Notes on Göttsche's conjecture about the χ_y genus

I explain here how to lift the proof of Theorem 5 of [3] for a fixed curve from Euler characteristic to the χ_y genus (1) as predicted by L. Göttsche. I then address Göttsche's conjecture for the χ_y genus of families of curves. The natural result (4) here is weaker than what Göttsche conjectures. Göttsche's conjecture, however, is for families arising from a linear series on a surface. The form (4) does appear optimal for all flat families of reduced, irreducible Gorenstein curves.

Göttsche's conjecture must rely on special surface geometry. Using constructions with $K3$ surfaces, abelian surfaces, and their covers, I sketch an improvement (7) of (4) which is much closer (but not equal to) Göttsche's conjecture.

A. Fixed curve

Let C be a complete, reduced, and irreducible curve with Gorenstein singularities. The standard example is a reduced and irreducible curve in a nonsingular complete surface. Let $g_{\mathsf{geom}}(C) \leq g_{\mathsf{arith}}(C)$ be the geometric and arithmetic genera of C.

Let $P_{n,C}$ be the moduli space of stable pairs on C. Since C is reduced, irreducible, and Gorenstein [3], we have an isomorphism between the moduli of stable pairs and the Hilbert scheme of points of C,

$$P_{n,C} \cong \mathrm{Hilb}(C, n+g-1)\ .$$

Theorem 5 of [3] expresses the generating function for Euler characteristics of $P_{n,C}$ in BPS form:

$$\sum_{n\in\mathbb{Z}} (-1)^n e(P_{n,C}) q^n = \sum_{r=g_{\mathsf{geom}}}^{g_{\mathsf{arith}}} n_{r,C}\ q^{1-r}(1+q)^{2r-2}\ .$$

Since the alternating sign is not of interest to us here, we rewrite the result as

$$\sum_{n\in\mathbb{Z}} e(P_{n,C}) q^n = \sum_{r=g_{\mathsf{geom}}}^{g_{\mathsf{arith}}} \tilde{n}_{r,C}\ q^{1-r}(1-q)^{2r-2}$$

where the integers $\tilde{n}_{r,C}$ are redefined to include the sign.

Let $\chi_y(P_{n,C}) \in \mathbb{Z}[y]$ be the χ_y genus obtained from the virtual Hodge polynomial $s(P_{n,C}) = \sum_{i,j} b_{ij} u^i v^j$,

$$\chi_y(P_{n,C}) = s\big(P_{n,C}\big)(-y,-1) = \sum_{i,j} (-1)^{i+j} b_{ij} y^i \ .$$

For a nonsigular curve X_r of genus r, we easily find

$$\sum_{n\in\mathbb{Z}} \chi_y(P_{n,X_r})\ q^n = q^{1-r}(1-q)^{r-1}(1-yq)^{r-1} \ .$$

We will prove

$$\sum_{n\in\mathbb{Z}} \chi_y(P_{n,C})\ q^n = \sum_{r=g_{\mathsf{geom}}}^{g_{\mathsf{arith}}} N_{r,C}(y)\ q^{1-r}(1-q)^{r-1}(1-yq)^{r-1} \tag{1}$$

for polynomials $N_{r,C}(y)$ which of course specialize as

$$N_{r,C}(1) = \tilde{n}_{r,C} \ .$$

B. Proof

The proof of (1) follows Theorem 5 of [3] in all the main steps. The crucial point is the correspondence

$$\begin{aligned} F &\mapsto Hom_C(F, \omega_C) \ , \\ \chi(F) &\mapsto -\chi(F) \ . \end{aligned}$$

where F is a rank 1 torsion free sheaf on C.

We denote the generating series of the χ_y genera for the moduli spaces of stable pairs on C by

$$\mathsf{Z}_C(q,y) = \sum_{n\in\mathbb{Z}} \chi_y(P_{n,C})\ q^n \ .$$

Then, we have

$$q^{g-1}\mathsf{Z}_C(q,y) = \sum_{n\in\mathbb{Z}} \chi_y(P_{n,C})\ q^{n+g-1} \ .$$

In the language of motivic integration [2], we have

$$q^{g-1}\mathsf{Z}_C(q,y) = \sum_{n\in\mathbb{Z}} \chi_y(J_{n,C})\ q^n \int_{F\in J_{n,C}} \frac{y^{h^0(F)}-1}{y-1} d\mu_J \tag{2}$$

where $J_{n,C}$ is the degree n compactified Jacobian (via rank 1 torsion free sheaves) of C. Rewriting, we find

$$y^{1-g}q^{1-g}\mathsf{Z}_C(q,y) = \sum_{n\in\mathbb{Z}} \chi_y(J_{n+2g-2,C})\, q^n \int_{F\in J_{n+2g-2,C}} y^{1-g}\frac{y^{h^0(F)}-1}{y-1}d\mu_J\ .$$

On the other hand, we can substitute $q\mapsto (yq)^{-1}$ in equation (2),

$$(yq)^{1-g}\mathsf{Z}_C\left(\frac{1}{yq},y\right) = \sum_{n\in\mathbb{Z}} \chi_y(J_{-n,C})\ (yq)^n \int_{F\in J_{-n,C}} \frac{y^{h^0(F)}-1}{y-1}d\mu_J\ .$$

Since there exists a line bundle of degree 1 on C, there are isomorphisms

$$J_{a,C}\cong J_{b,C}$$

for all $a,b\in\mathbb{Z}$. In particular, the χ_y genus $\chi_y(J)$ is independent of degree. Finally, by Riemann-Roch, we calculate the difference,

$$y^{1-g}q^{1-g}\mathsf{Z}_C(q,y) - (yq)^{1-g}\mathsf{Z}_C\left(\frac{1}{yq},y\right) = \sum_{n\in\mathbb{Z}} \chi_y(J)\ q^n\frac{y^n-y^{1-g}}{y-1}\ .$$

Taking only the non-negative powers of q, we obtain

$$\begin{aligned}\left[\mathsf{Z}_C(q,y) - \mathsf{Z}_C\left(\frac{1}{yq},y\right)\right]_{q\geq 0} &= \sum_{n\geq 0}\chi_y(J)\ q^n\frac{y^n-1}{y-1}\\ &= \chi_y(J)\frac{q}{(1-q)(1-yq)}.\end{aligned}$$

The interesting terms of Z_C occur with q degrees between $1-g_{\mathsf{arith}}$ and $g_{\mathsf{arith}}-1$. The terms below vanish, and the terms above are given by

$$\left[\chi_y(J)\frac{q}{(1-q)(1-yq)}\right]_{q>g_{\mathsf{arith}}-1}\ .$$

Using the invariance of

$$q^{1-r}(1-q)^{r-1}(1-yq)^{r-1}$$

under $q\mapsto (yq)^{-1}$ for $r>0$, an elementary argument shows the existence of polynomial $N_{r,C}(y)$ satisfying

$$\sum_{n\in\mathbb{Z}}\chi_y(P_{n,C})\ q^n = \sum_{r=0}^{g_{\mathsf{arith}}} N_{r,C}(y)\ q^{1-r}(1-q)^{r-1}(1-yq)^{r-1} \tag{3}$$

A crucial point here is the justification of the motivic integration step. The fibers of

$$P_{n,C}\to J_{n+g-1,C}$$

are projective spaces of various dimensions. We can stratify $J_{n+g-1,C}$ so that the projective space fibers are of constant dimension. There are two ways to argue.

- First, since a universal bundle can be found on $J_{n+g-1,C}$ (by selecting a nonsingular point), the projective space fibrations are obtained from projectivizations of vector bundles and hence are Zariski locally free.
- Second, for any projective space fibration (perhaps not Zariski locally trivial), the Serre polynomial (and thus also the χ_y genera) is nevertheless obtained by the product rule *as if the fibration were Zariski locally trivial.* The proof of the last claim is obtained by establishing the triviality of the local system in cohomology associated to a fibration of projective spaces. Then, we argue as in Theorem 5.4 of [1].

When we consider moving curves, only the second argument will apply,

Finally, to cut out the lower terms of (3), we use the same argument as in [3] — motivically decomposing the moduli of pairs as pieces from the smooth part of C and the singular part. The singular part can be put in a curve of geometric genus 0. Using the simple motivic structure, we find (1) holds:

$$\sum_{n\in\mathbb{Z}} \chi_y(P_{n,C})\ q^n = \sum_{r=g_{\mathsf{geom}}}^{g_{\mathsf{arith}}} N_{r,C}(y)\ q^{1-r}(1-q)^{r-1}(1-yq)^{r-1}$$

The above discussion lifts also to the full Serre polynomial. In fact, using the Zariski local triviality of the projective space fibrations, we can lift the result fully motivically. These lifting results were known previously to V. Shende.

C. Families

We now consider flat families of reduced, irreducible curves with Gorenstein singularities,

$$\pi : \mathcal{C} \to B\ .$$

Let g_{arith} be the (constant) arithmetic genus of the family. Let $P_{n,\pi}$ be the relative family of stable quotients. I claim the following result holds:

$$(4) \qquad \sum_{n\in\mathbb{Z}} \chi_y(P_{n,\pi})\ q^n = \sum_{r=0}^{g_{\mathsf{arith}}} N_{r,\pi}(y)\ q^{1-r}(1-q)^{r-1}(1-yq)^{r-1}$$

The proof of (4) is not formal since the fixed curve result (1) can not be integrated over the base (χ_y genus is more subtle than Euler characteristic). One has to instead repeat the above argument for the entire family. Besides the observation that local triviality of projective bundle is unnecessary, another issues arises. For a fixed curve, the Jacobians are all isomorphic since there exists a degree 1 line bundle. The relative Jacobians of π *need not be isomorphic*. But we are only interested in their Serre polynomial. I claim the Serre polynomials are equal: they can be computed via the associated local systems of cohomology on B which *are* isomorphic (along with the relevant Hodge structure). This concludes the proof of (4).

What I havent shown is that if the minimum geometric genus which occurs in the family π is g_{geom}, then the lower $r < g_{\mathsf{geom}}$ terms drop out of (4). This is in general false, but might be true in for complete families obtained from surfaces (as conjectured by Göttsche).

Göttsche's conjecture. *Let S be a nonsingular projective surface. Let $\pi : \mathcal{C} \to \mathbb{P}^k$ be the universal curve over a linear series satisfying*

(i) *all curves in the system are reduced, irreducible*

(ii) *the moduli space $P_{n,\pi}$ is nonsingular .*

Then, we can extract the χ_y genus of $P_{n,\pi}$ from

$$(5) \qquad \sum_{n\in\mathbb{Z}} \chi_y(P_{n,\pi})\ q^n = \sum_{r=g_{\mathsf{arith}}-k}^{g_{\mathsf{arith}}} N_{r,\pi}(y)\ q^{1-r}(1-q)^{r-1}(1-yq)^{r-1}$$

The linear series in Göttsche's conjecture need not be complete. Condition (ii) need hold only for the single n in question. Under the hypotheses given, the $N_{r,\pi}$ are easily seen to be determined from the 4 topological invariants

$$c_2(S),\ c_1^2(S),\ c_1(S)L,\ L^2$$

where L is the class of the linear series. Since no curves of geometric genus less than $g_{\mathsf{arith}}-k$ will appear for a general linear series, Göttsche's conjecture would follow from the following stronger statement (under the same hypotheses):

$$(6)\qquad \sum_{n\in\mathbb{Z}} \chi_y(P_{n,\pi})\; q^n = \sum_{r=g_{\mathsf{geom}}}^{g_{\mathsf{arith}}} N_{r,\pi}(y)\; q^{1-r}(1-q)^{r-1}(1-yq)^{r-1}$$

where g_{geom} is the minimun geometric genus which appears in the linear system. While Göttsche has considerable numerical evidence for the original form, whether (6) is true is unclear.

To kill the bottom terms in (4) seems to require further constraints. If $g_{\mathsf{geom}} = 0$, then there is nothing to kill. So we can try to prove the vanishing of the lower terms as before by transferring to the $g_{\mathsf{geom}} = 0$ case. In effect, this means looking for lots of linear systems of reduced, irreducible curves on surfaces of dimension g and $g_{\mathsf{arith}} = g$. Primitive classes on $K3$ surfaces provide examples. Are there enough other examples?

D. Examples

Let $S \to \mathbb{P}^1$ be the rational elliptic surface. Here, S is the blow-up of $\mathbb{P}^2$ at 9 points. Let $\delta \subset \mathbb{P}^1$ be the 12 points corresponding to the nodal fibers. Let

$$\pi : \mathcal{C} \to \mathbb{P}^1 \setminus \delta$$

be the family of nonsingular elliptic curves obtained by removing the nodal fibers.

We can easily calculate the χ_y genera of relative moduli spaces of stable pairs for π. The moduli space $P_{n,\pi}$ is empty for $n < 0$. We have

$$P_{0,\pi} = \mathbb{P}^1 \setminus \delta\ , \quad P_{1,\pi} = \mathcal{C}\ .$$

For $n > 1$, we have the fibration

$$P_{n,\pi} \to \mathcal{C}$$

as a $\mathbb{P}^{n-1}$ bundle. Remember π has a section, so there is no difficulty in identifying the elliptic fibers with their Jacobians of any degree. We calculate

$$\sum_{n\in\mathbb{Z}} \chi_y(P_{n,\pi})\ q^n =$$
$$y-11+(y^2-2y+1)q+(y^2-2y+1)(y+1)q^2+\ldots =$$
$$(y^2-2y+1)\frac{q}{(1-q)(1-yq)}+(y-11)\ .$$

Fitting the form of (4), we find

$$N_{0,\pi}=y-11, \quad N_{1,\pi}=y^2-2y+1\ .$$

In particular, the $N_{0,\pi}$ appears even though the minimal geometric genus in the family π is 1. Hence, (4) can not be strengthened without extra hypotheses. The complications occur here because of monodromy issues for the local system associated to the family π. It seems likely that (4) is optimal without further constraints.

An objection could be made that the base B in the last example is not complete. If

$$\pi : \mathcal{C} \to B$$

is a complete nontrivial family of nonsingular genus g curves, B. Fantechi and I found that more than just the $N_{g,\pi}$ polynomial must occur.

A quick summary of the calculation is as follows. There exists complete nontrivial families

$$\pi : \mathcal{C} \to B$$

of nonsingular genus 4 curves. Let

$$\ell = \int_B c_1(\mathbb{E}_4) > 0,$$

where $\mathbb{E}_4$ is the Hodge bundle over M_4. The above positivity is because the first Chern class is ample on M_4. Certainly $N_{4,\pi}(y)=\chi_y(B)$ by examining the q^{-3} coefficient of (4) and the identification $P_{-3,\pi}=B$. If only $N_{4,\pi}(y)$ appears in (4), then we would have

$$\chi_y(P_{-2,\pi}) = \chi_y(B)\cdot\chi_y(C_4)$$

where C_4 is a nonsingular genus 4 curve. In particular, $\chi_y(P_{-2,\pi})$ would be independent of ℓ.

Since $P_{-2,\pi} = \mathcal{C}$, we can simply compute the χ_y genus. One coefficient is

$$\begin{aligned} \chi(\mathcal{O}_{\mathcal{C}}) &= \chi(\pi_* \mathcal{O}_{\mathcal{C}}) - \chi(R^1 \pi_* \mathcal{O}_{\mathcal{C}}) \\ &= 1 - g(B) - \chi(\mathbb{E}_4^*) \\ &= \ell - 3(1 - g(B)) \end{aligned}$$

which surely depends upon ℓ (as ℓ certainly does not determine $g(B)$). Hence, $N_{4,\pi}(y)$ can not be the only term of (4) even though all fibers of π have geometric genus 4.

It is known that complete families of nonsingular curves of general type must have positive genus, so do not determine a pencil on a surface. We return to the conclusion that Göttsche's conjecture must rely on special properties of linear systems on surfaces.

E. Families of curves on surfaces

Let us consider linear systems of dimension k on surfaces

$$\pi : \mathcal{C} \to \mathbb{P}^k$$

satisfying the hypotheses of Göttsche's conjecture. We easily find 2 families:

(i) curves of primitive class L satisfying $L^2 = 2k - 2$ on $K3$ surfaces of Picard rank 1,

(ii) curves of primitive class L satisfying $L^2 = 2k + 2$ on Abelian surfaces of Picard rank 1 .

For another family, consider a $K3$ surface of type (i). Let D_2 be general nonsingular divisor in class $2L$, and let

$$\epsilon : S \to K3$$

be the double cover ramified along D_2. We have:

(iii) curves of primitive class $\epsilon^*(L)$ satisfying $\epsilon^*(L)^2 = 2(2k-2)$ on the double cover S.

For the same $K3$, let D_4 be general nonsingular divisor in class $4L$, and let

$$\epsilon : T \to K3$$

be the double cover ramified along D_4. We have:

(iv) curves of primitive class $\epsilon^*(L)$ satisfying $\epsilon^*(L)^2 = 2(2k-2)$ on the double cover T.

The topological invariants $(c_2(S), c_1^2(S), c_1(S)L, L^2)$ can easily be computed for the 4 families. We get the vectors

$$(24, 0, 0, 2k-2),$$
$$(0, 0, 0, 2k+2),$$
$$(40+8k, 2(2k-2), 2(2k-2), 2(2k-2)),$$
$$(16+32k, 8(2k-2), 4(2k-2), 2(2k-2)).$$

The determinant is $-1536(k+1)(k-1)^2$ which vanishes for positive k only at $k = 1$ (which is anyway a trivial case where Göttsche's conjecture is certainly true).

The highest genus which appears here is of course in type (iv). We calculate the genus there to be

$$2g-2 = \epsilon^*(L)(\epsilon^*(L) + K_T) = 12k - 12$$

so $g = 6k-5$. Using (4) and the topological determination of the $N_{r,\pi}$ (under the hypotheses of Göttsche's conjecture), we should obtain a result like:

$$\sum_{n\in\mathbb{Z}} \chi_y(P_{n,\pi})\ q^n = \sum_{r=g_{\mathsf{arith}}-6k}^{g_{\mathsf{arith}}} N_{r,\pi}(y)\ q^{1-r}(1-q)^{r-1}(1-yq)^{r-1}\ . \tag{7}$$

Here we have used the curves on surface condition to obtain the additional vanishing. To prove (7), a careful induction should be made. I do not claim a proof.

Rahul 20/11/2011

References

[1] E. Getzler and R. Pandharipande, *The Betti numbers of* $\overline{M}_{0,n}(\mathbb{P}^r, d)$, JAG.

[2] M. Kapranov, The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups, arXiv:math/0001005.

[3] R. Pandharipande and R. Thomas, *Stable pairs and BPS invariants*, JAMS.