Lehn and Verlinde formulas for moduli spaces of sheaves on surfaces

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joint work with Martijn Kool

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Let S smooth projective surface

Hilbert scheme of points:

$S^{[n]} = Hilb^n(S) = \{\text{zero dim. subschemes of degree } n \text{ on } S\}$

Vector bundle V on $S \Longrightarrow$ tautological vb $V^{[n]}$ on $S^{[n]}$

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generating functions for Segre integrals $\int_{S^{[n]}} c_{2n}(V^{[n]})$
$+$ their conj. relation to Verlinde numbers $\chi(S^{[n]}, \mu(L) \otimes E^{\otimes r})$.

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$Hilb^n(S)$ is a moduli space of rank 1 stable sheaves on S

Aim: Extend above results to moduli spaces of sheaves $M_S^H(\rho, c_1, c_2)$ of higher rank:
Use Mochizuki's formula computing virtual inters. numbers on $M_S^H(\rho, c_1, c_2)$ in terms of inters. numbers on $S^{[n]}$.

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$S^{[n]}$ is smooth projective, of dimension $2n$
Closely related to symmetric power
$S^{(n)} = S^n/(\text{perm. of factors})$
$\pi : S^{[n]} \to S^{(n)},\ Z \mapsto supp(Z)$ is a crepant resolution.

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Universal subscheme:

$$Z_n(S) = \{(x, [Z]) \mid x \in Z\} \subset S \times S^{[n]}$$

$p : Z_n(S) \to S^{[n]}, \quad q : Z_n(S) \to S$ projections
Fibre $p^{-1}([Z]) = Z$.

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Tautological sheaves: V vector bundle of rank r on S
$V^{[n]} := p_* q^*(V)$ vector bundle of rank rn on $S^{[n]}$
$V^{[n]}([Z]) = H^0(V|_Z)$, in particular $\mathcal{O}_S^{[n]}([Z]) = H^0(\mathcal{O}_Z)$

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Line bundles on $S^{[n]}$: $\mathrm{Pic}(S^{[n]}) = \mu(\mathrm{Pic}(S)) \oplus \mathbb{Z}E$ with $E = \det(\mathcal{O}_S^{[n]}$ and $\mu(L)$ pullback from $S^{(n)}$ of equiva. pushforward of $L \boxtimes \ldots, \boxtimes L$ from S^n to $S^{(n)}$. We have

$$\det(V^{[n]}) = \mu(\det(V)) \otimes E^{\otimes \mathrm{rk}(V)}, \quad V \in K(S)$$

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Want formulas for

$$\chi(S^{[n]}, \mu(L) \otimes E^{\otimes r}) \quad \textbf{Verlinde formula}$$

$$\int_{S^{[n]}} c_{2n}(V^{[n]}) \quad \textbf{Lehn formula}$$

Tool:

Theorem (Ellingsrud-G-Lehn)

Let $P(x_1, \ldots, x_{2n}, y_1, \ldots, y_n)$ *polynomial. Put*

$$P[S^{[n]}, L] := \int_{S^{[n]}} P(c_1(S^{[n]}), ..., c_{2n}(S^{[n]}), c_1(L^{[n]}), \ldots, c_n(L^{[n]}))$$

There is a polynomial $\widetilde{P}(x, y, z, w)$*, such that for all surfaces* S*, all line bundles* L *on* S *we have*

$$P[S^{[n]}, L] = \widetilde{P}(K_S^2, \chi(\mathcal{O}_S), LK_S, K_S^2).$$

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Usually have sequence of polynomials $P_n(x_1,...x_{2n},y_1,\ldots,y_n)$, $n\geq 0$, "nicely organized", then

$$\sum_{n\geq 0} P_n[S^{[n]},L]x^n = A_1(x)^{L^2}A_2(x)^{LK_S}A_3(x)^{K_S^2}A_4(x)^{\chi(O_S)}$$

for universal power series $A_1,\ldots,A_4\in\mathbb{Q}[[x]]$

For L a line bundle on S consider the top Segre class

$$\int_{S^{[n]}} s_{2n}(L^{[n]}) = \int_{S^{[n]}} c_{2n}((-L)^{[n]})$$

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Conjecture (Lehn 1999)

$$\sum_{n=0}^{\infty} \int_{S^{[n]}} s_{2n}(L^{[n]}) z^n = \frac{(1-w)^a(1-2w)^b}{(1-6w+6w^2)^c},$$

with the change of variable

$$z = \frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3},$$

with $a = LK_S - 2K_S^2$, $b = (L-K_S)^2 + 3\chi(\mathcal{O}_S)$,
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Theorem (Marian-Oprea-Pandharipande, Voisin)

Lehn's conjecture is true.

Marian-Oprea-Pandharipande consider a generalized Lehn formula: a formula for $\sum_{n\geq 0}\int_{S^{[n]}} c_{2n}(\alpha^{[n]})z^n, \quad \alpha\in K(S)$

Theorem (Marian-Oprea-Pandharipande)

For any $s\in\mathbb{Z}$, there exist $V_s, W_s, X_s, Y_s, Z_s \in \mathbb{Q}[[z]]$ s.th. for any $\alpha\in K(S)$ of rank s on S, we have

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They showed explicit expressions for Y_s, Z_s for $s\in\{-2,-1,0,1,2\}$, and conjecture that Y_s, Z_s are algebraic functions for all $s\in\mathbb{Z}$

Consider the generating series $\sum_{n=0}^{\infty} w^n \chi(S^{[n]}, \mu(L) \otimes E^{\otimes r})$.

Theorem (Ellingsrud-G-Lehn)

For any $r \in \mathbb{Z}$, there exist $g_r, f_r, A_r, B_r \in \mathbb{Q}[[w]]$ such that for any $L \in Pic(S)$, we have

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With the change of variables $w = v(1+v)^{r^2-1}$, we have

$$g_r(w) = 1 + v, \quad f_r(w) = \frac{(1+v)^{r^2}}{1 + r^2 v}.$$

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Serre duality implies $A_r = B_{-r}/B_r$ for all r. Furthermore, $A_r = B_r = 1$ for $r = 0, \pm 1$. In general the A_r, B_r are unknown.

We have seen

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c(\alpha^{[n]}) = V_s^{c_2(\alpha)}\, W_s^{c_1(\alpha)^2}\, X_s^{\chi(\mathcal{O}_S)}\, Y_s^{c_1(\alpha)K_S}\, Z_s^{K_S^2}, \quad s = \mathrm{rk}(\alpha)$$

$$\sum_{n=0}^{\infty} w^n\, \chi(S^{[n]}, \mu(L) \otimes E^{\otimes r}) = g_r^{\chi(L)}\, f_r^{\frac{1}{2}\chi(\mathcal{O}_S)}\, A_r^{LK_S}\, B_r^{K_S^2},$$

with $V_s,\ W_s,\ X_s \in \mathbb{Q}[[z]]$, $f_r,\ g_r \in \mathbb{Q}[[w]]$ known algebraic functions, and $Y_s,\ Z_s \in \mathbb{Q}[[z]]$, $A_r,\ B_r \in \mathbb{Q}[[w]]$ unknown

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Based on strange duality there is a conjectural relation between these two generating functions

Conjecture (Johnson, Marian-Oprea-Pandharipande)

For any $r \in \mathbb{Z}$, we have

$$A_r(w) = W_s(z) \, Y_s(z), \quad B_r(w) = Z_s(z),$$

where $s = 1 - r$ and $w = v(1+v)^{r^2-1}$, $z = t(1+(1-s)t)^{1-s}$, and $v = t(1+rt)^{-1}$.

Aim: Find analogues of all these results for higher rank moduli spaces
Let (S, H) polarized surface. A torsion free coherent sheaf $\mathcal{E}$ on S is called H-semistable, if for all subsheaves $\mathcal{F} \subset \mathcal{E}$, we have

$$\frac{\chi(S, \mathcal{F} \otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{F})} \leq \frac{\chi(S, \mathcal{E} \otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{E})}, \quad n \gg 0$$

For $\rho \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S, \mathbb{Z})$, and $c_2 \in H^4(S, \mathbb{Z})$, let
$M := M_S^H(\rho, c_1, c_2)$ moduli space of rank ρ H-semistable sheaves on S with Chern classes c_1, c_2

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Note: via $Z \mapsto I_Z$, we have $S^{[n]} = M_S^H(1, 0, n)$.
Assume M contains no strictly semistable sheaves
For simplicity also assume there exists a universal sheaf $\mathcal{E}$ on $S \times M$, (i.e. $\mathcal{E}|_{S \times \{[E]\}} = E$)

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$M = M_S^H(\rho, c_1, c_2)$ has a perfect obstruction theory of expected dimension

$$\mathrm{vd}(M) := 2\rho c_2 - (\rho - 1)c_1^2 - (\rho^2 - 1)\chi(\mathcal{O}_S)$$

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In particular

- it carries a virtual class $[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}(M)}(M)$
- has a virtual Tangent bundle $T_M^{\mathrm{vir}} \in K^0(M)$
- has a virtual structure sheaf $\mathcal{O}_M^{\mathrm{vir}} \in K_0(S)$
  For any $V \in K^0(M)$ the virtual holomorphic Euler characteristic of V is $\chi^{\mathrm{vir}}(M, V) = \chi(M, V \otimes \mathcal{O}_M^{\mathrm{vir}})$

For any class $\alpha \in K^0(S)$, we define

$$\mathrm{ch}(\alpha_M) := -\mathrm{ch}(\pi_{M!}(\pi_S^*\alpha \cdot \mathcal{E} \cdot \det(\mathcal{E})^{-\frac{1}{\rho}}))$$

On $M := M_S^H(1,0,n) \cong S^{[n]}$, we have $\alpha_M = \alpha^{[n]}$

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On $M := M_S^H(1,0,n) \cong S^{[n]}$, we have $\alpha_M = \alpha^{[n]}$
For any $\sigma \in H^k(S,\mathbb{Q})$ the μ-class of Donaldson theory is

$$\mu(\sigma) := \left(c_2(\mathcal{E}) - \frac{\rho-1}{2\rho}c_1(\mathcal{E})^2\right)/\mathrm{PD}(\sigma) \in H^k(M,\mathbb{Q}),$$

For $\alpha \in K^0(S), L \in H^2(S,\mathbb{Z})$, and $pt \in H^4(S,\mathbb{Z})$ the Poincaré dual of a point, the *virtual Segre number* of M is

$$\int_{[M]^{\mathrm{vir}}} c(\alpha_M) \exp\left(\mu(L) + u\mu(pt)\right) \in \mathbb{Q}[u]$$

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and S has a smooth connected canonical divisor
Write $\varepsilon_\rho := \exp(2\pi i/\rho)$ and $[n] := \{1, \dots, n\}$. For any $J \subset [n]$, write $|J|$ for its cardinality and $\|J\| := \sum_{j\in J} j$

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Conjecture

Let $\rho \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$. There exist V_s, W_s, X_s, Q_s, R_s, $T_s \in \mathbb{C}[[z]]$, $Y_{J,s}$, $Z_{J,s}$, $S_{J,s} \in \mathbb{C}[[z^{\frac{1}{2}}]]$, s.th. for all $J \subset [\rho - 1]$ for all S as above, any $\alpha \in K^0(S)$ with $\mathrm{rk}(\alpha) = s$ and $L \in \mathrm{Pic}(S)$ we have that

$$\int_{[M_S^H(\rho, c_1, c_2)]^{\mathrm{vir}}} c(\alpha_M) \exp(\mu(L) + u\,\mu(pt))$$

is the coefficient of $z^{\frac{1}{2}\mathrm{vd}(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_S)+K_S^2}\, V_s^{c_2(\alpha)}\, W_s^{c_1(\alpha)^2}\, X_s^{\chi(\mathcal{O}_S)}\, e^{L^2 Q_s + (c_1(\alpha)L)R_s + u\,T_s}$$

$$\sum_{J \subset [\rho-1]} (-1)^{|J|\chi(\mathcal{O}_S)}\, \varepsilon_\rho^{\|J\| K_S c_1}\, Y_{J,s}^{c_1(\alpha)K_S}\, Z_{J,s}^{K_S^2}\, e^{(K_S L) S_{J,s}}.$$

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With $z=t(1+(1-\frac{s}{\rho})t)^{1-\frac{s}{\rho}}$, we have

$$V_s(z)=(1+(1-\tfrac{s}{\rho})t)^{\rho-s}(1+(2-\tfrac{s}{\rho})t)^{s},$$

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$$Q_s(z)=\tfrac{1}{2}t(1+(1-\tfrac{s}{\rho})t),\quad R_s(z)=t,\quad T_s(z)=\rho t(1+\tfrac{1}{2}(1-\tfrac{s}{\rho})(2-\tfrac{s}{\rho})t).$$

Furthermore, $Y_{J,s}$, $Z_{J,s}$, $S_{J,s}$ are all algebraic functions

Conjecture

$\int_{[M]^{\mathrm{vir}}} c(\alpha_M)\exp(\mu(L)+u\,\mu(pt))$ is the coefficient of $z^{\frac{1}{2}\mathrm{vd}(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_S)+K_S^2}\, V_s^{c_2(\alpha)}\, W_s^{c_1(\alpha)^2}\, X_s^{\chi(\mathcal{O}_S)}\, e^{L^2 Q_s+(c_1(\alpha)L)R_s+u\,T_s}$$

$$\sum_{J\subset[\rho-1]}(-1)^{|J|\chi(\mathcal{O}_S)}\,\varepsilon_\rho^{\|J\|K_S c_1}\, Y_{J,s}^{c_1(\alpha)K_S}\, Z_{J,s}^{K_S^2}\, e^{(K_S L)S_{J,s}}.$$

With $z=t(1+(1-\frac{s}{\rho})t)^{1-\frac{s}{\rho}}$, we have

$$V_s(z)=(1+(1-\tfrac{s}{\rho})t)^{\rho-s}(1+(2-\tfrac{s}{\rho})t)^s,$$

$$W_s(z)=(1+(1-\tfrac{s}{\rho})t)^{\frac{1}{2}(s-1-\rho)}(1+(2-\tfrac{s}{\rho})t)^{\frac{1}{2}(1-s)},$$

$$X_s(z)=(1+(1-\tfrac{s}{\rho})t)^{\frac{1}{2}(s^2-(\rho+\frac{1}{\rho})s)}(1+(2-\tfrac{s}{\rho})t)^{-\frac{1}{2}s^2+\frac{1}{2}}(1+(1-\tfrac{s}{\rho})(2-\tfrac{s}{\rho})t)^{-\frac{1}{2}},$$

$$Q_s(z)=\tfrac{1}{2}t(1+(1-\tfrac{s}{\rho})t),\quad R_s(z)=t,\quad T_s(z)=\rho t(1+\tfrac{1}{2}(1-\tfrac{s}{\rho})(2-\tfrac{s}{\rho})t).$$

Furthermore, $Y_{J,s}$, $Z_{J,s}$, $S_{J,s}$ are all algebraic functions

The fact that $R_s(z)=t$ explains the variable change: z counts the virtual dimension; t counts $c_1(\alpha)L$

For $S^{[n]}$ we have $Y_s=Y_{\emptyset,s}$, $Z_s=Z_{\emptyset,s}$

Determinant bundles: Let $c \in K(S)$ be the class of $E \in M = M_S^H(\rho, c_1, c_2)$ and $K_c := \{v \in K(S)\ :\ \chi(S, c \otimes v) = 0\}$
For $\alpha \in K_c$ put with $\pi_S : S \times M \to S$, $\pi_M : S \times M \to M$ projections

$$\lambda(\alpha) := \det\left(\pi_{M!}\left(\pi_S^*\alpha \cdot [\mathcal{E}]\right)\right)^{-1} \in \mathrm{Pic}(M)$$

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Fix $r \in \mathbb{Z}$, $L \in \mathrm{Pic}(S) \otimes \mathbb{Q}$ with $\mathcal{L} := L \otimes \det(c)^{-\frac{r}{\rho}} \in \mathrm{Pic}(S)$ take $v \in K_c$ such that $\mathrm{rk}(v) = r$ and $c_1(v) = \mathcal{L}$, put

$$\mu(L) \otimes E^{\otimes r} := \lambda(v) \in \mathrm{Pic}(M).$$

On $M_S^H(1, 0, n) \cong S^{[n]}$ this is previous definition of $\mu(L) \otimes E^{\otimes r}$
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Relation to Donaldson μ class in cohom.: $\mu(c_1(L)) = c_1(\mu(L))$
Denote by $\mathcal{O}_M^{\mathrm{vir}}$ the virtual structure sheaf of M
The *virtual Verlinde numbers* of S are the virtual holomorphic Euler characteristics

$$\chi^{\mathrm{vir}}(M, \mu(L) \otimes E^{\otimes r}) := \chi(M, \mu(L) \otimes E^{\otimes r} \otimes \mathcal{O}_M^{\mathrm{vir}})$$

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and S has a smooth connected canonical divisor
Write $\varepsilon_\rho := \exp(2\pi i/\rho)$ and $[n] := \{1, \dots, n\}$. For any $J \subset [n]$, write $|J|$ for its cardinality and $\|J\| := \sum_{j \in J} j$

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Conjecture

Let $\rho \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$. There exist $A_{J,r}, B_{J,r} \in \mathbb{C}[[w^{\frac{1}{2}}]]$ for all $J \subset [\rho - 1]$ such that $\chi^{\mathrm{vir}}(M, \mu(L) \otimes E^{\otimes r})$ equals the coefficient of $w^{\frac{1}{2}\mathrm{vd}(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_S)+K_S^2}\, G_r^{\chi(L)}\, F_r^{\frac{1}{2}\chi(\mathcal{O}_S)} \sum_{J \subset [\rho-1]} (-1)^{|J|\chi(\mathcal{O}_S)}\, \varepsilon_\rho^{\|J\| K_S c_1}\, A_{J,r}^{K_S L}\, B_{J,r}^{K_S^2}.$$

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Here $G_r(w) = 1 + v, \quad F_r(w) = \dfrac{(1+v)^{\frac{r^2}{\rho^2}}}{1 + \frac{r^2}{\rho^2} v}$ *with* $w = v(1+v)^{\frac{r^2}{\rho^2}-1}$
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Furthermore, $A_{J,r}$, $B_{J,r}$ are all algebraic functions.

This conjecture is true for $K3$ surfaces

Virtual Serre duality $\chi^{\mathrm{vir}}(M, L) = (-1)^{\mathrm{vd}(M)} \chi^{\mathrm{vir}}(M, K_M^{\mathrm{vir}} \otimes L^{-1})$ gives

Conjecture

For any $\rho > 0$, we have

$$\frac{B_{J,-r}(w^{\frac{1}{2}})}{B_{J,r}(-w^{\frac{1}{2}})} = G_r(w)^{\binom{\rho}{2}} A_{J,r}(-w^{\frac{1}{2}})^{\rho}$$

$$A_{J,-r}(w^{\frac{1}{2}}) = \frac{1}{A_{J,r}(-w^{\frac{1}{2}}) G_r(w)^{\rho-1}}$$

for all $J \subset [\rho - 1]$ and $r \in \mathbb{Z}$.

We get the following analogue of the Segre-Verlinde correspondence for Hilbert schemes

Conjecture

For any $\rho \in \mathbb{Z}_{>0}$ *and* $r \in \mathbb{Z}$, *for all* $J \subset [\rho - 1]$, *we have*

$$A_{J,r}(w^{\frac{1}{2}}) = W_{\rho-r}(z)\, Y_{J,\rho-r}(z^{\frac{1}{2}}), \quad B_{J,r}(w^{\frac{1}{2}}) = Z_{J,\rho-r}(z^{\frac{1}{2}}),$$

with

$$w = v(1+v)^{\frac{r^2}{\rho^2}-1}, \quad z = t(1+(1-\tfrac{s}{\rho})t)^{1-\frac{s}{\rho}}, \quad v = t(1+\tfrac{r}{\rho}t)^{-1}.$$

We conjecturally determined the $Y_{J,s}$, $Z_{J,s}$, $S_{J,s}$ as algebraic functions for $\rho = 2$, $s = -1, \dots, 5$, $\rho = 3$, $s = 0, \dots, 6$, and $\rho = 4$, $s = 0, 4$, and the $A_{J,s}$, $B_{J,s}$ corresponding to them under the Segre-Verlinde correspondence

We conjecturally determined the $Y_{J,s}$, $Z_{J,s}$, $S_{J,s}$ as algebraic functions for $\rho = 2$, $s = -1, \ldots, 5$, $\rho = 3$, $s = 0, \ldots, 6$, and $\rho = 4$, $s = 0, 4$, and the $A_{J,s}$, $B_{J,s}$ corresponding to them under the Segre-Verlinde correspondence
Below we list these functions for $M_S^H(\rho, c_1, c_2)$, and $\mathrm{rk}(\alpha) = s$ with
$\rho = 2$, $s = 1, -1$ $\quad \rho = 3$, $s = 1$, $\quad \rho = 4$, $s = 0$.

Rank $\rho = 2$

for $W = Y, Z, S$ we have $W_{\emptyset,s}(-z^{\frac{1}{2}}) = W_{\{1\},s}(z^{\frac{1}{2}})$
We write $W_s := W_{\emptyset,s}$

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s=1: For $z = t(1 + \frac{1}{2}t)^{\frac{1}{2}}$, we have

$$Y_1 = (1+t)+t^{\frac{1}{2}}(1+\frac{3}{4}t)^{\frac{1}{2}}, \quad Z_1 = \frac{1 + \frac{3}{4}t - \frac{1}{2}t^{\frac{1}{2}}(1 + \frac{3}{4}t)^{\frac{1}{2}}}{1 + \frac{1}{2}t}, \quad S_1 = -\frac{1}{2}t+t^{\frac{1}{2}}(1+\frac{3}{4}t)^{\frac{1}{2}}.$$

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s=-1: For $z = t(1+\frac{3}{2}t)^{\frac{3}{2}}$ put

$$x = \frac{Y_{-1}}{(1+\frac{3}{2}t)^2}, \quad y = \frac{(1+\frac{3}{2}t)^3 Z_{-1}}{Y_{-1}^2}$$

They are solutions of

$$t^4x^4 - 2t^2(1+2t)x^3 + (1+\frac{3}{2}t)^2x^2 - 2(1+2t)x + 1 = 0$$
$$y^4 - 2(1+\frac{15}{4}t)y^3 + (1+\frac{5}{2}t)(1+\frac{15}{4}t)y^2 - t(1-\frac{5}{2}t)(1+\frac{15}{4}t)^2 = 0$$

Rank $\rho = 3$

s=1: For $z = t(1+\frac{2}{3}t)^{\frac{2}{3}}$, the power series $S_{\varnothing,1}$, $S_{\{1,2\},1}$, $S_{\{1\},1}$, $S_{\{2\},1}$ are the four solutions of

$$x^4 + 2tx^3 - (3t+t^2)x^2 - (3t^2+2t^3)x - (t^3 + \tfrac{2}{3}t^4) = 0.$$

$Y_{\varnothing,1}$, $Y_{\{1,2\},1}$, $Y_{\{1\},1}$, $Y_{\{2\},1}$ are the four solutions of

$$x^4 - (4+\tfrac{17}{3}t)(1+\tfrac{2}{3}t)^{\frac{1}{2}}x^3 + (6+18t+16t^2+\tfrac{31}{9}t^3)x^2 - (4+\tfrac{17}{3}t)(1+\tfrac{2}{3}t)^{\frac{7}{2}}x + (1+\tfrac{2}{3}t)^6 = 0.$$

$Z_{\varnothing,1}$, $Z_{\{1,2\},1}$, $Z_{\{1\},1}$, $Z_{\{2\},1}$ are the four solutions of

$$x^4 - 6\frac{1+\frac{10}{9}t}{1+\frac{2}{3}t}x^3 + \frac{(13+\frac{58}{3}t+\frac{55}{9}t^2)(1+\frac{10}{9}t)}{(1+\frac{2}{3}t)^3}x^2 + \frac{(4+\frac{5}{3}t)(1+\frac{10}{9}t)^2}{(1+\frac{2}{3}t)^3}(-3x+1) = 0.$$

Rank $\rho = 4$

s=0: For $z = t(1+t)$, we conjecturally have

$$S_{\varnothing,0} = (1+2^{\frac{1}{2}})t^{\frac{1}{2}}(1+t)^{\frac{1}{2}}, \quad S_{\{1\},0} = t^{\frac{1}{2}}(1+t)^{\frac{1}{2}},$$

$$Z_{\varnothing,0} = 2(2+2^{\frac{1}{2}}), \quad Z_{\{1\},0} = 2,$$

$$Y_{\varnothing,0} = \frac{(1+t)^4((1+t)^{\frac{1}{2}}+t^{\frac{1}{2}})}{(1+2t)^{\frac{1}{2}}((1+2t)-2^{\frac{1}{2}}t^{\frac{1}{2}}(1+t)^{\frac{1}{2}})},$$

$$Y_{\{1\},0} = \frac{(1+t)^4((1+t)^{\frac{1}{2}}+t^{\frac{1}{2}})}{(1+2t)^{\frac{1}{2}}(1+(1-i)t)}.$$

The other power series are obtained by $z^{\frac{1}{2}} \mapsto -z^{\frac{1}{2}}$, $2^{\frac{1}{2}} \mapsto -2^{\frac{1}{2}}$ and $i \mapsto -i$

Main tool: Mochizuki's formula:
Compute intersection numbers on $M = M_S^H(\rho, c_1, c_2)$ in terms of intersection numbers on Hilbert scheme of points.

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On $S \times M$ have $\mathcal{E}$ universal sheaf
i.e. if $[E] \in M$ corresponds to a sheaf E on S then $\mathcal{E}|_{S\times[E]} = E$.
For $\alpha \in H^k(S)$, put

$$\tau_i(\alpha) := \pi_{M_*}(c_i(\mathcal{E})\pi_S^*(\alpha)) \in H^{2i-4+k}(M)$$

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Let $P(\mathcal{E})$ be any polynomial in the $\tau_i(\alpha)$
Mochizuki's formula expresses $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$ in terms of intersec. numbers on $S^{[n_1]} \times S^{[n_2]} \times \ldots \times S^{[n_\rho]}$, and Seiberg-Witten invariants.

$$\int_{[M]^{\mathrm{vir}}} c(\alpha_M)\exp(\mu(L)+u\mu(pt)), \quad \chi^{\mathrm{vir}}_{-y}(M,\mu(L)+E^{\otimes r})$$

can both be expressed as $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$, for suitable polyn. P, so can reduce computation to Hilbert schemes.

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For $\chi^{\mathrm{vir}}_{-y}(M,\mu(L)+E^{\otimes r})$ use **virtual Riemann-Roch formula**

Theorem (Fantechi-G., Kapronov Ciocan-Fontanine)

For $V\in K^0(M)$ have

$$\chi^{\mathrm{vir}}(M,V)=\int_{[M]^{\mathrm{vir}}}\mathrm{ch}(V)\mathrm{td}(T^{\mathrm{vir}}_M).$$

Seiberg-Witten invariants:
invariants of differentiable 4-manifolds
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In general for alg. surfaces they are easy to compute, e.g. if $b_1(S) = 0$, $p_g(S) > 0$ and $|K_S|$ contains smooth connected curve, then SW cl. of S are $0, K_S$ with

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This is the reason for assumption $|K_S|$ contains smooth connected curve, otherwise our results look more complicated: They are expressed in terms of the Seiberg-Witten inv. of S

For simplicity look at case $\rho = 2$, $s = 1$, $\alpha = L \in \mathrm{Pic}(S)$ and compute $\int_{[M]^{\mathrm{vir}}} c(L_M)$.

$S^{[n_1]} \times S^{[n_2]} = \{$pairs (Z_1, Z_2) of subsch. of deg. (n_1, n_2) on $S\}$

Work on $S \times S^{[n_1]} \times S^{[n_2]}$, projection p to $S^{[n_1]} \times S^{[n_2]}$

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Two universal sheaves: Let $a \in \mathrm{Pic}(S)$

1. $\mathcal{I}_i(a)$ sheaf on $S \times S^{[n_1]} \times S^{[n_2]}$ with $\mathcal{I}_i(a)|_{S\times(Z_1,Z_2)} = I_{Z_i} \otimes a$
2. $\mathcal{O}_i(a)$, vector bundle of rank n_i on $S^{[n_1]} \times S^{[n_2]}$, with fibre $\mathcal{O}_i(a)(Z_1, Z_2) = H^0(\mathcal{O}_{Z_i} \otimes a)$

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For a vector bundle E of rank r and variable s put

$$c_i(E \otimes s) = \sum_{k=0}^{i} \binom{r-i}{k} s^{i-k} c_k(E), \quad Eu(E) = c_r(E)$$

Mochizuki formula

For sheaves $\mathcal{E}_1, \mathcal{E}_2$ on $S \times S^{[n_1]} \times S^{[n_2]}$ put

$$Q(\mathcal{E}_1, \mathcal{E}_2) = Eu(-RHom_p(\mathcal{E}_1, \mathcal{E}_2) - RHom_p(\mathcal{E}_2, \mathcal{E}_1))$$

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For $a_1, a_2 \in \mathrm{Pic}(S)$ put

$$\Psi(a_1, a_2, L, n_1, n_2, s) = \frac{P(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s) Eu(\mathcal{O}_1(a_1)) Eu(\mathcal{O}_2(a_2) \otimes s^2}{Q(\mathcal{I}_1(a_1) \otimes s^{-1}, \mathcal{I}_2(a_2) \otimes s) \cdot (2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}$$

$$A(a_1, a_2, L, c_2, s) = \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[s, s^{-1}]$$

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In our case $\int_{[M]^{\mathrm{vir}}} c(L_M)$, we have essentially

$$P(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s) = c(\mathcal{O}_1((a_1-a_2)/2+L) \otimes s^{-1}) c(\mathcal{O}_2((a_2-a_1)/2+L) \otimes s)$$

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$$Q(\mathcal{E}_1, \mathcal{E}_2) = Eu(-RHom_p(\mathcal{E}_1, \mathcal{E}_2) - RHom_p(\mathcal{E}_2, \mathcal{E}_1))$$

For $a_1, a_2 \in \mathrm{Pic}(S)$ put

$$\Psi(a_1, a_2, L, n_1, n_2, s) = \frac{P(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s) Eu(\mathcal{O}_1(a_1)) Eu(\mathcal{O}_2(a_2) \otimes s^2}{Q(\mathcal{I}_1(a_1) \otimes s^{-1}, \mathcal{I}_2(a_2) \otimes s) \cdot (2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}$$

$$A(a_1, a_2, L, c_2, s) = \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[s, s^{-1}]$$

In our case $\int_{[M]^{\mathrm{vir}}} c(L_M)$, we have essentially

$$P(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s) = c(\mathcal{O}_1((a_1-a_2)/2+L) \otimes s^{-1}) c(\mathcal{O}_2((a_2-a_1)/2+L) \otimes s)$$

Theorem (Mochizuki)

Assume $\chi(E) > 0$ for $E \in M_H^S(c_1, c_2)$. Then

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} P(\mathcal{E}) = \sum_{\substack{c_1=a_1+a_2 \\ a_1H<a_2H}} SW(a_1) \mathrm{Coeff}_{s^0} A(a_1, a_2, L, c_2, s)$$

Universality: Put

$$Z_S(a_1, a_2, L, s, q) = \sum_{n_1, n_2 \geq 0} \int_{S^{[n_1]} \times S^{[n_2]}} A(a_1, a_2, L, a_1 a_2 + n_1 + n_2, s) q^{n_1+n_2}$$

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Proposition

There exist univ. functions $A_1(s,q),\dots,A_{11}(s,q)\in\mathbb{Q}[s,s^{-1}][[q]]$ *s.th.* $\forall_{S,a_1,a_2,L}$

$$Z_S(a_1,a_2,L,s,q)=F_0(a_1,a_2,L,s)A_1^{a_1^2}A_2^{a_1a_2}A_3^{a_2^2}A_4^{a_1K_S}A_5^{a_2K_S}A_6^{K_S^2}A_7^{\chi(\mathcal{O}_S)}\cdot A_8^{L^2}A_9^{LK_S}A_{10}^{La_1}A_{11}^{La_2},$$

(where $F_0(a_1,a_2,L,s)$ *is some explicit elementary function).*

Proof: Modification of the cobordism argument for Hilbert schemes of points

$A_1(s,q),\dots A_{11}(s,q)$ are determ. by value of $Z_S(a_1,a_2,L,s,q)$ for 11 triples (S,a_1,a_2,L) (S surface, $a_1,a_2,L\in \mathrm{Pic}(S)$) s.th. corresponding 11-tuples

$$(a_1^2, a_1a_2, a_2^2, a_1K_S, a_1K_S, K_S^2, \chi(O_S)), L^2, LK_S, La_1, La_1)$$

are linearly independent.

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$(\mathbb{P}^2,\mathcal{O},\mathcal{O},\mathcal{O}),(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O},\mathcal{O},\mathcal{O}),(\mathbb{P}^2,\mathcal{O}(1),\mathcal{O},\mathcal{O}),(\mathbb{P}^2,\mathcal{O},\mathcal{O}(1),\mathcal{O}),$
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In this case S is a smooth toric, i.e. have an action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints,
Action of T lifts to action on $S^{[n]}$ still with finitely many fixpoints described by partitions, compute by equivariant localization.
This computes $Z_S(a_1, a_2, L, s, q)$ in terms of combinatorics of partitions.

We determined $Z_S(a_1, a_2, \ldots, a_\rho, \alpha, L, s, q)$

- for $\rho = 2$ modulo q^{11}
- for $\rho = 3$ modulo q^9
- for $\rho = 4$ modulo q^8

This shows the conjectures e.g. for the blowup of a K3 surface in a point for

- for $\rho = 2$ up to virtual dimension 16
- for $\rho = 3$ up to virtual dimension 14
- for $\rho = 4$ up to virtual dimension 6

Let X be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$
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Fibre $E(p_i)$ of X at fixp. p_i has basis of eigenvect. for T-action $E(p_i) = \bigoplus_{k=1}^{r} \mathbb{C} v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

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Equivariant chern class of fibre at fixpoint:

$$c^T(E(p_i)) = (1+c_1^T(E(p_i))+\ldots+c_r^T(E(p_i)) = \prod_{i=1}^r (1+n_i\epsilon_1+m_i\epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$$

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Let $P(c(E)))$polynomial in Chern classes of E, of degree $d = \dim(X)$

Theorem (Bott residue formula)

$$\int_{[X]} P(c(E)) = \sum_{k=1}^{e} \frac{P(c^T(E(p_k)))}{c_d^T(T_X(p_k))}$$

(does not depend on ϵ_1, ϵ_2)

For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

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$Z \in (\mathbb{P}^2)^{[n]}$ is T-invariant $\Longrightarrow Z = Z_0 \sqcup Z_1 \sqcup Z_2 \quad supp(Z_i) = p_i$.
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Easy: Z is T-invariant $\iff I_Z \in k[x, y]$ is gen. by monomials
Can write

$$I_Z = (y^{n_0}, xy^{n_1}, ..., x^r y^{n_r}, x^{r+1}) \quad (n_0, \ldots, n_r) \text{ partition of } n$$

Fixpoints on $(\mathbb{P}^2)^{[n]}$ are in bijections with triples (P_0, P_1, P_2) of partitions of 3 numbers adding up to n.

Need to compute things like $c(\mathcal{O}^{[n]})$
$\mathcal{O}^{[n]}$ vector bundle on $(\mathbb{P}^2)^{[n]}$ with fibre $\mathcal{O}^{[n]}(Z) = H^0(\mathcal{O}_Z)$

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Then the fibre $\mathcal{O}^{[n]}(Z) = H^0(\mathcal{O}_Z) = \mathbb{C}[x,y]/(y^4, xy^2, x^2y, x^3)$
Thus basis of eigenvectors of fibre for T action is

$$\begin{matrix} 1 & y & y^2 & y^3 \\ x & xy & & \\ x^2 & & & \end{matrix} \quad \text{with eigenvalues} \quad \begin{matrix} 1 & t_2 & t_2^2 & t_2^3 \\ t_1 & t_1 t_2 & & \\ t_1^2 & & & \end{matrix}$$

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Thus

$$c^T(\mathcal{O}^{[n]}(Z)) = (1+\epsilon_2)(1+2\epsilon_2)(1+3\epsilon_2)(1+\epsilon_1)(1+\epsilon_1+\epsilon_2)(1+2\epsilon_1).$$