g-1)}{|B_{2g}||B_{2g-2}|} \cdot \lambda_{g-1}\lambda_{g-3}\,,$$

$$\mathsf{taut}\left([\mathcal{A}_3 \times \mathcal{A}_{g-3}]\right) = \frac{1}{45360} \cdot \frac{g(g-1)(g-2)}{|B_{2g}||B_{2g-2}||B_{2g-4}|} \cdot \lambda_{g-1}(\lambda_{g-4}^2 - \lambda_{g-3}\lambda_{g-5})\,,$$

$$\mathsf{taut}\left([\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_{g-3}]\right) = \frac{1}{90} \cdot \frac{g(g-1)(g-2)}{|B_{2g}||B_{2g-2}||B_{2g-4}|} \cdot \lambda_{g-1}\lambda_{g-2}\lambda_{g-4}\,,$$

$$\mathsf{taut}\left(\left[\underbrace{\mathcal{A}_1 \times \ldots \times \mathcal{A}_1}_{k} \times \mathcal{A}_{g-k}\right]\right) = \left(\prod_{i=g-k+1}^{g} \frac{i}{6|B_{2i}|}\right) \lambda_{g-1} \cdots \lambda_{g-k}\,.$$

At the moment, the only product locus which we have proven to have a non-vanishing non-tautological part is

$$[\mathcal{A}_1 \times \mathcal{A}_5] \notin R^5(\mathcal{A}_6)\,.$$

But we expect most products to have interesting non-tautological parts.

The product loci are the simplest to consider, but there are many other Noether-Lefschetz loci with extra line bundles (and more general loci with extra algebraic Hodge classes).

The study of the projections of these is wide open (but see my Leiden lecture notes for conjectures related to some extension loci).






Acknowledgements

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