$A_g$, $M_g^c$, and $\text{Hilb}(C^2,d)$

Rahul Pandharipande
ETH Zürich

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Extended notes updated
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joint with
S. Canning
S. Molcho
D. Oprea
A. Pixton
H. H. Tseng
A. Triñar López

including results of
F. Greer - C. Lian
Various compactifications,

Perhaps no winner yet,

but Satake is convenient
I. Moduli of abelian varieties

\[ \text{Sp}(2g, \mathbb{R}) \sim \mathcal{H}_g \quad \text{Siegel upper half space (contractible)} \]

\[ A_g = \mathcal{H}_g \left/ \text{Sp}(2g, \mathbb{R}) \right. \]

model for
\[ B \text{Sp}(2g, \mathbb{R}) \]
up to finite Stabilizers

we have:
\[ H^* (A_g) = H^* \text{Sp}(2g, \mathbb{R}) , \]
all cohomology taken with \( \mathbb{Q} \)-coefficients.
IE is defined by

\[ \text{Abelian variety of } \dim g \]

Then \( \text{IE} = \Delta^* (\Omega^1_\pi) \)

\[ \text{rank } g \]

Lambda classes: \( \lambda_i = c_i (\text{IE}) \),

\[ \text{ch } R_{\pi_*} \Theta_x = \text{ch } (1 - \text{IE}^v + \wedge^2 \text{IE}^v \cdots) \].

Borel 1974: Stable cohomology of \( \text{Sp}(\mathbb{Z}) \)

generated by \( \lambda_i \).
Following van der Geer, define tautological classes:

- \( \mathcal{R} H^*(A_g) \subset H^*(A_g) \) \text{ cohomology}
  - subalgebra generated by all \( \lambda_i = c_i(\mathcal{E}) \),

- \( \mathcal{R}^*(A_g) \subset C H^*(A_g) \) \text{ algebraic cycles}
  - subalgebra generated by all \( \lambda_i = c_i(\mathcal{E}) \).

**Theorem (van der Geer 1996)**

\[ \mathcal{R} H^*(A_g) = \mathcal{R}^*(A_g) \text{ with presentation} \]

\[ \mathbb{Q}[\lambda_1, \ldots, \lambda_g] \]

\[ (\lambda_g = 0, \ c(\mathcal{E} \oplus \mathcal{E}^*) = 1) \]

\[ (1+\lambda_1+\lambda_2+\cdots+\lambda_g) \cdot (1-\lambda_1+\lambda_2-\cdots-(-1)^g \lambda_g) = 1. \]
As a consequence, \( H^*(A_g) \) is a Gorenstein ring with socle

\[ H^{(g)}(A_g) \cong \mathbb{Q} \cdot \lambda_1^{(g)} \cong \mathbb{Q} \cdot \lambda_1 \lambda_2 \ldots \lambda_{g-1}. \]

Many open questions:

1. Calculate \( H^*_\text{Sp}(2g, \mathbb{Z}) \) in unstable ranges
2. Calculate \( CH^*(A_g) \)
3. Calculate \( H^*_\text{Sp}(2g, \mathbb{Z}) \) with \( \mathbb{Z} \)-coefficients

all very difficult. We will go in a different direction.
We have $R^*(A_j) \subset CH^*(A_j)$

and $R^*(M_{g_j}^{ct}) \subset CH^*(M_{g_j}^{ct})$.

Is there a canonical projection

$CH^*(M_{g_j}^{ct}) \xrightarrow{Pr_{M_{g_j}^{ct}}} R^*(M_{g_j}^{ct})$ ?

There is no proposal for,

but for $CH^*(A_j) \xrightarrow{Pr_A} R^*(A_j)$,

we believe $Pr_A$ exists!
The idea uses compactification:

\[ A_g \subset A_g^{pc} \quad \text{Perfect Core Compactification} \]

Some facts:

(i) Hodge bundle extends canonically

\[ IE \subset IE \]

\[ \downarrow \quad \downarrow \]

\[ A_g \subset A_g^{pc} \]

(ii) \( H^*(A_g^{pc}) \) def \( \text{Subalgebra generated by all} \)

\[ \lambda_i = c_i(IE) \]
Theorem (van der Geer 1996)

• \( R^*(A_g^{pc}) = \frac{\mathbb{Q}[\lambda_1, \ldots, \lambda_g]}{(c(E \otimes E^e) = 1)} \)

\[ (1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g) \cdot (1 - \lambda_1 + \lambda_2 - \cdots - (i)\lambda_g) = 1. \]

• \( R^*(A_g^{pc}) \) is a Gorenstein ring

with socle

\[ R^{(g^+)}_{(2)}(A_g^{pc}) \cong \mathbb{Q} \cdot \lambda_1^{(g^+)} \cong \mathbb{Q} \cdot \lambda_1 \lambda_2 \cdots \lambda_{g-1} \lambda_g. \]

Using integration on \( A_g^{pc} \)

and the duality of \( R^*(A_g^{pc}) \),
We define a projection:

\[ \text{CH}^*(A^\text{pc}_j) \xrightarrow{\text{Pr}_{A^\text{pc}j}} \mathcal{R}^*(A^\text{pc}_j) \]

\[ \text{Pr}_{A^\text{pc}j} (\alpha \in \text{CH}^*(A^\text{pc}_j)) = \beta \in \mathcal{R}^*(A^\text{pc}_j) \]

where \( \forall \gamma \in \mathcal{R}^*(A^\text{pc}_j), \)

\[ \int_{A^\text{pc}_j} \alpha \cdot \gamma = \int_{A^\text{pc}_j} \beta \cdot \gamma, \]

\( \beta \) exists and is unique.

\[ \text{Pr}_{A^\text{pc}j} \text{ is a projection} \]
Idea for constructing

\[ \text{CH}^*(A_j) \xrightarrow{\text{Pr}_A} \mathbb{R}^*(A_j) \]

via \( \text{Pr}_{A,pc} \):

\[ \text{Pr}_A (\alpha \in \text{CH}^*(A_j)) = \text{Pr}_{A,pc} (\overline{\alpha} \in \text{CH}^*(A_{j,pc})) \]

but closure not canonical!

Not clear that \( \text{Pr}_A \) is well defined.

Conjecture (Canning-Oprea-P 2023):

\[ \lambda_g \mid_{A_{j,pc}, A_j} = 0 \in \text{CH}^*(A_{j,pc}, A_j) \]

Conjecture \( \Rightarrow \) \( \text{Pr}_A \) is well defined.
Using the Conjecture, there is another path to the projection:

for $\alpha \in \text{Ch}^*(A_g)$ and $\gamma \in R^*(A_g)$, define a pairing

\[
\langle \alpha, \gamma \rangle_{A_g} = \int \bar{\alpha} \cdot \gamma \cdot \lambda_g_{A_g}
\]

Conjecture $\Rightarrow$ $\langle \alpha, \gamma \rangle_{A_g}$ is well defined.

Exercise: $\langle \alpha, \gamma \rangle_{A_g} = \langle Pr_A(\alpha), \gamma \rangle_{A_g}$

$\forall \alpha \in \text{Ch}^*(A_g)$ and $\forall \gamma \in R^*(A_g)$
Update (19 November 2023):

The vanishing conjecture

\[ \alpha_g | \mathcal{A}_g^{pc} = 0 \in \mathcal{CH}^*(\mathcal{A}_g^{pc}, \mathcal{A}_g) \]

is true!

- Argument by Sam Molcho constructing trivial quotients of IE on the boundary via residue maps.
- Another path suggested by Ben Moonen using boundary geometry and rigidity results from Faltings–Chai.

Proofs work for all sufficiently fine toroidal compactifications: \( \mathsf{Pr}_A \) exists and is independent of choice of compactification.

arXiv: 2401.15768
III. Noether–Lefschetz loci

The simplest NL locus to consider is

$$A_1 \times A_{g-1} \to A_g.$$  

We assume $g \geq 2$

We define a twisted generalization by the following construction:

Let $NL_d \to A_g$ be the $d \geq 1$

module of pairs:

$$NL_d \ni \left[ E \to \chi \right]$$

Condition: $E \cdot \Theta = d$

theta divisor of $\chi$

1 dim Subgroup, PPAV of dimension $g$

an elliptic curve
In case \( d = 1 \), \( \text{NL}_1 = A_1 \times A_{g-1} \).

Easy to see:

\[
\dim \text{NL}_d = \dim A_g - (g-1)
\]

so \( [\text{NL}_d] \in CH^{g-1}(A_g) \).

The main topic of the lecture is the computation

\[
\Pr_A ([\text{NL}_d]) \in R^*_A(A_g).
\]

There are 2 immediate issues:

- \( \Pr_A \) depends on the vanishing Conjecture.
- Even if we assume the Conjecture, it is not clear how to integrate the classes of the closures in \( A^\text{pc}_g \).
Theorem (Cannings-Opred-P 2023):

If the vanishing conjecture holds,

\[ P_{\mathcal{A}} ( [ A_1 \times A_{g,1} ] ) = \frac{g}{6 |B_{2g}|} \lambda_{g-1}. \]

An interesting direction:

What is the projection of the Schottky locus,

\[ P_{\mathcal{A}} ( \text{Tor}^* [ M_g^{ct} ] ) \in \Lambda^* (A_g) ? \]
A different question

We consider now a different question:

\[ [NL_d] \in R^*(A_g), \]

What could it be?

- Since \( P_{r_A} \) is well-defined, the answer to the question is:

\[ [NL_d] \in R^*(A_g) \Rightarrow [NL_d] = P_{r_A}([NL_d]). \]
Proposition (Camenik-Opredel-P 2023):

If \([NL_d] \in R^*(A_g)\), then

\[
\begin{bmatrix}
NL_d
\end{bmatrix} = c_{g,d} \cdot \lambda_{g-1}.
\]

\(\uparrow\)

Scalar in \(\mathbb{Q}\)

Proof: We have \([NL_d] \in R^*(A_g)\).

Moreover \(\lambda_{g-1} \cdot [NL_d] = 0\),

Since \(NL_d \in \begin{bmatrix}
E & \chi
\end{bmatrix}\)

\(\downarrow\)

\[
A_1 \times A_{g-1}^{\text{Pol}} \ni \begin{bmatrix}
E
\end{bmatrix} \times \begin{bmatrix}
\chi \\
E
\end{bmatrix}
\]

\(\uparrow\) non principal polarization
and \( \lambda_{g-1} \mid \mathcal{N}_d \) is pulled-back from \( \lambda_{g-1} \mid A_1 \times A_{g-1}^{\text{pol}} \),

and \( \lambda_{g-1} \mid A_1 \times A_{g-1}^{\text{pol}} = 0 \)

because \( c(1E) \mid A_1 = 0 \)

and \( \lambda_{g-1} \mid A_{g-1}^{\text{pol}} = 0 \).

Finally, \( Q \cdot \lambda_{g-1} c \in R^{g-1}(A_g) \)

is the annihilator of \( \lambda_{g-1} \) in \( R^{g-1}(A_g) \). \( \square \)
I. Integration

We have seen

\[ [NL_d] \in R^*(A_g) \Rightarrow [NL_d] = c_{g,d} \cdot \lambda_{g-1}. \]

The question is now what is the scalar \( c_{g,d} \)?

The idea is to pull-back via Torelli:

\[ \text{Tor} : M_g^{ct} \rightarrow A_g , \]

\[ \text{Tor}^* \left( [NL_d] \right) \in R^{g-1}(M_g^{ct}) , \]

\[ \text{Tor}^* \left( \lambda_{g-1} \right) \in R^{g-1}(M_g^{ct}) . \]
Then we can calculate $c_{g,d}$ by the $\lambda_g$ - evaluation on $M_{g}^{ct}$:

$$\int_{\overline{M}_{g}} \text{Tor}^* \left( \left[ NL_d \right] \right) \cdot \lambda_{g-2} \lambda_g = c_{g,d}$$

$$\int_{\overline{M}_{g}} \lambda_{g-2} \lambda_{g-1} \lambda_g$$

Computed by Faber-P (1999)

$$\int_{\overline{M}_{g}} \lambda_{g-2} \lambda_{g-1} \lambda_g = \frac{1}{2 \cdot (2g-2)!} \frac{1}{2g} \frac{1}{2^{g-2}}$$
But how are we going to calculate

\[
\int \text{Tor}^* \left( \left[ NL_d \right] \right) \cdot \mathfrak{g}_{g-2} \mathfrak{g}_g \ ?
\]

\[\bar{\mathcal{M}}_g\]

This requires a miracle provided by stable maps and the quantum cohomology of \(\text{Hilb}(\mathbb{C}^2, d)\).
VI Stable maps

Consider the fiber product:

\[ \text{Tor}^{-1}(NL_d) \rightarrow NL_d \rightarrow M_g^{ct} \rightarrow \text{Tor} \rightarrow A_g \]

We will add a marked point:

\[ \text{Tor}_i^{-1}(NL_d) \rightarrow NL_d \rightarrow M_{g,i}^{ct} \rightarrow \text{Tor}_i \rightarrow A_g \]
\[(2g-2) \cdot \int_{\overline{M}_g} \overline{\text{Tor}}^* \left( \left[ N_{L_d} \right] \right) \cdot \mathfrak{a}_{g-2} \mathfrak{a}_g\]

\[
\ll
\int_{\overline{M}_{g,1}} \overline{\text{Tor}}_{i}^* \left( \left[ N_{L_d} \right] \right) \cdot \psi_i \cdot \mathfrak{a}_{g-2} \mathfrak{a}_g
\]

Since on \( M_{g,1}^{ct} \), we have

\[
\overline{\text{Tor}}_{i}^* \left( \left[ N_{L_d} \right] \right) = \alpha^* \left( \overline{\text{Tor}}^* \left( \left[ N_{L_d} \right] \right) \right)
\]

where \( \alpha : M_{g,1}^{ct} \overset{\text{forgetful}}{\longrightarrow} M_g^{ct} \).
Let $M_{1,1}^{ct} = M_{1,1}$ be the models of pointed nonsingular elliptic curves:

\[
\begin{array}{c}
\pi^{ct} \\
\downarrow \\
M_{1,1}
\end{array}
\]

Let $M_{g,1}^{ct}(\pi^{ct}, d) \xrightarrow{ev} M_{g,1}^{ct}$ be the Grothendieck $\pi$-relative space of stable maps to the fibers of $\pi$. 
$\mathcal{M}_{g,1}^{ct} (\pi_{ct}, d)$ has a virtual class of dimension marked point

\[ \text{virdim} = 1 + 1 + 2g - 2 = 2g. \]

\[ \dim \mathcal{M}_{g,1} \]

unpointed maps to a fixed elliptic fiber

$\text{Tor}_i^* \left( \left[ N \mathcal{L}_d \right] \right)$ is an intersection class on $\text{Tor}_i^* (N \mathcal{L}_d)$ of dimension

\[ \text{virdim} = 3g - 3 + 1 - (g - 1) = 2g - 1. \]

\[ \dim \mathcal{M}_{g,1}^{ct} \]

\[ \text{codim of } N \mathcal{L}_d \]
compact type
\[
\left[ f : (C, p) \to (E, q) \right] \in \mathcal{M}_{g,1}^{ct} (\pi^ct, d)
\]

there is a discrete invariant:
\[ f^* : \text{Jac}_o(E) \to \text{Jac}_o(C), \]

\[
\text{Jdeg} f = \frac{d}{|\ker f^*|}.
\]

\[ \text{Jdeg}_f \in \{ 1, 2, \ldots, d \} \text{ must divide } d \]

and is a discrete invariant of \( f \),

\[
\mathcal{M}_{g,1}^{ct} (\pi^ct, d) = \left/ \mathcal{M}_{g,1}^{ct} (\pi^ct, d) \right. \big/ \text{Jdeg} \hat{d}.
\]

maps with \( \text{Jdeg} f = \hat{d} \)
Another way to think about $\deg f$:

A degree $d$ stable map

$$f : (C, p) \to (E, q)$$

Compact type

factors uniquely as

$$(C, p) \xrightarrow{g} (\hat{E}, \hat{q}) \xrightarrow{h} (E, q)$$

where $h : \hat{E} \to E$ is group homomorphism of elliptic curves

and $g^* : \text{Jac}_0(\hat{E}) \to \text{Jac}_0(C)$

is injective.
Then \( |\ker f^*| = \deg(h) \)

and \( d = \deg(g) \cdot \deg(h) \).

So we have

\[
\text{Jdeg } f = d \bigg/ |\ker f^*| = \deg(g).
\]

The disjoint decomposition

\[
\mathcal{M}_{g,1}^{\mathrm{ct}}(\pi^c, d) = \bigsqcup_{\text{Jdeg } \hat{d}} \mathcal{M}_{g,1}^{\mathrm{ct}}(\pi^c, d)^{\hat{d}}
\]

has principal part \( \mathcal{M}_{g,1}^{\mathrm{ct}}(\pi^c, d)^{\hat{d}} \).

The lower parts \( \mathcal{M}_{g,1}^{\mathrm{ct}}(\pi^c, d)^{\hat{d} < d} \)

can be studied via \( \mathcal{M}_{g,1}^{\mathrm{ct}}(\pi^c, \hat{d}) \).
Theorem (Cannings–Opredel-P, Pixton 2023)

There is an isomorphism of DM stacks

\[ \ev^{-1}(q)^d \cong \Tor_1^p(NL_d) \]

\[ \cap \]

\[ M_{g,1}^{ct}(\pi, d)^d \]

Here \( \ev^{-1}(q)^d \) is the locus of map where the evaluation of the marking on the domain equals the zero point \( q \) of the elliptic target.
To be useful we must also match the virtual classes:

\[
\text{vir dim } \text{ev}^{-1}(q)^d = \text{vir dim } \mathcal{M}_{g,1}^{\text{ct}}(\pi^\text{ct},d)^d - 1 = 2g - 1,
\]

\[
\left[\text{ev}^{-1}(q)^d\right]^{\text{ct,vir}} = \text{ev}^*(q) \cap \left[\mathcal{M}_{g,1}^{\text{ct}}(\pi^\text{ct},d)^d\right]^{\text{vir}}.
\]

Conjecture (Canning-Oprea-P, Pixton 2023)

under the above isomorphism,

\[
\left[\text{ev}^{-1}(q)^d\right]^{\text{ct,vir}} = \text{Tor}_i^\times \left( \left[ NL_d \right] \right).
\]
Update Feb 2024:
Francois Greer and Carl Lian can prove
\[
\left[ e V^{-1} ( q )^d \right]_{\text{ct,vir}}^\text{ct,vir} = \text{Tor}^*_1 \left( \left[ NL_d \right] \right)
\]

exactly in the required form using
a matching of obstruction theories.

Update April 2024:
The Greer - Lian proof can be found here:


arXiv: 2404.10826
An important property of

\[
\left[ \text{ev}^{-1}(q^d) \right]^{\text{ct, vir}} \in A_{2g-1}(\mathcal{M}_{g,1}^{\text{ct}}(\pi^ct, d^d))
\]

is the existence of a canonical extension to \( \overline{\mathcal{M}}_{g,1}(\pi^ct, d) \):

\[
\left[ \text{ev}^{-1}(q) \right]^{\text{vir}} \in A_{2g-1}(\overline{\mathcal{M}}_{g,1}(\pi, d))
\]

where \( \overline{\mathcal{M}}_{g,1}(\pi, d) \xrightarrow{\text{ev}} \mathcal{M}_{g,1} \xrightarrow{\pi} \mathcal{M}_{1,1} \)

over \( \partial \in \overline{\mathcal{M}}_{g,1} \), we have log stable maps
The complement
\[ \overline{\mathcal{M}}_{g,1}(π, d) \setminus \mathcal{M}_{g,1}^{ct}(π^{ct}, d) \]
is mapped by \( \mathcal{E} \) to the complement \( \overline{\mathcal{M}}_{g,1} \setminus \mathcal{M}_{g,1}^{ct} \).

Over \( \mathcal{M}_{g,1} \), this is by definition.

Over the point \([\mathcal{O}] \in \overline{\mathcal{M}}_{g,1} \),
the claim is more interesting:

there are no curves with compact type domains which map to \( \mathcal{O} \) with degree \( d \geq 1 \) by the definition of log maps.
We conclude:

\[
\int_{\overline{M}_{g,1}} \text{Tor}_* \left( [N \mathcal{L}_d] \right) \cdot \psi_1 \cdot \psi_{g-2} \psi_g \\
\int_{\overline{M}_{g,1}} \epsilon_* \left[ e_{\nu}^{-1}(q^d) \right]^{\text{vir}} \cdot \psi_1 \cdot \psi_{g-2} \psi_g \\
\int_{\overline{M}_{g,1}} \epsilon_* \left[ e_{\nu}^{-1}(q^d \cdot \psi_1) \right]^{\text{vir}} \cdot \psi_{g-2} \psi_g
\]

cotangent line now on $\overline{M}_{g,1} \left( \pi, d \right)$, no correction terms since there are no maps of positive degree $\mathbb{P}^1 \to E$. 
We now use the extension:

\[
\sum \ell^{-\deg \left( \frac{d}{\delta} \right)} \cdot \int_{\overline{M}_g, i} \left[ e_{\nu}^{-1}(q) \cdot \psi_i \right]^{\text{vir}} \cdot \mathfrak{g}_{g-2} \mathfrak{g}_d
\]

\[
\sum \ell^{-\deg \left( \frac{d}{\delta} \right)} \cdot \int_{\overline{M}_g, i} \left[ e_{\nu}^{-1}(q) \cdot \psi_i \right]^{\text{vir}} \cdot \mathfrak{g}_{g-2} \mathfrak{g}_d
\]

\[
\text{Count of } (E, \hat{q}) \rightarrow (E, q),
\]

\[
\sigma(x) = \sum_{\ell \mid x} \ell
\]
Hence the integrals

\[ \int_{\bar{M}_{g,1}} \epsilon_* \left[ \text{ev}^{-1}(q) \cdot \psi \right] \cdot \alpha_{g-2} \alpha_g^{\text{vir}} \]

and the integrals

\[ \int_{\bar{M}_{g,1}} \epsilon_* \left[ \text{ev}^{-1}(q) \cdot \psi \right] \cdot \alpha_{g-2} \alpha_g^{\text{vir}} \]

are related inductively by

a simple invertible transformation.
We will now calculate

$$\int_{\bar{M}_{g,1}} \left[ \text{ev}^{-1}(q) \cdot \psi \right]^{\text{vir}} \cdot \mathcal{A}_{g-2, \mathcal{A}_g}$$

$$\int_\mathcal{M}_{9,1} \mathcal{T}_i(q) \cdot \mathcal{A}_{g-2, \mathcal{A}_g}$$

$$\left[ \mathcal{M}_{9,1}(\pi, d) \right]^{\text{vir}}$$

using the idea of the

GW/\mathcal{H} correspondence Okounkov-P (2006)

A new issue is the families geometry.
\[
\int T_i(q) \, \mathfrak{g}_{g-2} \, \mathfrak{g}_g \quad [\overline{\mathcal{M}}_{g,1}(\pi, d)]^{\text{vir}} \quad \| \quad \langle T_i(q) \, \mathfrak{g}_{g-2} \, \mathfrak{g}_g \rangle_{g,d}^\pi \quad 0
\]

GW/H correspondence equation is found by degeneration of every fiber of

\[
\begin{array}{c}
\Xi \\
\pi \\
\downarrow \\
\overline{\mathcal{M}}_{g,1}
\end{array} \quad q
\]

to the normal cone of \( q \).
The resulting equation is

\[
\left< T_1 (g) \right| g_{g-2} g_g \right>_{g,d}^{\pi_0} = \frac{1}{2g} \sigma(d) \cdot (2g-2) \cdot \int \frac{c(E^\gamma)}{1 - \gamma} \frac{\mathcal{M}_{g-1,1}}{M_{g-1,1}}
\]

\[
+ \left< g_{g-2} g_g \right| (2) \right>_{g,d}^{\pi_0}
\]

\text{relative condition}

\text{integral evaluated to equal}

\[
\frac{|B_{2g-2}|}{(2g-2)(2g-2)!} \quad \text{(Faber-P 1999)}
\]
VIII

Quantum Cohomology of $\text{Hilb}(\mathbb{C}^2, d)$

\[
\frac{(t_1 + t_2)^2}{t_1 t_2} \sum \left< \frac{\alpha_{g-2, g}}{(2)} \right>_{g, e}^\pi \cdot \text{Part}(d-e) \\
2 \leq e \leq d
\]

\[
- \frac{1}{24} \frac{(t_1 + t_2)}{t_1 t_2} \sum \left< (2) \right>_{g, e}^{E \times \mathbb{C}^2} \cdot \overset{\sim}{\text{Part}}(d-e) \\
2 \leq e \leq d
\]

invertible relation

\[
\left< (2) \right>_{g, d}^{\pi \times \mathbb{C}^2}
\]

Possibly disconnected
t, t_2 weight on $\mathbb{C}^2$

(No degree 0 connected components)
The above relation is the Connected / disconnected equation (together with basic Hodge identities).

There are several terms to explain:

- \( \text{Part}(l) = \# \text{ of partitions of } l \)
  - \( \text{Part}(0) = 1 \)
  - \( \text{Part}(1) = 1 \)
  - \( \text{Part}(2) = 2 \)

A well-known property is

\[
\text{Hur}_E^l = \text{Part}(l) \quad \text{for } l \geq 1
\]

\( \text{Aut-weighted Count of} \)
\( \text{Possibly disconnected} \)
\( \text{unramified covers of } E = \varnothing \)
\( \text{of degree } l \)
\[ \overset{\sim}{\text{Part}}(l) \overset{\text{def}}{=} \overset{\sim}{\text{Hur}}_E^l \quad \text{for } l \geq 1 \]

\[ \overset{\sim}{\text{Part}}(0) = 0 \]

\[ \overset{\sim}{\text{Part}}(1) = 1 \]

\[ \overset{\sim}{\text{Part}}(2) = 1 + \frac{3}{2} = \frac{5}{2} \]

\[ \frac{1}{2} \cdot 1 \cdot 2 \]

\[ \overset{\sim}{\text{Part}} \quad \text{Aut-Weighted Count of} \]
\[ \overset{\sim}{\text{Connected Covers of }} E = \emptyset \]
\[ \overset{\sim}{\text{Possibly disconnected}} \]
\[ \overset{\sim}{\text{unramified covers of }} \]
\[ \overset{\sim}{\text{degree }} l \quad \text{where each cover} \]
\[ \overset{\sim}{\text{is weighted also by the}} \]
\[ \overset{\sim}{\text{number of connected components}}. \]
\[ P(x) = \sum_{l=0}^{\infty} x^l \text{Part}(l), \]

\[ \widetilde{P}(x) = \sum_{l=0}^{\infty} x^l \widetilde{\text{Part}}(l). \]

\[ \mathcal{F}(x, y) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} x^l y^k \text{Hur}_{E}^{l, k} \]

*Auto-weighted Count of Possibly disconnected unramified covers of $E = \Theta$ of degree $l$ with $k$ connected components*

\[ \mathcal{F}(x, y) = \exp(y \log P(x)) \]

\[ \widetilde{P}(x) = \frac{d}{dy} \mathcal{F}(x, y) \bigg|_{y=1} \]

\[ = P(x) \cdot \log P(x) \]

\[ = x + \frac{5}{2} x^2 + \frac{29}{6} x^3 + \frac{109}{12} x^4 + \frac{907}{60} x^5 + ... \]
\( \langle (2) \rangle^{E \times \Phi^2 \circ}_{g,d} \) denotes the connected \GW\ theory to a fixed target \( E \times \Phi^2 \)

The connected/disconnected calculus yields:

\[ \langle (2) \rangle^{E \times \Phi^2 \circ}_{g,d} = \sum_{2 \leq e \leq d} \langle (2) \rangle^{E \times \Phi^2 \circ}_{g,e} \cdot \text{Part}(d-e) \]

So we can easily compute \( \langle (2) \rangle^{E \times \Phi^2 \circ}_{g,d} \)

from \( \langle (2) \rangle^{E \times \Phi^2 \circ}_{g,d} \).
- \sum_{g \in \mathbb{Z}} n^{2g-3} \left\langle (2) \right\rangle_{g,d}^{E \times \mathcal{C}^2} \quad \Rightarrow \\
\quad (-i) \cdot \text{Trace} (\mathcal{M}_{D,d}) \
\quad \text{after } -q = e^{im} \
\quad \Rightarrow \\
\quad (-i) \cdot \text{Tr}_d \cdot (t_1 + t_2)
Let \( D = c_1(\mathcal{O}/_{\mathcal{I}}) \in \mathcal{H}^2(\text{Hilb}(\psi^2, k)) \)

Let \( M_{D,k} \) be the operator of quantum multiplication

\[
M_{D,k} = D \ast : \mathcal{H}^*(\text{Hilb}(\psi^2, k)) \rightarrow \mathcal{H}^*(\text{Hilb}(\psi^2, k)).
\]

\(\updownarrow\) computed explicitly by Okounkov-P (2010)

Let \( \text{Tr}_k = \frac{1}{t_1+t_2} \text{Trace} \left( M_{D,k} \right) \),

\[
M_D = (t_1+t_2) \sum_r \left( \frac{(-q^r+1)}{r} - \frac{1}{2} \frac{(-q^r+1)}{(-q^r-1)} \right) \alpha_r \alpha'_r 
\]

+ off diagonal terms.
By the GW/Hilb correspondence (for $\pi$)

$$
\sum_{g \geq 0} u^{2g-3} \langle (2) \rangle^\pi \cdot \mathcal{H}(\mathcal{O}_X^2)
$$

Tseng-P (2019)

$$
= (-i) \cdot \sum_{n \geq 0} q^n \langle (2) \rangle_{\text{Hilb}(\mathcal{O}_X^2)}^{1, \beta_n}
$$

genus 1

GW invariants in curve class

$\beta_n$ of degree

$$
\gamma = \sum_{\beta_n} c_1(\mathcal{O}_{\mathcal{I}})
$$

Tautological bundle on $\text{Hilb}(\mathcal{O}_X^2)$
The last step is to evaluate

$$
\langle (2) \rangle_{\text{Hilb}(\mathbb{F}, d)} = \sum_{n \geq 0} q^n \langle (2) \rangle_{1, \beta_n}^{\text{Hilb}(\mathbb{F}, d)}
$$

H.-H. Tseng and I found a conjectural answer:

**Conjecture (H.-H. Tseng - P 2023)**

$$
- \langle (2) \rangle_{\text{Hilb}(\mathbb{F}, d)} = -\frac{1}{24} \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left( \text{Tr}_d + \sum_{k=2}^{d-1} \frac{\zeta(k)}{d-k} \text{Tr}_X \right).
$$
Example $d=2$:

\[
(t_1 + t_2)^2 \left\langle \frac{\lambda_{g-2} \lambda_g}{t_1 t_2} \right| (2) \right\rangle^\pi_{g,2} = \left\langle (2) \right\rangle^\pi \times \Phi^2 \cdot g,2
\]

Convention:
- $g$ terms are summed as
  \[
  \sum_{g \geq 0} u^{2g-3} \cdots
  \]
- $-q = \exp(i\pi)$

Example $d=3$:

\[
(t_1 + t_2)^2 \left\langle \frac{\lambda_{g-2} \lambda_g}{t_1 t_2} \right| (2) \right\rangle^\pi_{g,3} + (t_1 + t_2)^2 \left\langle \frac{\lambda_{g-2} \lambda_g}{t_1 t_2} \right| (2) \right\rangle^\pi_{g,2}
\]

\[
- \left( - \frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} (-i) \cdot \mathcal{T}_{r_2}(q) \right)
\]

\[
\equiv \left\langle (2) \right\rangle^\pi \times \Phi^2 \cdot g,3
\]
Example $d = 4$:

\[
\left( \frac{t_1 + t_2}{t, t_2} \right)^2 \left\langle \lambda_3 \lambda_2 | (2) \right\rangle_{g, 4}^{\pi \circ} + \left( \frac{t_1 + t_2}{t, t_2} \right)^2 \left\langle \lambda_3 \lambda_2 | (2) \right\rangle_{g, 3}^{\pi \circ} + \left( \frac{t_1 + t_2}{t, t_2} \right)^2 \left\langle \lambda_3 \lambda_2 | (2) \right\rangle_{g, 2}^{\pi \circ} \cdot 2
\]

\[-\frac{1}{24} \left( \frac{t_1 + t_2}{t, t_2} \right) \left\langle (2) \right\rangle_{g, 3}^{E \times \Omega^2 \circ} \cdot \widetilde{\text{Part}} (1)
\]

\[-\frac{1}{24} \left( \frac{t_1 + t_2}{t, t_2} \right) \left\langle (2) \right\rangle_{g, 2}^{E \times \Omega^2 \circ} \cdot \widetilde{\text{Part}} (2)
\]

\[
\left\langle (2) \right\rangle_{g, 4}^{\pi \times \Omega^2 \circ}
\]
We simplify as

\[
\left(\frac{t_1 + t_2}{t_1, t_2}\right)^2 \langle \chi_{g-2} \chi_g \mid (2) \rangle_{g, 4} \pi \circ \left(\frac{t_1 + t_2}{t_1, t_2}\right)^2 \langle \chi_{g-2} \chi_g \mid (2) \rangle_{g, 3} \pi \circ \\
+ \left(\frac{t_1 + t_2}{t_1, t_2}\right)^2 \langle \chi_{g-2} \chi_g \mid (2) \rangle_{g, 2} \pi \circ \cdot 2
\]

\[
- \left(-\frac{1}{24} \left(\frac{t_1 + t_2}{t_1, t_2}\right)^2 (-i) \left( \text{Tr}_3 - \text{Tr}_2 \right) \right) \cdot 1
\]

\[
- \left(-\frac{1}{24} \left(\frac{t_1 + t_2}{t_1, t_2}\right) (-i) \text{Tr}_2 \cdot \frac{5}{2} \right)
\]

\[\pi \times \mathcal{C}^2 \circ \]

\[
\langle (2) \rangle_{g, 4}
\]
Projection of $\text{NL}_d$

By definition:

$$\Pr_A([\text{NL}_d]) \in \mathbb{R}^{g-1}(A_g).$$

Let $\delta_{g,d} \in \text{CH}^{g-1}(A_g)$,

$$\delta_{g,d} \in \ker(\Pr_A),$$

be the non tautological part:

$$[\text{NL}_d] = \Pr_A([\text{NL}_d]) + \delta_{g,d}.$$
By definition of $P_{\mathcal{A}}$,

$$<\delta_{g_d}, \gamma>_{A_g} = \int \delta_{g_d} \cdot \gamma \cdot \lambda_g$$

non-canonical closure

lifting of $\lambda$ classes

$$= 0$$

for all $\gamma \in R^{(g) - (g-1)}(A_g)$.

We have seen before that

$$\lambda_{g-1} \cdot [N_{L_d}] = 0 \in R^{2g-2}(A_g).$$
So we have

\[ 0 = \lambda_{g-1} \cdot \Pr_A \left( \left[ \text{NL}_d \right] \right) \]
\[ + \lambda_{g-1} \cdot \delta_{g,d} . \]

Certainly

\[ \lambda_{g-1} \cdot \Pr_A \left( \left[ \text{NL}_d \right] \right) \in R^{2g-2} (A_g) . \]

Claim: \[ \lambda_{g-1} \cdot \delta_{g,d} \in \ker (\Pr_A) \]

Proof: \[ \langle \lambda_{g-1} \cdot \delta_{g,d}, \gamma \rangle_{A_g} = \langle \delta_{g,d}, \lambda_{g-1} \cdot \gamma \rangle_{A_g} \]
\[ \forall \gamma \in R_{\left(\frac{g}{2}\right) - (2g-2)} (A_g) \]
Therefore, since

\[ R^{2g-2}_j (A_j) \cap \ker (P_{r_A}) = 0, \]

\[ \lambda_{g-1} \cdot P_{r_A} (\left[ NL_d \right]) = 0, \]

\[ \lambda_{g-1} \cdot S_{g,d} = 0. \]

As before, we conclude

\[ P_{r_A} (\left[ NL_d \right]) = \hat{C}_{g,d} \cdot \lambda_{g-1}. \]
If $[NL_d] \in R^*(A_g)$, then

\[ \hat{C}_{g,d} = C_{g,d} \]

defined by projection

Computed previously using $\text{Hilb}(\mathbb{C}^2, k)$

Conjecture (Canning-OPrance-P 2023)

for all $g \geq 2$, $d \geq 1$:

\[ \hat{C}_{g,d} = C_{g,d} \]

Probably $g = 1$ also works with careful definitions as a degenerate case.
The $d = 1$ case follows from

\begin{equation}
\text{Theorem (Canning-Oprea-P 2023):}
\end{equation}

If the vanishing conjecture holds,

\[ P_{r_A} \left( [A_1 \times A_{g-1}] \right) = \frac{g}{6|B_{2g}|} \lambda_{g-1}. \]

Together with the calculation of \( C_{g,1} \).

In general, we have

\[ [N_{L_d}] = \hat{C}_{g,d} \cdot \lambda_{g-1} + \delta_{g,d} \]

with \( \delta_{g,d} \in \text{Ker} \left( P_{r_A} \right) \)

and \( \alpha \cdot \delta_{g,d} = 0 \).
\( \forall a \in R^*(A_g) \text{ satisfying } a \cdot \lambda_{g-1} = 0, \)
\( (a \in \text{Ann}(\lambda_{g-1})) \).

In order to prove
\( \hat{C}_{g,d} = C_{g,d}, \)
we must show
\[
\int \text{Tor}^* \left( \delta_{g,d} \right) \cdot \lambda_{g-2} \lambda_{g} = 0.
\]
\( \overline{M}_g \)

I see two possible paths to prove.
I point out that

\[ \text{Tor}^*_x \left( M^C_{g} \right) \in \mathcal{H}^*(\mathcal{A}_g) \]

\[ \downarrow \]

\[ \int \text{Tor}^* \left( \delta_{g,d} \right) \cdot \tau_{g-2} \tau_g = 0. \]

\[ \overline{M}_g \]

But there is not much reason to believe that \( \text{Tor}^*_x \left( M^C_{g} \right) \) is tautological.
The best reason to believe
\[ \hat{C}_{g,d} = C_{g,d} \]
is a conjecture by Aitor:

Conjecture (Iribar López 2024)
\[ \text{CH}^*(A_g) \xrightarrow{\text{Pr}_A} \mathcal{R}^*(A_g) \]
is a ring homomorphism.

What limited evidence that we
have supports this claim
(at least for the subring of \text{CH}^*(A_g)
generated by NL and Jacobian loci.)
Update April 2024 (by Aitor)

Using the equation (which we know now)

$$\int \text{Tor}^* \left( \delta_{g,d} \right) \cdot \mathcal{A}_{g-2} \mathcal{A}_g = 0$$

and boundary arguments by Pixton,

the homomorphism property is established in the following case:

Let $T \in R^* (M^c_g)$ be any class.

Then we have

$$\text{pr}_A \left( \text{Tor}^* T \cdot [NL_d] \right)$$

so

$$\text{pr}_A \left( \text{Tor}^* T \right) \cdot \text{pr}_A \left( [NL_d] \right).$$
Calculation of the projection of $\text{NL}_d$

by Aitor Iribar López:

We have already proven

$$P_{\mathcal{A}}([\text{NL}_d]) = \hat{c}_{g,d} \cdot \lambda_{g-1} e \in \mathcal{H}^*(\mathcal{A}_g)$$

---

**Theorem A (Iribar López 2024)**

$$\hat{c}_{g,d} = d^{2g-1} \prod_{p \mid m} (1 - p^{-2g+2}) \cdot \frac{g}{\text{c} \mid \mathcal{B}_{2g} \mid}$$

---

Aitor's proof uses the geometry of the moduli of abelian varieties with level structures.
Let $c_{g,d}$ be computed using the conjectural formula for $\langle \text{Hilb} (\mathbb{F}^2, e) \rangle ^{\text{Hilb} (\mathbb{F}^2, e)}$, $2 \leq e \leq d$.

Theorem B* (Iribe López 2024)

for all $g \geq 2$, $d \geq 1$:

$\hat{c}_{g,d} = c_{g,d}$.

* here denotes the dependence on the conjectural formula for $\text{Hilb} (\mathbb{F}^2)$. 
Aifor's results yield the following implication

Conjecture (Iribar López 2024)

\[ CH^*(\mathcal{A}_g) \xrightarrow{Pr_A} \mathbb{R}^*(\mathcal{A}_g) \]

is a \textcolor{red}{ring} homomorphism.

\[\Downarrow\]

Conjecture (H.-H. Tseng-P 2023)

\[- \left\langle \binom{2}{(2)} \right\rangle^H_{\text{Hilb}(\mathbb{P}^2, d)} = \]

\[-\frac{1}{24} \cdot \frac{(t_1 + t_2)^2}{t_1 t_2} \left( Tr_d + \sum_{k=2}^{d-1} \frac{\binom{d}{d-k}}{d-k} Tr_k \right).\]
Appendix: Update March 2024

There is a new path to prove:

Conjecture (H.-H. Tseng - P 2023)

\[- \left\langle \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \right\rangle_{\text{Hill}(\Phi^2, d)} = \]

\[- \frac{1}{24} \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left( \text{Tr}_d + \sum_{k = 2}^{d-1} \frac{6(d-k)}{d-k} \text{Tr}_k \right). \]

We have seen that calculating the following Connected Gromov-Witten integral is sufficient:
\[ \int \underbrace{T_i(q) \mathfrak{g}_{g-2} \mathfrak{g}}_{\overline{\mathcal{M}_{g,1}(\pi, d)}} \]

\[ \langle T_i(q) \mathfrak{g}_{g-2} \mathfrak{g} \rangle_{g,d}^{\pi_0} \]

Here \( \overline{\mathcal{M}_{g,1}} \) is the moduli of pointed nonsingular elliptic curves and

\[ \begin{array}{c}
\overline{\mathcal{M}_{g,1}} \\
\pi \downarrow \\
\pi \end{array} \]

\[ \begin{array}{c}
\mathcal{E} \\
q & \text{zero section}
\end{array} \]
The first idea is to switch to an elliptically fibered $K3$ surface:

$$
\begin{array}{c}
S \\
\pi_S \\
\mathbb{P}^1
\end{array}
\xrightarrow{q}
24 \text{ nodal fibers}
$$

The fibers of $\pi_S$ are 1-pointed stable genus 1 curves.

The induced morphism

$$
\mathbb{P}^1 \rightarrow \overline{M}_{1,1}
$$

is of degree 48.
Then we have

\[ \int T_1(q) \prod_{g-2}^g [\overline{M}_{g,1}(\pi, d)]^{\text{vir}} \]

\[ \frac{1}{48} \int T_1(q) \prod_{g-2}^g [\overline{M}_{g,1}(\pi_5, d)]^{\text{vir}} \]

- The second idea is to use K3 vanishing.
Consider the integral:

\[ \int \mathcal{T}_1(q) \mathcal{Z}_{g-2} = 0 \]

Integrand \( \dim = 2 + g - 2 \)
\[ = g \]

Vanishing of standard K3 virtual class

\([\overline{M}_{g,1}(S, d)]^{vir}\)

Standard virtual class

\(d\) times fiber class of \(\pi_S\),

\(d > 0\).

The above vanishing will give us a nontrivial relation.
Claim A:
\[
\int T_i(q) \mathfrak{a}_{g-2} e(IE^v \otimes \text{Tan}_{\mathbb{P}^1})
\]
\[
\left[ \overline{M}_{g,1}(\pi_5, d) \right]_{\text{vir}} \parallel
\]
\[
\int T_i(q) \mathfrak{a}_{g-2}
\]
\[
\left[ \overline{M}_{g,1}(S, d) \right]_{\text{vir}}.
\]

Corollary:
\[
\int T_i(q) \mathfrak{a}_{g-2} e(IE^v \otimes \text{Tan}_{\mathbb{P}^1}) = 0.
\]
\[
\left[ \overline{M}_{g,1}(\pi_5, d) \right]_{\text{vir}}
\]
Proof: There is a morphism

$$\overline{M}_{g,1}(\pi_5, d) \to \overline{M}_{g,1}(S, d)$$

which is an isomorphism of DM stacks away from the 24 nodal fibers of $\pi_5$. Moreover, away from the 24 nodal fibers, the obstruction theory of $\overline{M}_{g,1}(\pi_5, d)$ augmented by $\operatorname{IEV} \otimes \operatorname{Tr}_{\pi_5}$ matches the standard obstruction theory of $\overline{M}_{g,1}(S, d)$. 
The entire issue is about the nodal fibers

\[
\begin{array}{c}
S \\
\downarrow \pi_S \\
\mathbb{P}^1
\end{array}
\xrightarrow{\eta} \begin{array}{c}
S' \\
\downarrow \\
\mathbb{P}^1'
\end{array}
\] .

We use here the degeneration to the normal cone of the divisor \( \alpha \in S \) of nodal fibers, a standard technique, but a complication here is that \((S, \alpha)\) requires \( \log GW \) (since \( \alpha \) is singular).
We study the normal cone

$$\mathcal{X} = \text{Bl}(S \times \mathcal{C}, a \times 0)$$

$\downarrow$

$\mathcal{C}$

$\mathcal{X}$ has a single singularity
(a 3-fold double point)

over each point $p \times 0$

where $p \in \mathcal{C}$ is a node.

The main observation here:

We can avoid all log complication

by studying $\mathcal{X}^0 \subset \mathcal{X}$.

The nonsingular locus
The reason that the non-compact log geometry $X^0 \subset X$ can be used here is that the curve classes are fibers and have intersection 0 with $\alpha$.

Said differently: the moduli spaces of log stable maps to the log degeneration $X^0$ are compact. Then the usual degeneration calculus of relative GW theory can be used.
After degeneration, the equality of Claim A is clear since the geometric differences of the moduli spaces vanish.

A second proof of Claim A would follow by constructing a connection for the obstruction theory on $\overline{\mathcal{M}}_{g,1}(\Pi_5, d)$ obtained by combining the fiberwise deformation with $\mathcal{E}^\vee \otimes \mathcal{P}^\perp$. 
The third step is to expand

\[ e(I\mathbb{E} \otimes \text{Tan}_{p^1}) = (-1)^g \lambda_g + (-1)^{g-1} \lambda_{g-1} \cdot [z_{p^1}] \]

so we obtain

\[ 0 = \int T_1(q) \, \mathcal{Z}_{g-2} \cdot e(I\mathbb{E} \otimes \text{Tan}_{p^1}) \]

\[ \left[ \overline{\mathcal{M}}_{g,1}(\pi_5, d) \right]^{\text{vir}} \]

\[ = \int T_1(q) \, \mathcal{Z}_{g-2} \cdot (-1)^g \lambda_g \]

\[ \left[ \overline{\mathcal{M}}_{g,1}(\pi_5, d) \right]^{\text{vir}} \]

\[ + 2 \int T_1(q) \, \mathcal{Z}_{g-2} \cdot (-1)^{g-1} \lambda_{g-1} \]

\[ \overline{\mathcal{M}}_{g,1}(E, d) \]

^{\text{vir}}
After rewriting, we find

\[ \int T_i(q) \, \pi_{g-2} \pi_g \]

\[ \left[ \overline{\mathcal{M}}_{g,1}(\pi, d) \right]^\text{vir} \]

\[ \frac{1}{24} \int T_i(q) \, \pi_{g-2} \pi_{g-1} \]

\[ \left[ \overline{\mathcal{M}}_{g,1}(E, d) \right]^\text{vir} \]

fixed elliptic target
The last step in the evaluation of the latter integral by Pixton (2008):

\[
\sum_{d \geq 0} Q^d \int \mathcal{T}_1(q) \begin{bmatrix} \overline{M}_{g,1}(E,d) \end{bmatrix}^{vir} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}
= |B_{2g-2}| \cdot \binom{2g}{2} \sum_{n \geq 1} \frac{6_{2g-1}^{(n)} Q^n}{2g \cdot 2g!}
\]

where

\[
C_{2g}(Q) = \frac{-B_{2g}}{2g \cdot 2g!} + \frac{2}{2g!} \sum_{n \geq 1} 6_{2g-1}^{(n)} Q^n
\]
in other words

\[ C_{2g}(Q) = - \frac{B_{2g}}{2g \cdot 2g!} E_{2g}(Q) \].

See page 32 of


for the results of Pixton.
Claim B: The evaluation

\[
\sum_{d \geq 0} Q^d \int T_1(q) \overline{\mathbb{M}_{g,1}(\bar{\Pi}, d)}^{vir} \]

\[= \frac{1}{24} \left| B_{2g-2} \right| \cdot \binom{2g}{2} C_{2g}(Q) \]

is equivalent to the conjectured formula for \( \left\langle (2) \right\rangle \).

Proof by Iriber López.
The status now is that all the claims related to

\[ \Pr_A \left( \left[ \text{NL}_d \right] \right) \in \mathbb{R}^{g-1}(A_g) \]

and the series \( \langle (2) \rangle \)

are proven:

\[ \Pr_A \left( \left[ \text{NL}_d \right] \right) = \hat{c}_{g,d} \cdot \lambda_{g-1}, \]

\[ \hat{c}_{g,d} = \prod_{p \mid d} (1 - p^{-2g+2}) \cdot \frac{g}{6 |B_{2g}|}, \]

\[ \text{by Iribar López} \]
\[ -\langle \mathcal{H}_{\text{ILB}}(\mathcal{P},d) \rangle_{1} = \]

\[-\frac{1}{24} \frac{(t_1+t_2)^2}{t_1 t_2} \left( Tr_d + \sum_{k=2}^{d-1} \frac{6(d-k)}{d-k} Tr_k \right), \]

[by claim A + B]

**Definition**

\[ C_{g,d} = \frac{\int \overline{\mathcal{M}}_g}{\overline{\mathcal{M}}_g} \cdot \overline{\mathcal{M}}_g \]

\[ \int \overline{\mathcal{M}}_g \cdot \overline{\mathcal{M}}_g \cdot \overline{\mathcal{M}}_g \]

\[ \hat{C}_{g,d} = C_{g,d} \text{ by Calculation of} \]

\[ \langle \mathcal{H}_{\text{ILB}}(\mathcal{P},d) \rangle_{1} \]
Many open directions remain. My favorites:

Conjecture (Iribar López 2024)

\[ \text{CH}^*(A_g) \xrightarrow{\text{Pr}_A} \mathcal{R}^*(A_g) \]

is a ring homomorphism.

- Study the extension of the diagram

\[ \text{Tor}_i^{-1}(NL_d) \rightarrow NL_d \]
\[ M_{g,1}^{ct} \rightarrow A_g \]
to the perfect cone compactifications

\[
\begin{array}{ccc}
\text{Tor}_i^{-1}(\overline{NL}_d) & \longrightarrow & \overline{NL}_d \\
\downarrow & & \downarrow \\
\overline{M}_{g,1} & \longrightarrow & \overline{A}_g
\end{array}
\]

• Calculate

\[
\left\langle \text{Hilb}^\chi_{(g,d)} \right. \left. \begin{array}{c} G_1, G_2, \ldots, G_n \end{array} \right. \bigg|_1
\]

for arbitrary partition insertions \( G_i \).
The End

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