Curve counts on $K3$ surfaces and modular forms

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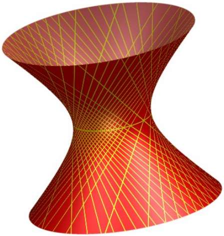
Quadric surface ($d = 2$)
ruled by lines:



If the four degree $d-1$ polynomials

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have no common solutions in $\mathbb{CP}^3$, then $F_d=0$ defines a nonsingular 2-dimensional variety

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Nonsingular cubic surface ($d = 3$) with 27 lines:

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Nonsingular hypersurfaces $S_4 \subset \mathbb{CP}^3$ of degree $d = 4$ are quartic $K3$ surfaces. For example, the Fermat quartic:

$$(x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0) \subset \mathbb{CP}^3 .$$

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The intersection pairing of S_4,

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$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

The intersection form is even.

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Kummer $K3$:

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We view a rational curve on S_4 as an algebraic map

$$\phi : \mathbb{CP}^1 \to \mathbb{CP}^3$$

defined by homogeneous polynomials $P_i \in \mathbb{C}[y_0, y_1]$ of degree e,

$$\mathbb{CP}^1 \ni [y_0, y_1] \overset{\phi}{\mapsto} [P_0(y_0, y_1), P_1(y_0, y_1), P_2(y_0, y_1), P_3(y_0, y_1)]\,,$$

which satisfies

$$F_4\big(P_0, P_1, P_2, P_3\big) = 0\ .$$

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Above calculation suggests S_4 contains no rational curves (number of conditions exceeds available dimensions by 1).

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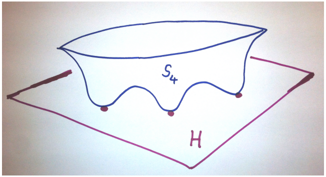

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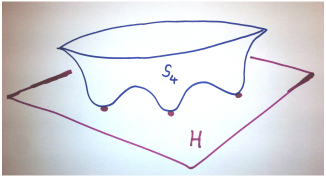

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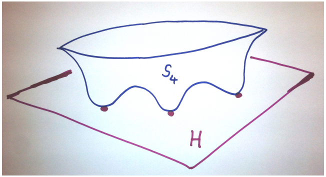

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The intersection $S_4 \cap H \subset H$ with the tri-tangent plane is a

quartic plane curve with 3 singularities,
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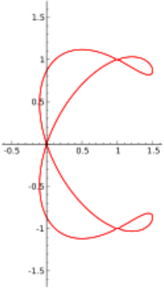

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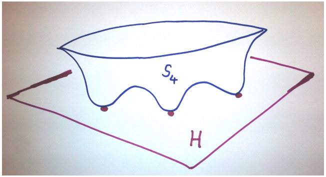

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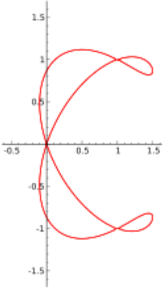

Perhaps S_4 does contain rational curves after all?

§III. Stable maps and the virtual fundamental class

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Let S be an algebraic $K3$ surface, and let

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be a nonzero effective curve class. The moduli space $\overline{M}_g(S,\beta)$ of genus g stable maps has expected dimension

$$\dim_{\mathbb{C}}^{\mathrm{vir}} \overline{M}_g(S,\beta) = \int_\beta c_1(S) + (\dim_{\mathbb{C}}(S) - 3)(1-g) = g - 1.$$

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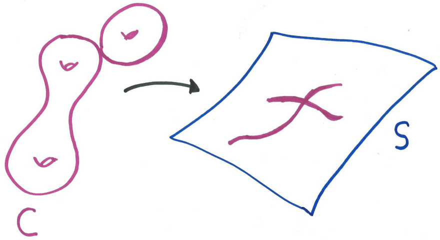

The obstruction space at the moduli point $[f : C \to S]$ is

$$\mathrm{Obs}_{[f]} = H^1(C, f^* T_S)$$

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However, there are curves on algebraic $K3$ surfaces.

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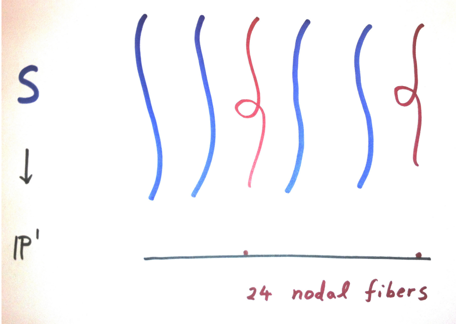
An elliptically fibered $K3$ surface has 24 nodal rational fibers.



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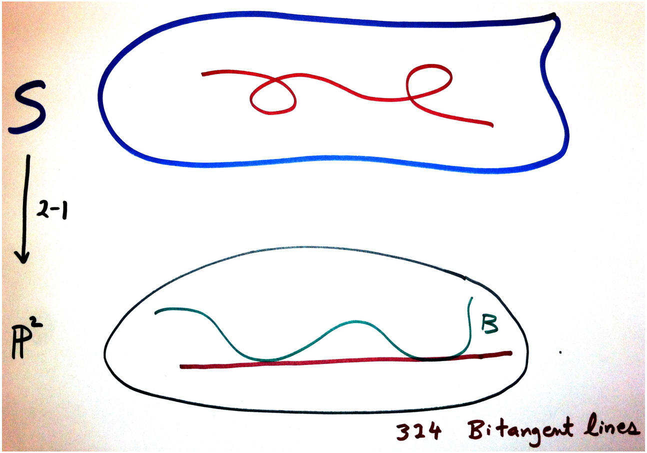

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The trivial piece of $\mathrm{Obs}_{[f]}$ can be removed. The result is a *reduced virtual class* invariant under deformations of S for which β remains in $\mathrm{Pic}(S)$,

$$\dim_{\mathbb{C}}^{\mathrm{red}} \overline{M}_g(S, \beta) = \dim_{\mathbb{C}}^{\mathrm{vir}} \overline{M}_g(S, \beta) + 1 = g \ .$$

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Define the reduced genus 0 counts of S in a primitive class $\beta \in \mathrm{Pic}(S)$ by:

$$\mathsf{N}_{0,h} = \int_{[\overline{M}_0(S,\beta)]^{\mathrm{red}}} 1 \ , \qquad \langle \beta, \beta \rangle = 2h - 2$$

Sensible since the reduced virtual dimension is 0 if $g = 0$.

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and conjectured in 1995:

$$\sum_{h\geq 0} \mathsf{N}_{0,h}\, q^{h-1} = \frac{1}{\Delta(q)} = \frac{1}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} \,,$$

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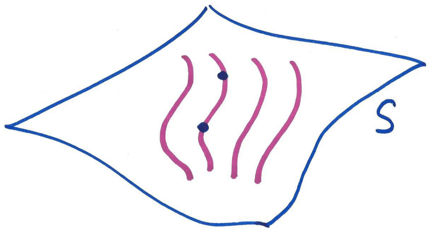

Define the Gromov-Witten invariants by

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma_i) \right\rangle_{g,\beta}^{S} = \int_{[\overline{M}_{g,n}(S,\beta)]^{red}} \prod_{i=1}^{n} \psi_i^{\alpha_i} \cup \mathrm{ev}_i^*(\gamma_i)\,,$$

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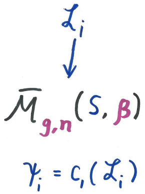

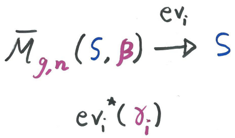

Define a generating series for the descendent theory of $K3$ surfaces:

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The ring QMod is naturally graded by weight (where E_{2k} has weight $2k$) and carries a filtration

$$\mathrm{QMod}_{\leq 2k} \subset \mathrm{QMod}$$

given by forms of weight $\leq 2k$.

Theorem (Maulik-P-Thomas, 2010)

The descendent potential is the Fourier expansion in q of a quasimodular form:

$$\mathsf{F}_g\big(\tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\big) \in \frac{1}{\Delta(q)}\,\mathrm{QMod}_{\leq 2g+2r}\,.$$

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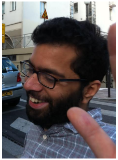

§VI. Conjectures for $S \times E$

The Calabi-Yau 3-fold $X = S \times E$ is a perfect place for counting.

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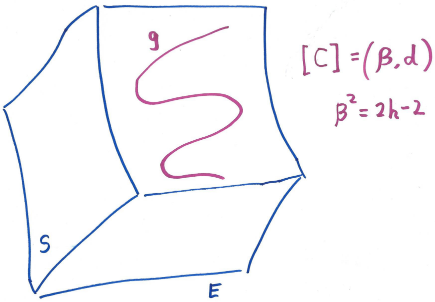

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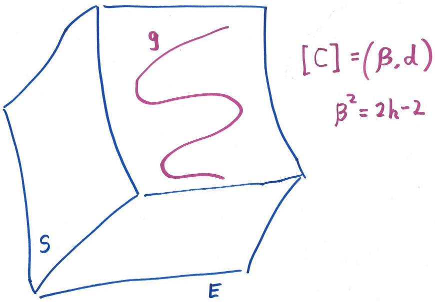

Define the count to be $\mathsf{N}^{X\bullet}_{g,h,d}$

Define the partition function:

$$\mathrm{N}^{X\bullet}(u, q, \tilde{q}) = \sum_{g\in\mathbb{Z}} \sum_{h\geq 0} \sum_{d\geq 0} \mathrm{N}^{X\bullet}_{g,h,d}\; u^{2g-2} q^{h-1} \tilde{q}^{d-1}.$$

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Conjecture (Oberdieck-P, 2014)

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Related to Katz-Klemm-Vafa (1998) study of heterotic duality, black hole counts of Dabholkar-Murthy-Zagier (2012).

The Igusa cusp form $\chi_{10}(\Omega)$ is a weight 10 Siegel modular form on

$$\Omega = \begin{pmatrix} \tau & z \\ z & \widetilde{\tau} \end{pmatrix} \in \mathbb{H}_2 \,,$$

where $\tau, \widetilde{\tau} \in \mathbb{H}_1$ lie in the Siegel upper half plane, $z \in \mathbb{C}$, and

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$\chi_{10}(\Omega)$ is a function of p, q, $\tilde{q}$.

Define the Jacobi theta function by

$$F(z,\tau) = u \exp\left(\sum_{k\geq 1}(-1)^k \frac{B_{2k}}{2k(2k!)} E_{2k} u^{2k}\right).$$

Define the Weierstrass $\wp$ function by

$$\wp(z,\tau) = -\frac{1}{u^2} + \sum_{k\geq 2}(-1)^k(2k-1)\frac{B_{2k}}{(2k)!} E_{2k} u^{2k-2}.$$

Define the coefficients $c(m)$ by

$$-24\wp(z,\tau)F(z,\tau)^2 = \sum_{n\geq 0}\sum_{k\in\mathbb{Z}} c(4n-k^2)p^k q^n.$$

Igusa cusp form $\chi_{10}(\Omega)$ following Gritsenko - Nikulin is

$$\chi_{10}(\Omega) = pq\tilde{q} \prod_{(k,h,d)} (1 - p^k q^h \tilde{q}^d)^{c(4hd-k^2)},$$

where the product is over all $k \in \mathbb{Z}$ and $h, d \geq 0$ satisfying one of:

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