$\mathbb{A}_g$, $\overline{\mathbb{M}}_g$, and $\text{Hilb}(\mathbb{C}^2, d)$

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joint with
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including results of
F. Greer - C. Lian
Compactifications

Extended Torelli

Torelli

moduli spaces of classifying spaces

$\text{Sp}(2g, \mathbb{H}) \xrightarrow{\text{hom}} \text{Map}(g)$

Groups

$X_3(g)$

$\downarrow$

$M_{k3}^g$

moduli of quasi-polarized $X_3$ surfaces

moduli of PPAV

$\mathcal{A}_g$

$\mathcal{P}_C$

$\overline{M}_g$

$M_g$

$\overline{M}_g$

moduli of smooth curves

Correspondence with linear sections as studied by Mukai

Various compactifications,

perhaps no winner yet,

but Satake is convenient
I. Moduli of abelian varieties

\[ \text{Sp}(2g, \mathbb{H}) \sim \mathcal{H}_g \quad \text{Siegel upper half space} \]

\[ A_j = \frac{\mathcal{H}_g}{\text{Sp}(2g, \mathbb{H})} \quad \text{(contractible)} \]

model for
\[ B \text{Sp}(2g, \mathbb{H}) \]

up to finite stabilizers

we have: \[ H^*(A_j) = H^*_\text{Sp}(2g, \mathbb{H}) \]

All cohomology taken with \( \mathbb{Q} \)-coefficients.
IE is defined by

\[ \Delta \xrightarrow{\pi} A_g \]

Then \[ IE = \Delta^* (\Omega^1_{\pi}) \]

\[ \text{rank } g \]

Lambda class: \[ \lambda_i = c_i (IE) \]

\[ \text{ch } R_{\pi_X} \mathcal{O}_\mathcal{X} = \text{ch } (1 - IE + \wedge^2 IE + \ldots) \]

Borel 1974: Stable cohomology of \( Sp(2) \)

generated by \( \lambda_i \).
Following van der Geer, define tautological classes:

- $RH^*(A_g) \subset H^*(A_g)$ \text{ cohomology}
  subalgebra generated by all $\lambda_i = c_i(E)$,

- $R^*(A_g) \subset CH^*(A_g)$ \text{ algebraic cycles}
  subalgebra generated by all $\lambda_i = c_i(E)$.

\underline{Theorem (van der Geer 1996)}

$RH^*(A_g) = R^*(A_g)$ with presentation

$$\mathbb{Q}[\lambda_1, \ldots, \lambda_g]$$

\[ \lambda_g = 0, \quad c(E \oplus E^*) = 1 \]

\[ (1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g) \cdot (1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g) = 1. \]
As a consequence, $\mathcal{R}^*(A_g)$ is a Gorenstein ring with socle

$$\mathcal{R}^{(g)}(A_g) \cong \mathbb{Q} \cdot \lambda_1^{(g)} \cong \mathbb{Q} \cdot \lambda_1 \lambda_2 \cdots \lambda_{g-1}.$$  

Many open questions:

- Calculate $H^*_{\text{Sp}(2g,\mathbb{Z})}$ in unstable ranges
- Calculate $\text{CH}^*(A_g)$
- Calculate $H^*_{\text{Sp}(2g,\mathbb{Z})}$ with $\mathbb{Z}$-coefficients

all very difficult. We will go in a different direction.
II. Projection

We have $\mathcal{R}^*(A_j) \subset CH^*(A_j)$

and $\mathcal{R}^*(M^c_j) \subset CH^*(M^c_j)$.

Is there a canonical projection

$CH^*(M^c_j) \xrightarrow{Pr_{M^c_j}} \mathcal{R}^*(M^c_j)$?

There is no proposal for ,

but for $CH^*(A_j) \xrightarrow{Pr_A} \mathcal{R}^*(A_j)$,

we believe $Pr_A$ exists!
The idea uses compactification:

\[ A_g \subset A_g^{pc} \]

Some facts:

(i) Hodge bundle extends canonically

\[ 1E \subset 1E \]

\[ \downarrow \quad \downarrow \]

\[ A_g \subset A_g^{pc} \]

(ii) \( R^* (A_g^{pc}) \overset{\text{def}}{=} CH^* (A_g^{pc}) \)

Subalgebra generated by all

\[ \lambda_i = c_i (1E) \].
Theorem (van der Geer 1996)

\[ \mathcal{H}^* \left( A_j^{pc} \right) = \frac{\mathbb{Q} \left[ \lambda_1, \ldots, \lambda_g \right]}{\left( c \left( E \otimes E^* \right) = 1 \right)} \]

\[ \left( 1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g \right) \cdot \left( 1 - \lambda_1 + \lambda_2 - \cdots - \lambda_g \right) = 1 \]

\[ \mathcal{H}^* \left( A_j^{pc} \right) \text{ is a Gorenstein ring} \]

with socle

\[ \mathcal{H}^{(g)} \left( A_j^{pc} \right) \cong \mathbb{Q} \cdot \lambda_{1}^{(g)} \cong \mathbb{Q} \cdot \lambda_1 \lambda_2 \cdots \lambda_{g-1} \lambda_{g} \]

Using integration on \( A_j^{pc} \)

and the duality of \( \mathcal{H}^* \left( A_j^{pc} \right) \).
We define a projection:

\[ \text{CH}^*(A_j^{pc}) \xrightarrow{Pr_{A_j^{pc}}} \mathcal{L}^*(A_j^{pc}) \]

\[ \text{Pr}_{A_j^{pc}} (\alpha \in \text{CH}^*(A_j^{pc})) = \beta \in \mathcal{L}^*(A_j^{pc}) \]

where \( \forall \gamma \in \mathcal{L}^*(A_j^{pc}) \),

\[ \int_{A_j^{pc}} \alpha \cdot \gamma = \int_{A_j^{pc}} \beta \cdot \gamma \],

\( \beta \) exists and is unique.

\[ \text{Pr}_{A_j^{pc}} \text{ is a projection} \]
Idea for constructing

\[ \text{CH}^*(A_j) \xrightarrow{\text{Pr}_A} \Lambda^*(A_j) \]

via \( \text{Pr}_{A_{pc}} : \)

\[ \text{Pr}_A (\alpha \in \text{CH}^*(A_j)) = \text{Pr}_{A_{pc}} (\overline{a} \in \text{CH}^*(A_{pc})) \bigg|_{A_j} \]

but closure not canonical!

Not clear that \( \text{Pr}_A \) is well defined.

**Conjecture (Canning-Oprea-P 2023):**

\[ \lambda_g \bigg|_{A_{pc} \setminus A_j} = 0 \in \text{CH}^*(A_{pc} \setminus A_j) \]

Conjecture \( \Rightarrow \text{Pr}_A \) is well defined.
Using the Conjecture, there is another path to the projection:

for $a \in \text{Ch}^*(A_g)$ and $\gamma \in \mathcal{R}^*(A_g)$, define a pairing

$$\langle a, \gamma \rangle_{A_g} = \int_{A_g^{pc}} a \cdot \gamma \cdot \lambda_g$$

Conjecture $\Rightarrow \langle a, \gamma \rangle_{A_g}$ is well defined.

Exercise: $\langle a, \gamma \rangle_{A_g} = \langle \text{Pr}_A(a), \gamma \rangle_{A_g}$

$\forall a \in \text{Ch}^*(A_g)$ and $\forall \gamma \in \mathcal{R}^*(A_g)$
Update (19 November 2023):

The vanishing conjecture

$$\mathcal{A}_g \bigg/ A_g^{pc} \cdot A_g = 0 \in \text{CH}^*(\mathcal{A}_g^{pc} \cdot A_g)$$

is true!

- Argument by Sam Molcho constructing trivial quotients of $1E$ on the boundary via residue maps.
- Another path suggested by Ben Moonen using boundary geometry and rigidity results from Faltings-Chai.

Proofs work for all sufficiently fine toroidal compactification $\text{pr}_A^*$ exists and is independent of choice of compactification.


arXiv: 2401.15768
III. Noether-Lefschetz loci

The simplest NL locus to consider is

\[ A_1 \times A_{g-1} \to A_g. \]

We assume \( g \geq 2 \).

We define a twisted generalization by the following construction:

Let \( NL_d \to A_g \) be the \( d \geq 1 \) model of pairs:

\[ NL_d \cong \left[ E \to X \right] \]

Condition: \( E \cdot \Theta = d \)

theta divisor of \( X \)

1 dim Subgroup, an elliptic curve

PPAV of dimension \( g \)
In case \( d=1 \), \( \mathcal{N}_{L_1} = A_1 \times A_{g-1} \).

Easy to see:

\[
\dim \mathcal{N}_{L_d} = \dim A_g - (g-1)
\]

so \( \{ \mathcal{N}_{L_d} \} \in CH^{g-1}(A_g) \).

The main topic of the lecture is the computation

\[
Pr_A(\{ \mathcal{N}_{L_d} \}) \in R^*(A_g).
\]

There are 2 immediate issues:

- \( Pr_A \) depends on the vanishing Conjecture.
- Even if we assume the Conjecture, it is not clear how to integrate the classes of the closures in \( A_g^{pc} \).
Theorem (Cannings-OPrea-P 2023):

If the vanishing conjecture holds,

\[ \Pr_A ([A_1 \times A_{g-1}]) = \frac{g}{6 |B_{g-1}|} \lambda_{g-1}. \]

An interesting direction:

What is the projection of the Schottky locus,

\[ \Pr_A (\text{Tor}_* [M_g^{ct}]) \in \kappa^*(A_g) ? \]
IV. A different question

We consider now a different question:

\[
\text{If } [N_{L_d}] \in R^*(A_g), \\
\text{what could it be?}
\]

- Since $P_{r_A}$ is well-defined, the answer to the question is:

\[
[N_{L_d}] \in R^*(A_g) \\
\Rightarrow \\
[N_{L_d}] = P_{r_A}([N_{L_d}]).
\]
Proposition (Cannings-Oprea-P 2023):

If $[NL_d] \in R^*(A_g)$, then

$$[NL_d] = c_{g,d} \cdot \lambda_{g-1} \cdot \square$$

A scalar in $\mathbb{Q}$.

**Proof:** We have $[NL_d] \in R^g(A_g)$.

Moreover, $\lambda_{g-1} \cdot [NL_d] = 0$.

Since $NL_d \in [E \leftrightarrow \chi]$

$\downarrow$

$A_1 \times A_{g-1}^{\text{Pol}} \in [E] \times [\chi/\overline{E}]$

A non principal polarization.
\[ \lambda_{g-1} \mid_{NL_d} \text{ is pulled-back} \]

from \[ \lambda_{g-1} \mid A_1 \times A_{g-1}^{\text{pol}} \]

and \[ \lambda_{g-1} \mid A_1 \times A_{g-1}^{\text{pol}} = 0 \]

because \[ c(\mathcal{E}_1) \mid A_1 = 0 \]

and \[ \lambda_{g-1} \mid A_{g-1}^{\text{pol}} = 0. \]

Finally, \[ Q \cdot \lambda_{g-1} \in R^{g-1}(A_g) \]

is the annihilator of \[ \lambda_{g-1} \] in \[ R^{g-1}(A_g) \]. \qed
V. Integration

We have seen

\[ [NL_d] \in R^*(A_g) \Rightarrow [NL_d] = c_{g,d} \cdot \lambda_{g-1}. \]

The question is now what is the scalar \( c_{g,d} \)?

The idea is to pull-back via Torelli:

\[
\text{Tor} : \ M_g^{ct} \rightarrow A_g ,
\]

\[
\text{Tor}^* \left( [NL_d] \right) \in R^{g-1} \left( M_g^{ct} \right),
\]

\[
\text{Tor}^* \left( \lambda_{g-1} \right) \in R^{g-1} \left( M_g^{ct} \right).
\]
Then we can calculate \( c_{g,d} \)

by the \( \mathfrak{g} \)-evaluation on \( \overline{M}_g^{ct} \):

\[
\int_{\overline{M}_g} \text{Tor}^* \left( \left[ N_{L_d} \right] \right) \cdot \mathfrak{g}_{g-2} \mathfrak{g}_g = c_{g,d}
\]

\[
\int_{\overline{M}_g} \mathfrak{g}_{g-2} \mathfrak{g}_{g-1} \mathfrak{g}_g = \frac{1}{2(2g-2)!} \frac{1}{2g} \frac{1}{2g-2}
\]

Computed by Faber-P (1999)
But how are we going to calculate

$$\int \text{Tor}^* \left( \left[ NL_d \right] \right) \cdot \mathfrak{g}_{g-2} \mathfrak{g}_g ? \mathfrak{M}_g$$

This requires a miracle provided by stable maps and the quantum cohomology of $\text{Hilb}(\mathbb{C}^2, d)$. We assume $g \geq 2$ and $d \geq 1$. 
VI Stable maps

Consider the fiber product:

\[ \text{Tor}^{-1}(NL_d) \rightarrow NL_d \]

\[ M_{g,ct} \rightarrow \text{Tor} \rightarrow A_g \]

We will add a marked point:

\[ \text{Tor}^{-1}(NL_d) \rightarrow NL_d \]

\[ M_{g,1} \rightarrow \text{Tor}_i \rightarrow A_g \]
\[
\text{dilaton} \downarrow \\
(2g-2) \cdot \int \overline{\mathcal{M}_g} \overline{\text{Tor}^* \left( \left[ \mathcal{N}_d \right] \right)} \cdot \mathbb{R}_{g-2}^2 \ g
\]

\[
\overline{\mathcal{M}_g} \left\| \right. \\
\int \overline{\text{Tor}_i^* \left( \left[ \mathcal{N}_d \right] \right)} \cdot \mathcal{V}_i \cdot \mathbb{R}_{g-2}^2 \ g
\]

Since on \( \mathcal{M}_{g,i} \), we have

\[
\text{Tor}_i^* \left( \left[ \mathcal{N}_d \right] \right) = \alpha^* \left( \text{Tor}^* \left( \left[ \mathcal{N}_d \right] \right) \right)
\]

where \( \alpha : \mathcal{M}_{g,i} \overset{\text{forgetful}}{\rightarrow} \mathcal{M}_g \).
Let \( M_{1,1}^{ct} = M_{1,1} \) be the model of pointed nonsingular elliptic curves:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\pi^{ct}} & M_{1,1} \\
\downarrow & & \downarrow \\
\end{array}
\]

\( q \) Zero Section

Let \( M_{g,1}^{ct} (\pi^{ct}, d) \) be the Grothendieck \( \pi \)-relative space of stable maps to the fibers of \( \pi \).
\[ \mathcal{M}_{g,1}^{\text{ct}}(\pi_{\text{ct}},d) \] has a virtual class of dimension.

\[ \text{virdim} = 1 + 1 + 2g-2 = 2g. \]

\[ \dim \mathcal{M}_{g,1} \]

\[ \text{unpointed maps to a fixed elliptic fiber} \]

\[ \text{Tor}^*_1([NL_d]) \] is a intersection class on \( \text{Tor}^{-1}(NL_d) \) of dimension.

\[ \text{virdim} = 3g-3 + 1 - (g-1) = 2g-1. \]

\[ \dim \mathcal{M}_{g,1}^{\text{ct}} \]

\[ \text{codim of } NL_d \]
there is a discrete invariant:

\[ f^*: \text{Jac}_0(E) \to \text{Jac}_0(C), \]

\[ \text{Jdeg}_f = \frac{d}{|\ker f^*|}. \]

\[ \text{Jdeg}_f \in \{1, 2, \ldots, d\} \] must divide \( d \)

and is a discrete invariant of \( f \),

\[ \mathcal{M}^\text{ct}_{g,1}(\pi^\text{ct}, d) = \bigsqcup \mathcal{M}^\text{ct}_{g,1}(\pi^\text{ct}, d)^{\hat{d}}. \]

maps with \( \text{Jdeg}_f = \hat{d} \)
Another way to think about $\text{Jdeg}_f$:

a degree $d$ stable map

$$f : (C, p) \rightarrow (E, q)$$

Compact type

factors uniquely as

$$(C, p) \xrightarrow{g} (\hat{E}, \hat{q}) \xrightarrow{h} (E, q)$$

Where $h : \hat{E} \rightarrow E$ is group homomorphism of elliptic curves and $g^* : \text{Jac}_0(\hat{E}) \rightarrow \text{Jac}_0(C)$ is injective.
Then \[ |\ker f^*| = \deg(h) \]
and \[ d = \deg(g) \cdot \deg(h) , \]
so we have
\[
\text{Jdeg } f = d / |\ker f^*| = \deg(g) .
\]
The disjoint decomposition
\[
\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d) = \bigsqcup_{\text{Jdeg } \hat{a}} \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^{\hat{a}}
\]
has principal part \( \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d) \).
The lower parts \( \mathcal{M}^{ct}_{g,1}(\pi^{ct}, \hat{d})^{\hat{d} < d} \)
can be studied via \( \mathcal{M}^{ct}_{g,1}(\pi^{ct}, \hat{d}) \).
Theorem (Canning-Oprea-Pixton 2023)

There is an isomorphism of DM stacks

\[ \text{ev}^{-1}(q)^{d} \cong \text{Tor}_{1}(NL_{d}) \]

\[ \circ \]

\[ \mathcal{M}_{g,1}^{\text{ct}}(\pi, d)^{d} \]

Here \( \text{ev}^{-1}(q)^{d} \) is the locus of maps where the evaluation of the marking on the domain equals the zero point \( q \) of the elliptic target.
To be useful we must also match the virtual classes:

\[
\text{vir} \dim \quad e^{-1}v(q^d) = \text{vir} \dim \quad M_{g,1}^{c_t} (\pi_{c_t}^*)^d - 1 \\
= 2g - 1,
\]

\[
\left[ e^{-1}v(q^d) \right]^{c_t, \text{vir}} = e^*v(q) \cap \left[ M_{g,1}^{c_t} (\pi_{c_t}^*)^d \right]^{\text{vir}}.
\]

**Conjecture (Canning-Oprea-P, Pixton 2023)**

under the above isomorphism,

\[
\left[ e^{-1}v(q^d) \right]^{c_t, \text{vir}} = \text{Tor}_i^* \left( \left[ NL_d \right] \right).
\]
Update Feb 2024:
Francois Greer and Carl Lian can prove

$$
\left[ eV^{-1}(q) \right]_{ct,vir}^d = \text{Tor}^*_i \left( \left[ NL_d \right] \right)
$$

effectively in the required form using a matching of obstruction theories.

Update April 2024:
The Greer-Lian proof can be found here:


arXiv: 2404.10826
An important property of

\[ [e^{-1}(q^d)]^{ct,vir} \in A_{2g-1}(\mathcal{M}^{ct}_{g,1}(\pi^c, d)) \]

is the existence of a canonical extension to \( \overline{\mathcal{M}}_{g,1}(\pi^c, d) \):

\[ [e^{-1}(q)]^{vir} \in A_{2g-1}(\overline{\mathcal{M}}_{g,1}(\pi, d)) \]

where \( \overline{\mathcal{M}}_{g,1}(\pi, d) \) ev

Over \[ \partial \in \overline{\mathcal{M}}_{g,1} \]
we have log stable maps

\[ q \]
\[ \pi \]
\[ \mu \]
\[ e \]
The complement

$$\overline{M}_{g,1}(\pi, d) \setminus M_{g,1}^{\text{ct}}(\pi^c, d)$$

is mapped by $\mathcal{E}$ to the complement

$$\overline{M}_{g,1} \setminus M_{g,1}^{\text{ct}}.$$ 

Over $M_{g,1}$, this is by definition.

Over the point $[\mathcal{X}] \in \overline{M}_{g,1}$, the claim is more interesting:

there are no curves with compact type domains which map to $\mathcal{X}$ with degree $d \geq 1$ by the definition of log maps.
We conclude:

\[
\int_{\overline{M}_{g,1}} \overline{T_{\text{vir}}^* \left( \left[ NL_d \right] \right)} \cdot \psi_i \cdot \gamma_{g-2} \gamma_g
\]

\[
\int_{\overline{M}_{g,1}} \overline{\varepsilon_*} \left[ \psi \right] \overset{\text{vir}}{d} \cdot \gamma_{g-2} \gamma_g
\]

\[
\int_{\overline{M}_{g,1}} \overline{\varepsilon_*} \left[ \psi \right] \overset{\text{vir}}{d} \cdot \gamma_{g-2} \gamma_g
\]

cotangent line now on \( \overline{M}_{g,1} \left( \pi, d \right) \), no correction terms since there are no maps of positive degree \( \mathbb{P}^1 \to E \).
We now use the extension:

\[
\int_{\tilde{\mathcal{M}}_g, s} \epsilon_\ast \left[ e_{\nu}^{-1}(q) \cdot \psi_1 \right]^{\text{vir}} \cdot \mathbb{Z}_{g-2} \mathcal{Z}_g
\]

\[
\sum_{\hat{\text{d}1d}} \circ \left( \frac{\partial}{\partial \hat{q}} \right) \cdot \int_{\tilde{\mathcal{M}}_g, s} \epsilon_\ast \left[ e_{\nu}^{-1}(q) \cdot \hat{\psi} \right]^{\text{vir}} \cdot \mathbb{Z}_{g-2} \mathcal{Z}_g
\]

Count of \((\hat{E}, \hat{q}) \rightarrow (E, q)\),

\[
\sigma(x) = \sum_{e \in x} e
\]
Hence the integrals

\[ \int_{\overline{M}_{g,1}} \left[ \text{ev}^{-1}(q)^d \cdot \psi \right] \cdot \overline{\mu}_{g-2, g} \]

and the integrals

\[ \int_{\overline{M}_{g,1}} \left[ \text{ev}^{-1}(q) \cdot \psi \right] \cdot \overline{\mu}_{g-2, g} \]

are related inductively by

a simple invertible transformation.
We will now calculate

\[ \int \xi^* \left[ \text{ev}^{-1}(q) \cdot \psi \right]_{\overline{M}_{g,1}}^{\text{vir}} \cdot \frac{\omega_{g-2} \omega_g}{2} \]

\[ \int \mathcal{T}_! (q) \cdot \frac{\omega_{g-2} \omega_g}{2} \]

\[ \left[ \overline{M}_{g,1}(\pi, d) \right]^{\text{vir}} \]

using the idea of the

GW/H correspondence Okounkov-P (2006)

A new issue is the families geometry.
\[
\int \mathcal{T}_1(q) \, \mathcal{M}_{g-2, 2g} \left[ \mathcal{M}_{g, 1}^{\text{vir}}(\pi, d) \right]^{\text{vir}}_{\text{vir}} \\
\left\{ \mathcal{T}_1(q) \, \mathcal{M}_{g-2, 2g} \right\}_{\text{vir}}^{\Pi_{0}}
\]

GW/H correspondence equation is found by degeneration of every fiber of

\[
\begin{array}{c}
\pi \\
\downarrow \\
\mathcal{M}_{g, 1}
\end{array}
\xrightarrow{\Phi} q
\]

to the normal cone of \( q \).
The resulting equation is

\[ \left< T, (g) \left| 2_{g-2} 2_g \right> \right>_{g,d}^\pi \]

\[ = \frac{1}{2^g} \sigma(d) \cdot (2g-2) \cdot \int \frac{\lambda_{g-2} \lambda_{g-1} c(E)}{1 - \lambda} \overline{M}_{g-1,1} \]

\[ + \left< 2_{g-2} 2_g \left| (2) \right> \right>_{g,d}^\pi \]

relative condition

integral evaluated to equal

\[ \frac{|B_{2g-2}|}{(2g-2) (2g-2)!} \]

Faber-P (1999)
\( \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \sum_{2 \leq e \leq d} \langle \frac{e^2}{2^g} \mid (2) \rangle^d \circ \cdot \text{Part} (d-e) \)

\( - \frac{1}{24} \left( \frac{t_1 + t_2}{t_1 t_2} \right) \sum_{2 \leq e \leq d} \langle (2) \rangle^E \circ \cdot \text{Part} (d-e) \)

invertible relation

\( \langle (2) \rangle^\pi \circ \mathbb{C}^2 \cdot \quad \) possibly disconnected

( no degree 0 connected components )

\( t_1, t_2 \) weight on \( \mathbb{C}^2 \)
The above relation is the
Connected / disconnected equation
(together with basic Hodge identities).

There are several terms to explain:

- \( \text{Part}(l) \) = \# of partitions of \( l \)
  \[
  \begin{align*}
  \text{Part}(0) &= 1 \\
  \text{Part}(1) &= 1 \\
  \text{Part}(2) &= 2
  \end{align*}
  \]

A well-known property is

\[
\text{Hur}_E^l = \text{Part}(l) \quad \text{for } l \geq 1
\]

\[\leftarrow \text{Aut-weighted Count of possibly disconnected unramified covers of } E = \emptyset \text{ of degree } l\]
\[ \tilde{\text{Part}}(l) \overset{\text{def}}{=} \tilde{\text{Hur}}_E^l \quad \text{for } l \geq 1 \]

\[ \tilde{\text{Part}}(0) = 0 \]

- Aut-weighted Count of possibly disconnected unramified covers of \( E = \mathbb{C} \) of degree \( l \) where each cover in weighted also by the number of connected components.

\[ \tilde{\text{Part}}(1) = 1 \]

\[ \tilde{\text{Part}}(2) = 1 + \frac{3}{2} = \frac{5}{2} \]

- Disconnected Cover
- Connected Covers

\[ \frac{1}{2} \cdot 1 \cdot 2 \]

- Connected Aut Count Components
Let \( P(x) = \sum_{l=0}^{\infty} x^l \text{Part}(l) \),

\( \widetilde{P}(x) = \sum_{l=0}^{\infty} x^l \widetilde{\text{Part}}(l) \).

\[ T(x, y) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} x^l y^k \text{Hur}_E^{l, k} \]

\( \text{Aut-weighted Count of possibly disconnected unramified covers of } E = \mathbb{Q} \)
\( \text{of degree } l \text{ with } K \text{ connected components} \)

\[ T(x, y) = \exp \left( y \log P(x) \right) \]

\( \widetilde{P}(x) = \frac{d}{dy} T(x, y) \bigg|_{y=1} \)

\[ = P(x) \cdot \log P(x) \]

\[ = x + \frac{5}{2} x^2 + \frac{29}{6} x^3 + \frac{109}{12} x^4 + \frac{907}{60} x^5 + \ldots \]
\[ \langle (2) \rangle_{g,d}^{E \times \mathcal{F}^2 \circ} \] denotes the connected GW theory to a fixed target \( E \times \mathcal{F}^2 \).

The connected/disconnected calculus yields:

\[ \langle (2) \rangle_{g,d}^{E \times \mathcal{F}^2 \circ} = \sum_{2 \leq e \leq d} \langle (2) \rangle_{g,e}^{E \times \mathcal{F}^2 \circ} \cdot \text{Part} (d-e) \]

So we can easily compute \( \langle (2) \rangle_{g,d}^{E \times \mathcal{F}^2 \circ} \) from \( \langle (2) \rangle_{g,d}^{E \times \mathcal{F}^2 \circ} \).
By the GW/\mathcal{M}_{\mathfrak{H}}\mathfrak{H} correspondance (fixed $E$)\quad Okounkov-P (2005)
Bryan-P

\[-\sum_{g \in \mathcal{Z}} n^{2g-3} \left\langle \begin{array}{c} (2) \\ g, d \end{array} \right\rangle^{E \times \mathcal{H}^2} \cdot \]
\[\equiv \]

\[(-i) \cdot \text{Trace} \left( \mathcal{M}_{d, d} \right) \quad \text{after} -g = e^{im} . \]

\[\equiv \]

\[(-i) \cdot \mathcal{T}_{r, d} \cdot (t_1 + t_2) \]
Let \( D = c_1(\mathcal{O}/\mathcal{I}) \in \mathcal{H}^2(\text{Hilb}(\mathbb{C}^2, k)) \)

Let \( M_{D, k} \) be the operator \( D = -\ (2) \) of quantum multiplication

\[
M_{D, k} = D \ast : \mathcal{H}^*(\text{Hilb}(\mathbb{C}^2, k)) \to \mathcal{H}^*(\text{Hilb}(\mathbb{C}^2, k)).
\]

Computed explicitly by Okounkov-P (2010)

Let \( Tr_k = \frac{1}{t_1 + t_2} \text{Trace} \left( M_{D, k} \right) \)

\[
M_D = (t_1 + t_2) \sum_r \left( \frac{(-q)^r}{r} - \frac{(-q)^r}{r} \right) \alpha_r \alpha_r
\]

+ off diagonal terms.
The last step is to evaluate

$$\langle (2) \rangle_{\text{Hilb}(\mathbb{F},d)} = \sum_{n=0}^{\infty} q^n \langle (2) \rangle_{1, \beta_n}$$

H.-H. Tseng and I found a conjectural answer:

Conjecture (H.-H. Tseng - P 2023)

$$- \langle (2) \rangle_{\text{Hilb}(\mathbb{F},d)} = - \frac{1}{24} \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left( \text{Tr}_d + \sum_{k=2}^{d-1} \frac{6(d-k)}{d-k} \text{Tr}_X \right)$$
Example $d = 2$:

\[
\left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left\langle \lambda_{g-2} \lambda_g \middle| \langle 2 \rangle \right\rangle_{g, 2} = \left\langle \langle 2 \rangle \right\rangle_{g, 2}^{\pi \times \Phi^2}.
\]

Convention:

$g$ terms are summed as

\[
\sum_{g \geq 0} \hat{u}^{2g-3} \ldots
\]

and $-q = \exp(i\pi)$

Example $d = 3$:

\[
\left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left\langle \lambda_{g-2} \lambda_g \middle| \langle 2 \rangle \right\rangle_{g, 3} + \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left\langle \lambda_{g-2} \lambda_g \middle| \langle 2 \rangle \right\rangle_{g, 2}
\]

\[
- \left( - \frac{1}{24} \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 (-i) \cdot \text{Tr}_2(q) \right)
\]

\[
\ll \left\langle \langle 2 \rangle \right\rangle_{g, 3}^{\pi \times \Phi^2}.
\]
Example \( d = 4 \):

\[
\left( \frac{t_1 + t_2}{t, t_2} \right)^2 \left\langle \frac{\gamma_3 - \gamma_4}{(2)} \right\rangle_{g, 4} + \left( \frac{t_1 + t_2}{t, t_2} \right)^2 \left\langle \frac{\gamma_3 - \gamma_3}{(2)} \right\rangle_{g, 3}^\circ + \left( \frac{t_1 + t_2}{t, t_2} \right)^2 \left\langle \frac{\gamma_3 - \gamma_3}{(2)} \right\rangle_{g, 2}^\circ \cdot 2
\]

\[-\frac{1}{24} \left( \frac{t_1 + t_2}{t, t_2} \right) \left\langle (2) \right\rangle_{g, 3}^\circ \cdot \widehat{\mathcal{P}}_\text{art} (1)\]

\[-\frac{1}{24} \left( \frac{t_1 + t_2}{t, t_2} \right) \left\langle (2) \right\rangle_{g, 2}^\circ \cdot \widehat{\mathcal{P}}_\text{art} (2)\]

\[
\left\langle (2) \right\rangle_{g, 4}^\circ \cdot \mathcal{E} \times \mathbb{C}^2
\]
We simplify as

\[
\begin{align*}
\frac{(t_1 + t_2)^2}{t_1t_2} & \left\langle \frac{\lambda_3^2 \lambda_3}{(2)} \right\rangle_{g,4}^\pi \cdot \frac{(t_1 + t_2)^2}{t_1t_2} \left\langle \frac{\lambda_3^2 \lambda_3}{(2)} \right\rangle_{g,4}^\pi \\
& + \frac{(t_1 + t_2)^2}{t_1t_2} \left\langle \frac{\lambda_3^2 \lambda_3}{(2)} \right\rangle_{g,2}^\pi \cdot 2 \\
& - \left( -\frac{1}{2} \frac{(t_1 + t_2)^2}{t_1t_2} \right) \cdot (-i) \left( \text{Tr}_3 - \text{Tr}_2 \right) \cdot 1 \\
& - \left( -\frac{1}{2} \frac{(t_1 + t_2)}{t_1t_2} \right) \cdot (-i) \text{Tr}_2 \cdot \frac{5}{2} \\
\end{align*}
\]

\[
\langle (2) \rangle_{g,4}^\pi \cdot \xi^2 \cdot \pi \cdot \Phi^2 
\]
Projection of $NL_d$

By definition:

$$P_{\mathcal{A}}([NL_d]) \in R^{q-1}(A_g).$$

Let $s_{g,d} \in CH^{q-1}(A_g)$,

$$s_{g,d} \in \ker(P_{\mathcal{A}}),$$

be the non tautological part:

$$[NL_d] = P_{\mathcal{A}}([NL_d]) + s_{g,d}.$$
By definition of $P_A$,

$$\langle \delta_{g,d}, \gamma \rangle_{A_g} = \int \overline{\delta_{g,d}} \cdot \gamma \cdot \lambda_g$$

lifting of $\gamma$ classes

$$= 0$$

for all $\gamma \in R^{(g_2) - (g-1)}(A_g)$.

We have seen before that

$$\lambda_{g-1} \cdot [N L_d] = 0 \in R^{2g-2}(A_g).$$
So we have

\[ 0 = \lambda_{g-1} \cdot \Pr_A \left( \begin{bmatrix} NL_d \end{bmatrix} \right) + \lambda_{g-1} \cdot S_{g,d} \]

Certainly

\[ \lambda_{g-1} \cdot \Pr_A \left( \begin{bmatrix} NL_d \end{bmatrix} \right) \in \mathbb{R}^{2g-2} (A_g) \]

Claim: \[ \lambda_{g-1} \cdot S_{g,d} \in \ker \left( \Pr_A \right) \]

Proof: \[ \langle \lambda_{g-1} \cdot S_{g,d}, \gamma \rangle_{A_g} = \langle S_{g,d}, \lambda_{g-1} \cdot \gamma \rangle_{A_g} \]

\[ \forall \gamma \in \mathbb{R}^{\binom{g}{2} - (2g-2)} (A_g) \]
Therefore, since

\[ R^{2g-2} (A_j) \cap \ker (\Pr_A) = 0, \]

\[ \lambda_{g-1} \cdot \Pr_A (\lbrack NL_d \rbrack) = 0, \]

\[ \lambda_{g-1} \cdot \delta_{g,d} = 0. \]

As before, we conclude

\[ \Pr_A (\lbrack NL_d \rbrack) = \hat{C}_{g,d} \cdot \lambda_{g-1}. \]
If \( [NL_d] \in \mathbb{R}^*(A_g) \), then
\[
\hat{C}_{g,d} = C_{g,d}
\]
defined by projection

Computed previously using \( \text{Hilb}(\mathbb{C}^2, k) \)

Conjecture (Canning - Oprea - P 2023)

for all \( g \geq 2, \ d \geq 1 \):
\[
\hat{C}_{g,d} = C_{g,d}
\]

Probably \( g = 1 \) also works with careful definitions as a degenerate case.
The $d = 1$ case follows from

\[ \Pr_A \left( [A_1 \times A_{g-1}] \right) = \frac{g}{6 |\beta_{2g}|} \lambda_{g-1}. \]

Theorem (Cavanna-Oprea-P 2023):
If the vanishing conjecture holds,

\[ \Pr_A \left( [A_1 \times A_{g-1}] \right) = \frac{g}{6 |\beta_{2g}|} \lambda_{g-1}. \]

Together with the calculation of $c_{g,1}$.

In general, we have

\[ [N L_d] = \hat{c}_{g,d} \cdot \lambda_{g-1} + \delta_{g,d} \]

with \( \delta_{g,d} \in \text{Ker}(\Pr_A) \)

and \( \alpha \cdot \delta_{g,d} = 0 \)
\( \forall \alpha \in R^*(A_g) \text{ satisfying } \alpha \cdot \gamma_{g-1} = 0, \)

\[
( \alpha \in \text{Ann} (\gamma_{g-1}) ) .
\]

In order to prove

\[ \hat{C}_{g,d} = C_{g,d} , \]

we must show

\[ \int \text{Tor}^* \left( \sigma_{g,d} \right) \cdot \gamma_{g-2} \gamma_g = 0 . \]

I see two possible paths to prove
I point out that

\[ \text{Tor}_*^* \left( \mathcal{M}_g^{ct} \right) \in \mathbb{L}^* \left( \mathcal{A}_g \right) \]

\[ \downarrow \]

\[ \int \left( \text{Tor}_*^* \left( \delta_{g,d} \right) \cdot \mathcal{A}_{g-2} \mathcal{A}_g = 0. \right) \]

\[ \overline{\mathcal{M}}_g \]

But there is not much reason to believe that \( \text{Tor}_*^* \left( \mathcal{M}_g^{ct} \right) \)

is tautological.
The best reason to believe

\[ \hat{C}_{g,d} = C_{g,d} \]

is a conjecture by Aitor:

Conjecture (Iribar López 2024)

\[ CH^*(A_g) \xrightarrow{\text{Pr}_A} R^*(A_g) \]

is a ring homomorphism.

What limited evidence that we have supports this claim.

(at least for the subring of \( CH^*(A_g) \) generated by \( NL \) and Jacobian loci.)
Update April 2024 (by Aitor)

Using the equation (which we know now)

\[
\int \text{Tor}^* \left( \delta_{g,d} \right) \cdot \alpha_{g-2} \alpha_g = 0
\]

\[\overline{\mu}_g\]

and boundary arguments by Pixton,

the homomorphism property is established

in the following case:

Let \( T \in R^* (M^\text{ct}_g) \) be any class.

Then we have

\[
\Pr_A \left( \text{Tor}_* T \cdot \left[ NL_d \right] \right)
\]

\[
\Pr_A \left( \text{Tor}_* T \right) \cdot \Pr_A \left( \left[ NL_d \right] \right).
\]
Calculation of the projection of $N_{L_d}$

by Aitor Iribar López:

We have already proven

$$Pr_A ([ N_{L_d} ]) \Rightarrow \hat{C}_{g,d} \cdot \lambda_{g-1} \in \Lambda^*(A_g)$$

Theorem A (Iribar López 2024)

$$\hat{C}_{g,d} = \prod_{p \mid d} (1 - \frac{-2g+2}{p}) \cdot \frac{g}{6|B_{2g}|}$$

Aitor's proof uses the geometry of the moduli of abelian varieties with level structures.
Let \( c_{g,d} \) be computed using the conjectural formula for \( \text{Hilb}(\mathbb{F}^2, e) \):

\[
\langle (2) \rangle_{\text{Hilb}(\mathbb{F}^2, e)}^{\text{Hilb}(\mathbb{F}^2, e)}, \quad 2 \leq e \leq d.
\]

**Theorem B** (Iriber López 2024)

For all \( g \geq 2, \ d \geq 1 \):

\[
\hat{c}_{g,d} = c_{g,d}.
\]

* here denotes the dependence on the conjectural formula for \( \text{Hilb}(\mathbb{F}^2) \).
Aifor's results yield the following implication

Conjecture (Iribar López 2024)

$$\text{CH}^*(A_g) \xrightarrow{Pr_A} R^*(A_g)$$

is a ring homomorphism.

Conjecture (H.-H. Tseng - P 2023)

$$-\left\langle \left\langle 2 \right\rangle \right\rangle_{\text{Hilb}(\mathbf{P}^2,d)} =$$

$$-\frac{1}{24} \frac{(t_1+t_2)^2}{t_1t_2} \left( \text{Tr}_d + \sum_{k=2}^{d-1} \frac{d-d-k}{d-k} \text{Tr}_k \right).$$
Appendix: Update March 2024

There is a new path to prove:

Conjecture (H.-H. Tseng - P 2023)

\[- \langle (2) \rangle_{1}^{\text{Hilb}(\mathcal{C}, d)} = \]

\[- \frac{1}{24} \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left( T_{r_d} + \sum_{k=2}^{d-1} \frac{6(d-k)}{d-k} T_{r_k} \right). \]

We have seen that calculating the following Connected Gromov-Witten integral is sufficient:
\[
\int T_1(q) \ \overline{\mathcal{M}}_{g-2,2} \bigg[ \overline{\mathcal{M}}_{g,1}(\pi, \ell) \bigg]^{\text{vir}} \ll \\
\left\langle T_1(q) \ \overline{\mathcal{M}}_{g-2,2} \right\rangle_{g,d}^{\pi}\bigg[ \overline{\mathcal{M}}_{g,1} \bigg]^{\pi_0}
\]

Here \( \overline{\mathcal{M}}_{g,1} \) is the moduli of pointed non-singular elliptic curves and

\[
\begin{array}{c}
\pi \\
\downarrow
\end{array}
\]

\( q \) \quad \text{zero section}
The first idea is to switch to an elliptically fibered $K3$ surface:

$$
\begin{array}{c}
S \\
\pi_S \\
\mathbb{P}^1
\end{array} \xrightarrow{q} \quad 24 \text{ nodal fibers}
$$

The fibers of $\pi_S$ are 1-pointed stable genus 1 curves.

The induced morphism

$$
\mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}
$$

is of degree 48.
Then we have

\[
\int T_1(q) \, \mathfrak{g}_{g-2} \, \mathfrak{g}_g
\]

\[
\left[ \overline{M}_{g,1}(\pi, d) \right]^{\text{vir}}
\]

\[
\downarrow
\]

\[
\frac{1}{48} \int T_1(q) \, \mathfrak{g}_{g-2} \, \mathfrak{g}_g
\]

\[
\left[ \overline{M}_{g,1}(\pi_5, d) \right]^{\text{vir}}
\]

• The second idea is to use K3 vanishing.
Consider the integral:

\[ \int T_1(q) \, \mathcal{A}_{g-2} = 0 \]

\[ [\overline{M}_{g,1}(S,d)]^{vir} \]

\[ d \text{ times fiber class of } \pi_g, \quad d > 0. \]

The above vanishing will give us a nontrivial relation.
Claim A:

\[ \int T_1(q) \, \gamma_{g-2} \, e(\text{IE}^v \otimes \text{Tan}_p) \]

\[ [\bar{M}_{g,1}(\pi_5, d)]^{\text{vir}} \parallel \]

\[ \int T_1(q) \, \gamma_{g-2} \]

\[ [\bar{M}_{g,1}(S, d)]^{\text{vir}} . \]

Corollary:

\[ \int T_1(q) \, \gamma_{g-2} \, e(\text{IE}^v \otimes \text{Tan}_p) = 0 . \]

\[ [\bar{M}_{g,1}(\pi_5, d)]^{\text{vir}} \]
Proof: There is a morphism

$$\overline{M}_{g,1}(\pi_5, d) \rightarrow \overline{M}_{g,1}(S, d)$$

which is an isomorphism of DM stacks away from the 24 nodal fibers of $\pi_5$. Moreover, away from the 24 nodal fibers, the obstruction theory of $\overline{M}_{g,1}(\pi_5, d)$ augmented by $\text{IE}^V \otimes \text{Tan}_{\rho}$ matches the standard obstruction theory of $\overline{M}_{g,1}(S, d)$. 
The entire issue is about the nodal fibers

\[
\begin{array}{c}
S \\
\downarrow \quad \pi_S \\
\mathbb{P}^1
\end{array}
\Rightarrow
\begin{array}{c}
\text{some} \\
\downarrow
\end{array}
\]

We use here the degeneration to the normal cone of the divisor \( \alpha \subset S \) of nodal fibers, a standard technique, but a complication here is that \((S, \alpha)\) requires \(\log GW\) (since \(\alpha\) is singular).
We study the normal cone

\[ X = Bl(S \times \mathbb{C}, \alpha \times 0) \]

\[ \downarrow \]

\[ \mathbb{C} \]

\[ X \] has a single singularity
(a 3-fold double point)
over each point \( p \times 0 \)
where \( p \in \alpha \) is a node.

The main observation here:
we can avoid all log complication
by studying \( X^\circ \backslash X \).

\[ t \text{ nonsingular locus} \]
The reason that the noncompact log geometry $\mathcal{X}^0 \subset \mathcal{X}$ can be used here is that the curve classes are fibers and have intersection 0 with $\alpha$. Said differently: the moduli spaces of log stable maps to the log degeneration $\mathcal{X}^0$ are compact. Then the usual degeneration calculus of relative GW theory can be used.
After degeneration, the equality of Claim A is clear since the geometric differences of the moduli spaces vanish.

A second proof of Claim A would follow by constructing a coaction for the obstruction theory on $\bar{M}_{g,1}(\pi_5,d)$ obtained by combining the fiberwise deformation with $\mathcal{E}^V \otimes \mathcal{P}^l$. 
• The third step is to expand
\[ e(\mathfrak{I}E^\vee \otimes \text{Tan}_{p^1}) = (-1)^g \lambda_g + (-1)^{g-1} \lambda_{g-1} \cdot [2\text{pt}] \]
so we obtain
\[ 0 = \int \tau_1(q) \bar{\alpha}_{g-2} \cdot e(\mathfrak{I}E^\vee \otimes \text{Tan}_{p^1}) \]
\[ \left[ \overline{\mathcal{M}}_{g,1}(\pi_5, d) \right]^{\text{vir}} \]
\[ = \int \tau_1(q) \bar{\alpha}_{g-2} \cdot (-1)^g \lambda_g \]
\[ \left[ \overline{\mathcal{M}}_{g,1}(\pi_5, d) \right]^{\text{vir}} \]
\[ + 2 \int \tau_1(q) \bar{\alpha}_{g-2} \cdot (-1)^{g-1} \lambda_{g-1} \cdot \overline{\mathcal{M}}_{g,1}(E, d)^{\text{vir}} \]
After rewriting, we find

\[
\int T_i(q) \quad \mathcal{A}_{g-2} \mathcal{A}_g \\
\left[ \overline{\mathcal{M}}_{g,1}(\pi, d) \right]^\text{vir}
\]

\[=\]

\[
\frac{1}{24} \int T_i(q) \quad \mathcal{A}_{g-2} \mathcal{A}_{g-1} \\
\left[ \overline{\mathcal{M}}_{g,1}(E, d) \right]^\text{vir}
\]

fixed elliptic target
The last step in the evaluation of the latter integral by Pixton (2008):

$$\sum_{d \geq 0} \mathcal{Q}^d \int T_1(q) \, \mathcal{Q}^{2g-2} \mathcal{Q}^{2g-1} \left[ \overline{M}_{g,1}(\mathcal{E}, d) \right]^{vir} = \left| B_{2g-2} \right| \cdot \binom{2g}{2} C_{2g}(\mathcal{Q})$$

where

$$C_{2g}(\mathcal{Q}) = - \frac{B_{2g}}{2g \cdot 2g!} + \frac{z}{2g!} \sum_{n \geq 1} \delta_{2g-1}^{(n)} \mathcal{Q}^n,$$
In other words,

\[ C_{2g}(Q) = \frac{-B_{2g}}{2g \cdot 2g!} E_{2g}(Q). \]

See page 32 of


for the results of Pixton.
Claim B: The evaluation

\[ \sum_{d \geq 0} Q^d \int T_1(q) \bar{\mathcal{M}}_{g,1}(\bar{\pi}, d) \] 

\[ \equiv \frac{1}{24} |B_{2g-2}| \cdot \binom{2g}{2} C_{2g}(Q) \]

is equivalent to the conjectured formula for \( \langle 2 \rangle_{\text{Hilb}(\mathbb{P}^2, d)} \).

Proof by Iriber López.
The status now is that all the claims related to

\[ P_{r_A}(\left[ NL_d \right]) \in \mathbb{R}^{g-1}(A_g) \]

and the series \( \langle(2)\rangle \)

are proven:

\[ \hat{C}_{g,d} \cdot \lambda_{g-1} = P_{r_A}(\left[ NL_d \right]) \]

\[ \hat{C}_{g,d} = \prod_{p \mid d} \left(1 - p^{-2g+2}\right) \cdot \frac{g}{c|\beta_{2g}|} \]

[by Iriber López]
\[-\left\langle (2) \right\rangle_{\text{Hilb}(\mathcal{H}, d)} =
\frac{1}{24} \left( \frac{t_1 + t_2}{t_1 t_2} \right)^2 \left( T_{d} + \sum_{k=2}^{d-1} \frac{c(d-k)}{d-k} T_{d-k} \right),\]

[by claim A + B]

definition
\[c_{g,d} = \frac{\int \overline{\mu}_g \cdot \overline{\alpha}_{g-2} \overline{\lambda}_g}{\int \overline{\mu}_g \cdot \overline{\alpha}_{g-2} \overline{\lambda}_{g-1} \overline{\lambda}_g},\]

\[\hat{c}_{g,d} = c_{g,d} \text{ by calculation of}\]
\[\left\langle (2) \right\rangle_{\text{Hilb}(\mathcal{H}, d)}\]
Many open directions remain.

My favorites:

**Conjecture (Iribar López 2024)**

\[
\text{CH}^*(A_g) \xrightarrow{\text{Pr}_A} \mathcal{R}^*(A_g)
\]

is a ring homomorphism.

- Study the extension of the diagram

\[
\begin{array}{ccc}
\text{Tor}_i^{-1}(NL_d) & \xrightarrow{} & NL_d \\
\downarrow & & \downarrow \\
M_g^{ct} & \xrightarrow{\text{Tor}_i} & A_g
\end{array}
\]
to the perfect cone compactifications

\[ \text{Tor}_i^{-1}(NL_d) \quad \text{to} \quad \text{NL}_d \]

\[ \overline{M}_{g,1} \quad \text{to} \quad \overline{A}_g \quad \text{Perfect cone} \]

- Calculate

\[ \langle 6_1, 6_2, \ldots, 6_n \rangle_{\text{Hilb}(\mathbb{P}^2, d)} \]

for arbitrary partition insertions 6_i.
The End

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