Counting curves on $K3$ surfaces

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§I. Virtual class

Let S be a nonsingular projective $K3$ surface, and let

$$\beta \in \mathrm{Pic}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S,\mathbb{C})$$

be a nonzero effective curve class. The moduli space $\overline{M}_g(S,\beta)$ of genus g stable maps has expected dimension

$$\dim_{\mathbb{C}}^{\mathrm{vir}} \overline{M}_g(S,\beta) = \int_\beta c_1(S) + (\dim_{\mathbb{C}}(S) - 3)(1-g) = g - 1.$$

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The obstruction space at the moduli point $[f : C \to S]$ is

$$\mathrm{Obs}_{[f]} = H^1(C, f^* T_S)$$

which admits a 1-dimensional trivial quotient,

$$H^1(C, f^* T_S) \cong H^1(C, f^* \Omega_S) \to H^1(C, \omega_C) = \mathbb{C} \ .$$

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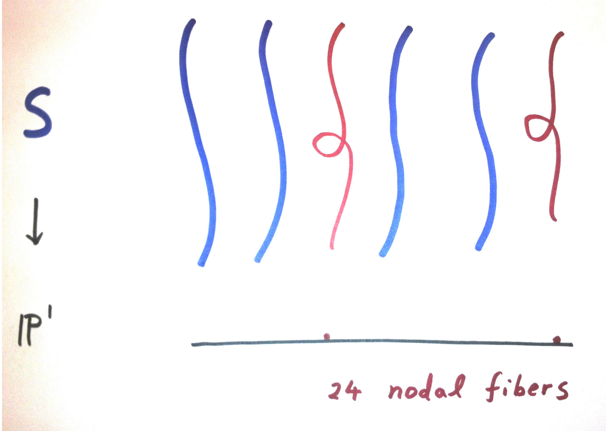

An elliptically fibered $K3$ surface has 24 nodal rational fibers.

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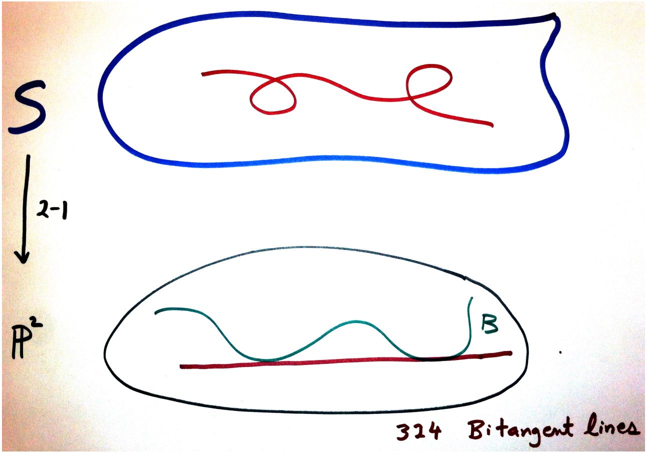

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The trivial piece of $\text{Obs}_{[f]}$ can be removed. The result is a *reduced virtual class* invariant under deformations of S for which β remains in $\text{Pic}(S)$,

$$\dim_{\mathbb{C}}^{\text{red}} \overline{M}_g(S, \beta) = \dim_{\mathbb{C}}^{\text{vir}} \overline{M}_g(S, \beta) + 1 = g \ .$$

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We define the reduced genus 0 invariants of S in class β by

$$\mathsf{N}_{0,\beta} = \int_{[\overline{M}_0(S,\beta)]^{\mathrm{red}}} 1 \ .$$

Sensible since the reduced virtual dimension is 0 if $g = 0$.

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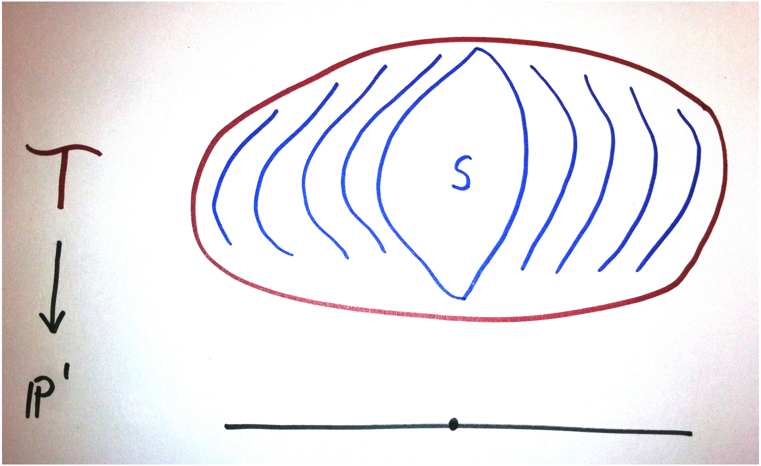

For $g > 0$, any Gromov-Witten integrand of degree g can be placed. The choice corresponding to the twistor contribution is

$$\mathsf{N}_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g$$

where λ_g is the top Chern of the Hodge bundle $\mathbb{E}_g$ with fiber

$$H^0(C, \omega_C) \quad \text{over} \quad [f : C \to S] \in \overline{M}_g(S, \beta) \ .$$

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If $\beta \in \mathrm{Pic}(S)$ is of divisibility 1, β is a *primitive* class. For primitive classes, Yau and Zaslow considered

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and conjectured in 1995:

$$\sum_{h \geq 0} \mathsf{N}_{0,1,h}\, q^{h-1} = \frac{1}{\Delta(q)} = \frac{1}{q \prod_{n=1}^{\infty}(1-q^n)^{24}} \,,$$

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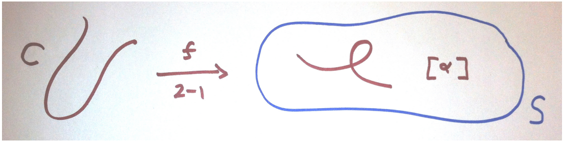
§III. Yau-Zaslow Conjecture : imprimitive classes

What about $N_{0,m,h}$ for $m > 1$?

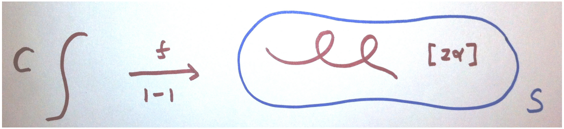
Let $\alpha \in \mathrm{Pic}(S)$ be primitive, consider stable maps to S in class 2α:

$$\overline{M}_0(S, 2\alpha) \;=\; \mathrm{Im}(\alpha) \sqcup \mathrm{Im}(2\alpha)\ .$$

$\mathrm{Im}(\alpha)$ is the moduli of maps which double cover an image curve in S of class α:



$\mathrm{Im}(2\alpha)$ is the moduli of maps which single cover an image curve in S of class 2α:



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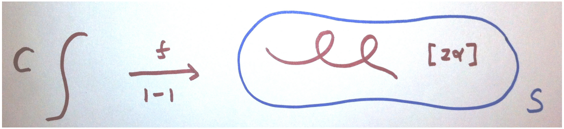

Certainly, we may write

$$N_{0,2\alpha} = \text{Cont}\,[\text{Im}(\alpha)] + \text{Cont}\,[\text{Im}(2\alpha)]$$

and guess

$$\text{Cont}\,[\text{Im}(\alpha)] \;\sim\; \frac{1}{8} N_{0,\alpha}$$

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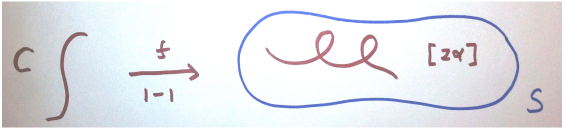

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The perspective motivates forming the combination

$$N_{0,2\alpha} - \frac{1}{8} N_{0,\alpha} = \; ? \; .$$

A miracle occurs: we find

$$N_{0,2\alpha} - \frac{1}{8}N_{0,\alpha} = N_{0,1,h_{2\alpha}}$$

where $2h_{2\alpha} - 2 = \int_S (2\alpha)^2$.

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Brief history: Gathmann checked a single instance of $m = 2$ in 2002. Lee-Leung checked the full $m = 2$ YZ formula in 2004.

Finally, a complete proof for all m was given in 2008 by Klemm, Maulik, P, and Scheidegger by a wild argument using:

- (rigorous) mirror symmetry for the STU model,
- GW/NL correspondence,
- Borcherds' results on Noether-Lefschetz relations,
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Yau-Zaslow concerned only genus 0. On to higher genus ...

§IV. Katz-Klemm-Vafa Conjecture

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be both effective and primitive. The Gromov-Witten potential is:

$$\mathsf{F}_\alpha(u,v) = \sum_{g \geq 0} \sum_{m > 0} \mathsf{N}_{g,m\alpha} \; u^{2g-2} v^{m\alpha}.$$

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The BPS counts $r_{g,m\alpha}$ are uniquely defined by the following equation:

$$\mathsf{F}_\alpha = \sum_{g\geq 0}\sum_{m>0} r_{g,m\alpha}\; u^{2g-2} \sum_{d>0} \frac{1}{d}\left(\frac{\sin(du/2)}{u/2}\right)^{2g-2} v^{dm\alpha}.$$

The BPS counts are defined for both primitive and divisible classes.

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The conjecture is rather surprising. From the definition, the divisibility m of β should matter.

Assuming the validity of Conjecture 1, let $r_{g,h}$ denote the BPS count associated to a class β with

$$\int_S \beta^2 = 2h - 2.$$

Conjecture 2. *The BPS counts $r_{g,h}$ are uniquely determined by the following equation:*

$$\sum_{g\geq 0}\sum_{h\geq 0}(-1)^g r_{g,h}(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^{2g}q^h = \prod_{n\geq 1}\frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}.$$

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In the limit $y \to 1$, the Yau-Zaslow formula for $g = 0$ is recovered. Conjectures 1 and 2 restricted to $g = 0$ provide the precise statement of the Yau-Zaslow formula for all m.

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The right side of Conjecture 2 is related to the generating series of Hodge numbers of the Hilbert schemes of points $\mathrm{Hilb}^n(S)$.

As a consequences of Conjecture 2, $r_{g,h}$ is an integer, $r_{g,h} = 0$ if $g > h$, and

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$r_{g,h}$	$h = 0$	1	2	3	4
$g = 0$	1	24	324	3200	25650
1		−2	−54	−800	−8550
2			3	88	1401
3				−4	−126
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Theorem (P-Thomas, 2014)

The count $r_{g,\beta}$ *depends upon* β *only through* $\int_S \beta^2 = 2h - 2$, *and the Katz-Klemm-Vafa formula holds:*

$$\sum_{g\geq 0}\sum_{h\geq 0}(-1)^g r_{g,h}\,(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2g} q^h =$$

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We obtain here a second proof of the complete Yau-Zaslow formula in $g=0$.

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The Picard group is of rank 3:

$$\mathrm{Pic}(\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}) \cong \mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \oplus \mathbb{Z}E \ ,$$

where L_1 and L_2 are the pull-backs of $\mathcal{O}(1)$ from the factors $\mathbb{P}^2$ and $\mathbb{P}^1$ and E is the exceptional divisor. The anticanonical class $3L_1 + 2L_2 - 2E$ is base point free.

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A nonsingular anticanonical $K3$ hypersurface $S \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ is naturally lattice polarized by L_1, L_2, and E. The lattice is

$$\Lambda = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

A general anticanonical Calabi-Yau 3-fold hypersurface,

$$X \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \times \mathbb{P}^1 \ ,$$

determines a 1-parameter family of anticanonical $K3$ surfaces in $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$,

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A fiber class $\beta \in H_2(X, \mathbb{Z})$ of X has degree (d_1, d_2, d_3),

$$d_1 = \int_\beta L_1 , \quad d_2 = \int_\beta L_2 , \quad d_3 = \int_\beta E .$$

Theorem (Maulik-P, 2007)

For an effective fiber class of degree (d_1, d_2, d_3),

$$n^{X}_{g,(d_1,d_2,d_3)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^{\pi}_{m,h,(d_1,d_2,d_3)}.$$

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- $n^X_{g,(d_1,d_2,d_3)}$ is the Gromov-Witten BPS count of X in the fiber class of degree (d_1, d_2, d_3),
- $NL^{\pi}_{m,h,(d_1,d_2,d_3)}$ is the Noether-Lefschetz number associated to the $K3$-fibration π.

The Noether-Lefschetz number $NL^{\pi}_{m,h,(d_1,d_2,d_3)}$ counts the number of $K3$ fibers S of π which carry a class

$$\beta \in \mathrm{Pic}(S)$$

of divisibility m, square $\int_S \beta^2 = 2h - 2$, and degree (d_1, d_2, d_3).

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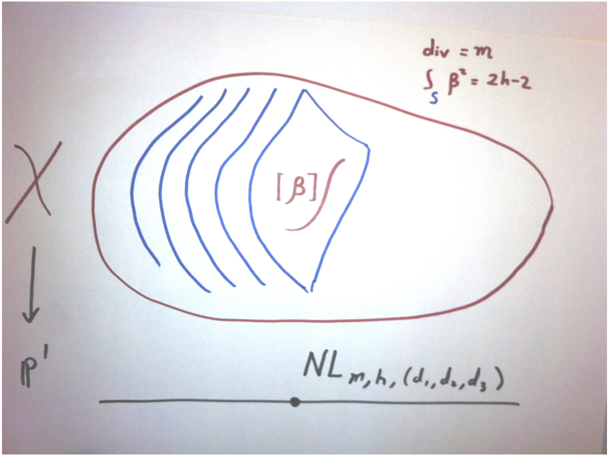

§V-2. The P/NL correspondence

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We consider the moduli space of stable pairs

$$[\mathcal{O}_X \xrightarrow{s} F] \in P_n(X, \beta)$$

where F is a pure sheaf supported on a Cohen-Macaulay subcurve of X, s is a morphism with 0-dimensional cokernel, and

$$\chi(F) = n, \quad [F] = \beta.$$

§V-2. The P/NL correspondence

We consider the moduli space of stable pairs

$$[\mathcal{O}_X \xrightarrow{s} F] \in P_n(X, \beta)$$

where F is a pure sheaf supported on a Cohen-Macaulay subcurve of X, s is a morphism with 0-dimensional cokernel, and

$$\chi(F) = n, \quad [F] = \beta.$$

The space $P_n(X, \beta)$ carries a virtual fundamental class of dimension 0 obtained from the deformation theory of complexes with trivial determinant in the derived category.

Theorem (P-Thomas, 2014)

For an effective fiber class of degree (d_1, d_2, d_3),

$$\widetilde{n}^X_{g,(d_1,d_2,d_3)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} \widetilde{r}_{g,m,h} \cdot NL^{\pi}_{m,h,(d_1,d_2,d_3)}.$$

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- $\widetilde{n}^{X}_{g,(d_1,d_2,d_3)}$ is the stable pairs BPS count of X in the fiber class of degree (d_1, d_2, d_3),
- $\widetilde{r}_{g,m,h}$ is the stable pairs analogue of the Gromov-Witten BPS count $r_{g,m,h}$,
- $NL^{\pi}_{m,h,(d_1,d_2,d_3)}$ is the Noether-Lefschetz number associated to the $K3$-fibration π as before.

§V-3. The GW/P correspondence

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Theorem (P-Pixton, 2012)

The Gromov-Witten and stable pairs BPS counts are equal,

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$$r_{g,m,h} = \widetilde{r}_{g,m,h} \, .$$

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Our choice of $K3$-fibration is used in the analysis of the linear equations!

The KKV conjecture has now been transformed purely into a question about the geometry of stable pairs.

§V-4. Stable pair geometry

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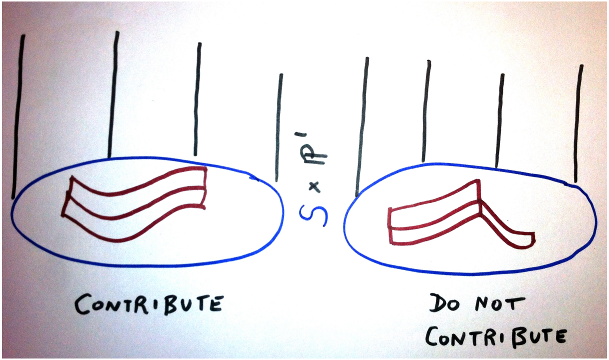
Localization with respect to $\mathbb{C}^*$, leads to fixed point calculations of $\widetilde{r}_{g,m,h}$. The crucial observation is that only *clean stackings* contribute.


$S \times \mathbb{P}^1$
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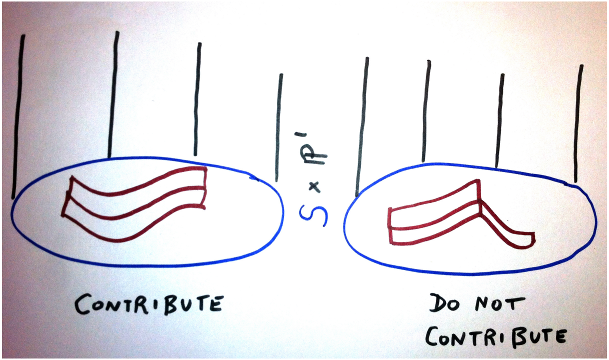
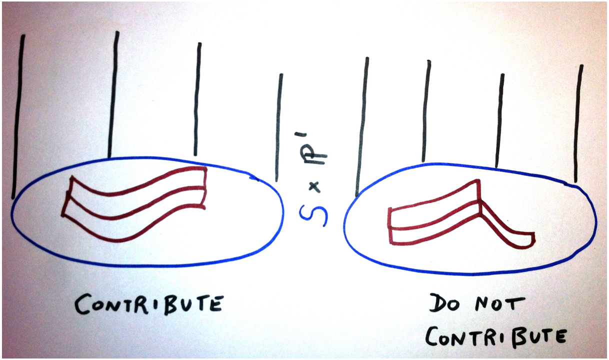
DO NOT CONTRIBUTE



The vanishing of the irregular stackings leads to a simple multiple cover structure for the $S \times \mathbb{C}$ reduced residue theory. The independence of $\widetilde{r}_{g,m,h}$ can be verified explicitly.



The vanishing of the irregular stackings leads to a simple multiple cover structure for the $S \times \mathbb{C}$ reduced residue theory. The independence of $\widetilde{r}_{g,m,h}$ can be verified explicitly.

Conjecture 2 of KKV is now reduced to the $m = 1$ case where the results are known by Maulik, P, Thomas (2010) via older sheaf theoretic calculations of Kawai-Yoshioka (2000).

§VI. Quartic surfaces

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Consider the family of $K3$ surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^3$:

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Let A and B be modular forms of weight $1/2$ and level 8,

$$A = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{8}}, \quad B = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{8}}.$$

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Let Θ be the modular form of weight $21/2$ and level 8 defined by

$$\begin{aligned} 2^{22}\Theta = \; & 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 \\ & -20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 \\ & -621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} \\ & -346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} \\ & -361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} \\ & -4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}. \end{aligned}$$

We may expand Θ as a series in $q^{\frac{1}{8}}$,

$$\Theta = -1 + 108q + 320q^{\frac{9}{8}} + 50016q^{\frac{3}{2}} + 76950q^2 \ldots .$$

Let $\Theta[m]$ denote the coefficient of q^m in Θ.

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Theorem (Maulik-P, 2007)

The Noether-Lefschetz numbers of the quartic pencil π are coefficients of Θ,

$$NL^{\pi}_{h,d} = \Theta\left[\frac{\triangle_4(h,d)}{8}\right],$$

where the discriminant is defined by

$$\triangle_4(h,d) = -\det\begin{vmatrix} 4 & d \\ d & 2h-2 \end{vmatrix} = d^2 - 8h + 8 .$$

By the GW/P correspondence, we obtain

$$n^X_{g,d} = \sum_{h=0}^{\infty} r_{g,h} \cdot \Theta\left[\frac{\triangle_4(h,d)}{8}\right],$$

as predicted by Klemm, Kreuzer, Riegler, and Scheidegger.

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Similar closed form solutions can be found for all the classical families of $K3$-fibrations.