The Hodge bundle, the universal 0-section, and the log Chow ring of the moduli space of curves

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January 2021

Abstract

We bound from below the complexity of the top Chern class λ_g of the Hodge bundle in the Chow ring of the moduli space of curves: no formulas for λ_g in terms of classes of degrees 1 and 2 can exist. As a consequence of the Torelli map, the 0-section over the second Voronoi compactification of the moduli of principally polarized abelian varieties also can not be expressed in terms of classes of degree 1 and 2. Along the way, we establish new cases of Pixton's conjecture for tautological relations.

In the log Chow ring of the moduli space of curves, however, we prove λ_g lies in the subalgebra generated by logarithmic boundary divisors. The proof is effective and uses Pixton's double ramification cycle formula together with a foundational study of the tautological ring defined by a normal crossings divisor. The results open the door to the search for simpler formulas for λ_g on the moduli of curves after log blow-ups.

Contents

1 Introduction

1.1 The Hodge bundle

Let $\overline{\mathcal{M}}_g$ be the moduli space of Deligne-Mumford stable curves, and let

$$\pi : \mathcal{C}_g \to \overline{\mathcal{M}}_g$$

be the universal curve with relative dualizing sheaf ω_π. The rank g Hodge bundle $\mathbb{E}_g$ on $\overline{\mathcal{M}}_g$ is defined by

$$\mathbb{E}_g = \pi_* \omega_\pi \,.$$

The study of the Chern classes of the Hodge bundle goes back at least to Mumford's Grothendieck-Riemann-Roch calculation [42] in the 1980s. Starting in the late 1990s, the connection of the Hodge bundle to the deformation theory of the moduli space of stable maps has led to an exploration of Hodge integrals in various contexts, see [2, 16, 17, 21, 36, 37, 38, 44, 46].

The top Chern class[1] of the Hodge bundle

$$\lambda_g = c_g(\mathbb{E}_g) \in \mathsf{CH}^g(\overline{\mathcal{M}}_g)$$

plays a special role for several reasons:

(i) Two *vanishing properties* hold:

$$\lambda_g^2 = 0 \in \mathsf{CH}^{2g}(\overline{\mathcal{M}}_g) \quad \text{and} \quad \lambda_g|_{\Delta_0} = 0 \in \mathsf{CH}^g(\Delta_0)\,,$$

where $\Delta_0 \subset \overline{\mathcal{M}}_g$ is the divisor of curves with a non-separating node. The first vanishing follows from the highest graded part of Mumford's relation

$$c(\mathbb{E}_g) \cdot c(\mathbb{E}_g^*) = 1\,,$$

and the second follows from the existence of a trivial quotient[2]

$$\mathbb{E}_g \twoheadrightarrow \mathbb{C}$$

determined by the residue at (a branch of) the node, see [18, Section 0.4].

(ii) The class $(-1)^g \lambda_g$ appears in the virtual fundamental class of the moduli of *contracted maps* in the Gromov-Witten theory of target curves. Since the *double ramification cycle* in the degree 0 case is defined via contracted maps, we have

$$\mathsf{DR}_{g,(0,\ldots,0)} = (-1)^g \lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,,$$

[1] All Chow classes are taken here with $\mathbb{Q}$-coefficients.

[2] The quotient is defined on the double cover of Δ_0 obtained by ordering the branches of the node.

where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable pointed curves. See [31, Sections 0.5.3 and 3.1].

Another basic consequence is the λ_g-formula [19],

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \dots, k_n} \cdot \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2} \lambda_g \,,$$

predicted by the Virasoro constraints for degree 0 maps to curves [20]. Here,

$$\psi_i = c_1(\mathbb{L}_i) \in \mathsf{CH}^1(\overline{\mathcal{M}}_{g,n})$$

is the Chern class of the cotangent line at the i^{th} point. The λ_g-formula plays a central role in the study of the tautological ring $\mathsf{R}^\star(\mathcal{M}_{g,n}^{\mathsf{ct}})$ of the moduli space of curves of compact type [47].

(iii) Again as an excess class, $(-1)^g \lambda_g$ appears fundamentally in the local Gromov-Witten theory of surfaces. For example, the Katz-Klemm-Vafa formula [34] proven in [39, 50] concerns integrals

$$\int_{[\overline{\mathcal{M}}_g(S,\beta)]^{\mathsf{red}}} (-1)^g \lambda_g$$

against the reduced virtual fundamental class of the moduli space of stable maps to $K3$ surfaces. For a recent study of the parallel problem for local log Calabi-Yau surfaces (with integrand $(-1)^g \lambda_g$), see [10].

(iv) The class $(-1)^g \lambda_g$ arises via the pull-back of the universal 0-section of the moduli space of *principally polarized abelian varieties* (PPAVs). Over the moduli space of compact type curves, the connection to PPAVs shows a third vanishing property:

$$\lambda_g|_{\mathcal{M}_g^{\mathsf{ct}}} = 0 \,,$$

see [56]. We will discuss PPAVs further in Section 1.2 below.

Our main results here concern the complexity of the class λ_g in the Chow ring. For $\overline{\mathcal{M}}_g$, we bound from below the complexity of formulas for

$$\lambda_g \in \mathsf{CH}^\star(\overline{\mathcal{M}}_g) \,.$$

As a consequence of the connection to the moduli of PPAVs, we also bound from below the complexity of formulas for the universal 0-section.

The log Chow ring of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$ is defined as a limit over all iterated blow-ups of boundary strata. The usual Chow ring is naturally a subalgebra

$$\mathsf{CH}^\star(\overline{\mathcal{M}}_g) \subset \mathsf{logCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \,.$$

The main positive result of the paper is the simplicity of λ_g in the log Chow ring. We prove

$$\lambda_g \in \mathsf{divlogCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \,,$$

where

$$\mathsf{divlogCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \subset \mathsf{logCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

is the subalgebra generated by logarithmic boundary divisors.

While λ_g in Chow is complicated, λ_g in log Chow is as simple as possible! We present several related open questions.

1.2 The 0-section

Let $\mathcal{A}_g$ be the moduli space of PPAVs of dimension g, and let

$$\pi : \mathcal{X}_g \to \mathcal{A}_g$$

be the universal abelian variety π equipped with a universal 0-section

$$s : \mathcal{A}_g \to \mathcal{X}_g\,.$$

The image of the 0-section determines an algebraic cycle class

$$Z_g \in \mathsf{CH}^g(\mathcal{X}_g)\,.$$

The second Voronoi compactification of $\mathcal{A}_g$ has been given a modular interpretation by Alekseev:

$$\mathcal{A}_g \subset \overline{\mathcal{A}}_g^{\mathsf{Alekseev}}\,.$$

Olsson [45] provided a modular interpretation for the normalization

$$\overline{\mathcal{A}}^{\mathsf{Olsson}} \to \overline{\mathcal{A}}_g^{\mathsf{Alekseev}}\,.$$

Our approach here will be equally valid for both $\overline{\mathcal{A}}^{\mathsf{Olsson}}$ and $\overline{\mathcal{A}}_g^{\mathsf{Alekseev}}$. We will simply denote the compactification by

$$\mathcal{A}_g \subset \overline{\mathcal{A}}_g\,,$$

where $\overline{\mathcal{A}}_g$ stand for either the space of Alekseev or the space of Olsson.

The four important properties[3] of the compactification $\overline{\mathcal{A}}_g$ which we will require are:

- The points of $\overline{\mathcal{A}}_g$ parameterize (before normalization) stable semiabelic pairs which are quadruples (G, P, L, θ) where G is a semiabelian variety, P is a projective variety equipped with a G-action, L is an ample line bundle on P, and $\theta \in H^0(P, L)$. The data (G, P, L, θ) satisfy several further conditions, see Section 4.2.16 of [45].

- There is a universal semiabelian variety

$$\overline{\pi} : \overline{\mathcal{X}}_g \to \overline{\mathcal{A}}_g$$

[3] We follow the notation of [45].

with a 0-section
$$\overline{s} : \overline{\mathcal{A}}_g \to \overline{\mathcal{X}}_g$$
corresponding to the semiabelian variety which is the first piece of data of a stable semiabelic pair (the rest of the pair data will not play a role in our study).

- The usual Torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$ extends canonically
$$\overline{\tau} : \overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g \,,$$
see [3].

- The $\overline{\tau}$-pullback to $\overline{\mathcal{M}}_g$ of $\overline{\mathcal{X}}_g$ is the universal family
$$\mathsf{Pic}^0_\epsilon \to \overline{\mathcal{M}}_g$$
parameterizing line bundles on the fibers of the universal curve
$$\epsilon : \mathcal{C}_g \to \overline{\mathcal{M}}_g$$
which have degree 0 *on every component* of the fiber [3].

The image of the 0-section $\overline{s}$ determines an operational Chow class
$$\overline{Z}_g \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{X}}_g)$$
since the image is an étale local complete intersection in $\overline{\mathcal{X}}_g$. The class $\overline{Z}_g$ is related to $(-1)^g\lambda_g$ via a pull-back construction. Let
$$t : \overline{\mathcal{M}}_g \to \mathsf{Pic}^0_\epsilon$$
be the 0-section defined by the trivial line bundle. By the properties of
$$\overline{\pi} : \overline{\mathcal{X}}_g \to \overline{\mathcal{A}}_g$$
discussed above,
$$\overline{\tau}^*\overline{s}^*(\overline{Z}_g) = t^*(t_*[\overline{\mathcal{M}}_g]) \,.$$
By the standard analysis of the vertical tangent bundle of Pic^0_ϵ,
$$t^*(t_*[\overline{\mathcal{M}}_g]) = (-1)^g\lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_g) \,.$$
We conclude
$$\overline{\tau}^*\overline{s}^*(\overline{Z}_g) = (-1)^g\lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_g) \,. \tag{1}$$

1.3 Complexity of the 0-section

The study the 0-section over $\mathcal{A}_g$ is related to the double ramification cycle (especially over curves of compact type), see Hain [24] and Grushevsky-Zakharov [22]. A central idea there is to use the beautiful formula

$$Z_g = \frac{\Theta^g}{g!} \in \mathsf{CH}^g(\mathcal{X}_g)\,, \tag{2}$$

where $\Theta \in \mathsf{CH}^1(\mathcal{X}_g)$ is the universal symmetric theta divisor trivialized along the 0-section. The proof of (2) in Chow uses the Fourier-Mukai transformation and work of Deninger-Murre [13], see [8, 57]. The article [22] provides a more detailed discussion of the history of (2).

We are interested in the following question: *to what extent is an equation of the form of* (2) *possible over* $\overline{\mathcal{A}}_g$*?* A result by Grushevsky and Zakharov along these lines appears in [23]. As before, let

$$\overline{Z}_g \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{X}}_g)$$

be the class of the 0-section $\overline{s}$. Grushevsky and Zakharov calculate the restriction $\overline{Z}_g|_{\mathcal{U}_g}$ of $\overline{Z}_g$ over a particular open set[4]

$$\mathcal{A}_g \subset \mathcal{U}_g \subset \overline{\mathcal{A}}_g$$

in terms of Θ, a boundary divisor $D \in \mathsf{CH}^1(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$, and a class

$$\Delta \in \mathsf{CH}^2(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})\,.$$

The result of Grushevsky-Zarkhov shows that while the naive extension of (2) does *not* hold over $\mathcal{U}_g$, the class $\overline{Z}_g|_{\mathcal{U}_g}$ lies in the subalgebra of $\mathsf{CH}^\star(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$ generated by classes of degrees 1 and 2. The formula of [23] is a useful extension of (2).

The divisor classes $\mathsf{CH}^1_{\mathsf{op}}(\overline{\mathcal{X}}_g)$ generate a subalgebra

$$\mathsf{divCH}^\star_{\mathsf{op}}(\overline{\mathcal{X}}_g) \subset \mathsf{CH}^\star_{\mathsf{op}}(\overline{\mathcal{X}}_g)\,.$$

The first bound from below of the complexity of the class of the 0-section is the following result.

Theorem 1 *For all $g \geq 3$, we have $\overline{Z}_g \notin \mathsf{divCH}^\star_{\mathsf{op}}(\overline{\mathcal{X}}_g)$.*

As a consequence, no divisor formula extending (2) is possible for $\overline{\mathcal{A}}_g$. Though not stated, the analysis of [23] over $\mathcal{U}_g$ can be used to show $\overline{Z}_g|_{\mathcal{U}_g}$ is *not* in the subalgebra of $\mathsf{CH}^\star(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$ generated by classes of degree 1. Theorem 1 can therefore also be obtained from [23].[5]

[4] $\mathcal{U}_g$ is the locus determined by semiabelian varieties of torus rank at most 1.

[5] We thank S. Grushevsky for correspondence about [23].

In fact, we can go further. Let

$$\mathsf{CH}^\star_{\leq k}(\overline{\mathcal{X}}_g) \subset \mathsf{CH}^*_{\mathsf{op}}(\overline{\mathcal{X}}_g)$$

be the subalgebra generated by all elements of degree at most k, so

$$\mathsf{divCH}^\star_{\mathsf{op}}(\overline{\mathcal{X}}_g) = \mathsf{CH}^\star_{\leq 1}(\overline{\mathcal{X}}_g)\,.$$

Theorem 2 *For all $g \geq 7$, we have $\overline{Z}_g \notin \mathsf{CH}^\star_{\leq 2}(\overline{\mathcal{X}}_g)$.*

By Theorem 2, the Grushevsky-Zakharov formula for $\overline{Z}_g|_{\mathcal{U}_g}$ will require corrections by higher degree classes when extended over $\overline{\mathcal{A}}_g$. We propose the following conjecture about the complexity of the class $\overline{Z}_g$.

Conjecture A. *No extension of* (2) *over $\overline{\mathcal{A}}_g$ for all g can be written in terms of classes of uniformly bounded degree.*

The pull-back relation (1) relates the complexity of the class

$$\lambda_g \in \mathsf{CH}^\star(\overline{\mathcal{M}}_g)$$

to the complexity of $\overline{Z}_g \in \mathsf{CH}^\star_{\mathsf{op}}(\overline{\mathcal{X}}_g)$. Theorems 1 and 2 will be immediate consequence of parallel[6] complexity bounds for λ_g.

1.4 Complexity of λ_g

The divisor classes $\mathsf{CH}^1(\overline{\mathcal{M}}_g)$ generate a subalgebra

$$\mathsf{divCH}^\star(\overline{\mathcal{M}}_g) \subset \mathsf{CH}^\star(\overline{\mathcal{M}}_g)\,.$$

The first bound from below of the complexity of λ_g is the following result.

Theorem 3 *For all $g \geq 3$, we have $\lambda_g \notin \mathsf{divCH}^\star(\overline{\mathcal{M}}_g)$.*

Via the pull-back relation (1), Theorem 3 immediately implies Theorem 1. The proof of Theorem 3, presented in Section 2, starts with explicit calculations in the tautological ring in genus 3 and 4 using the Sage package *admcycles* [12]. A boundary restriction argument is then used to inductively control all higher genera.

For the analogue of Theorem 2, let

$$\mathsf{CH}^\star_{\leq k}(\overline{\mathcal{M}}_g) \subset \mathsf{CH}^*(\overline{\mathcal{M}}_g)$$

be the subalgebra generated by all elements of degree at most k. A similar strategy (with a much more complicated initial calculation in genus 5) yields the following result which implies Theorem 2.

[6] In fact, we will prove in Section 2 stronger results in cohomology instead of Chow.

Theorem 4 *For all $g \geq 7$, we have $\lambda_g \notin \mathsf{CH}^\star_{\leq 2}(\overline{\mathcal{M}}_g)$.*

The proofs of Theorems 3 and 4 require new cases of Pixton's conjecture about the ideal of relations in the tautological ring

$$\mathsf{R}^\star(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{CH}^\star(\overline{\mathcal{M}}_{g,n}) \,.$$

Proposition 5 *Pixton's relations generate all relations among tautological classes in $\mathsf{R}^4(\overline{\mathcal{M}}_{4,1})$ and $\mathsf{R}^5(\overline{\mathcal{M}}_{5,1})$.*

While the above arguments become harder to pursue in general for $\mathsf{CH}^\star_{\leq k}(\overline{\mathcal{M}}_g)$, we expect the following to hold.

Conjecture B. *For fixed k, $\lambda_g \in \mathsf{CH}^\star_{\leq k}(\overline{\mathcal{M}}_g)$ holds only for finitely many g.*

Of course, Conjecture B implies Conjecture A.

1.5 Log Chow

Theorems 1-4 about the classes $\overline{Z}_g$ and λ_g are in a sense negative results since formula types are excluded. Our main positive result about λ_g concerns the larger log Chow ring

$$\mathsf{CH}^\star(\overline{\mathcal{M}}_g) \subset \mathsf{logCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \,.$$

The log Chow ring and the subalgebra

$$\mathsf{divlogCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

generated by logarithmic boundary divisors is defined carefully in Section 3. Our perspective, using limits over log blow-ups, requires the least background in log geometry. A more intrinsic approach to the definitions can be found in [7].

Theorem 6 *For all $g \geq 2$, we have $\lambda_g \in \mathsf{divlogCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$.*

Our proof of Theorem 6 is constructive: we start with Pixton's formula for the double ramification cycle for constant maps [31] and show each term lies in $\mathsf{divlogCH}^\star(\overline{\mathcal{M}}_g)$. In principle, bounds for the necessary log blow-ups are possible to obtain from the proof, but these will certainly not be optimal. Finding a minimal (or efficient) sequence of log-blows of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$ after which λ_g lies in the subalgebra of logarithmic boundary divisors is an interesting question.

A crucial part of the proof of Theorem 6 is the study in Section 5 of the logarithmic tautological ring,

$$\mathsf{R}^\star(X, D) \subset \mathsf{CH}^\star(X) \,,$$

defined by a normal crossings divisor $D \subset X$ in a nonsingular variety X. Tautological classes are defined here using the Chern roots of the normal bundle of logarithmic strata $S \subset X$. The precise definitions are given in Section 5.1.

We prove three main structural results about logarithmic tautological classes:

(i) $\mathsf{R}^\star(X,D) \subset \mathsf{divlogCH}^\star(X,D)$,

(ii) pull-backs of tautological classes under log blow-ups are tautological,

(iii) push-forwards of tautological classes under log blow-ups are tautological.

Our first proof of (i) is presented in Section 5.2 via an explicit analysis of *explosions*: sequences of blow-ups associated to logarithmic strata of X. A second approach to (i-iii), via the geometry of the Artin fan of (X,D), is given in Section 5.5. The Artin fan perspective is theoretically more flexible, but less explicit.

After Pixton's formula for the double ramification cycle for constant maps is shown to lie in $\mathsf{R}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$, property (i) implies Theorem 6. Since Pixton's formula and the proof of (i) are both effective, divisor expressions for λ_g are possible to compute. The result reveals the essential simplicity of λ_g and opens the door to the search for a simpler formula in divisors.

The proof of Theorem 6 yields a refined result: only logarithmic boundary divisors over

$$\Delta_0 \subset \overline{\mathcal{M}}_g$$

are needed to generate λ_g. The parallel result is also true for pointed curves:

$$\lambda_g \in \mathsf{divlogCH}^\star(\overline{\mathcal{M}}_{g,n}, \Delta_0)$$

for $2g-2+n>0$.

We have seen that $(-1)^g\lambda_g$ is a special case of the double ramification cycle. The general double ramification cycle

$$\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$$

is defined with respect to a vector of integers $A=(a_1,\ldots,a_n)$ satisfying

$$\sum_{i=1}^n a_i = 0\,.$$

In [28, Appendix A], the double ramification cycle was lifted to log Chow[7],

$$\widetilde{\mathsf{DR}}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})\,. \tag{3}$$

Motivated by Theorem 6, we conjecture a uniform divisorial property of the lifted double ramification cycle (3).

Conjecture C. *For all g and A, we have $\widetilde{\mathsf{DR}}_{g,A} \in \underline{\mathsf{div}}\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n})$ where*

$$\underline{\mathsf{div}}\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n})$$

[7] The paper [28] is primarily formulated in the language of the related bChow ring, which we discuss below and treat in detail in Section 7.

is the subalgebra generated by logarithmic boundary divisors together with the cotangent line classes $\psi_1, \ldots, \psi_n$.

Finally, we return to the Θ-formula (2) for Z_g. *Is an extension of the* Θ*-formula possible over* $\overline{\mathcal{M}}_g$ *in* $\mathsf{logCH}^\star(\overline{\mathcal{M}}_g)$*?* More specifically, can we find

$$\mathsf{T} \in \mathsf{logCH}^1(\overline{\mathcal{M}}_g)$$

which satisfies the following two properties?

(i) The restriction of T over the moduli of curves $\mathcal{M}_g^{\mathsf{ct}}$ of compact type is 0.

(ii) $(-1)^g \lambda_g = \frac{\mathsf{T}^g}{g!} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_g)$.

Property (i) is imposed since

$$\Theta|_{Z_g} = 0 \in \mathsf{CH}^1(Z_g)$$

by the trivialization condition for Θ. Unfortunately, the answer is *no* even for genus 2.

Proposition 7 *There does not exist a class* $\mathsf{T} \in \mathsf{logCH}^1(\overline{\mathcal{M}}_2)$ *satisfying the restriction property (i) and*

$$(-1)^2\lambda_2 = \frac{\mathsf{T}^2}{2!} \in \mathsf{logCH}^2(\overline{\mathcal{M}}_2)\,.$$

The Θ-formula for $(-1)^g\lambda_g$ can *not* be extended in a straightforward way in $\mathsf{CH}^g(\overline{\mathcal{M}}_g)$ or $\mathsf{logCH}^g(\overline{\mathcal{M}}_g)$. However,

$$\lambda_g \in \mathsf{logCH}^g(\overline{\mathcal{M}}_g)$$

is a degree g polynomial in the logarithmic boundary divisors over $\Delta_0 \subset \overline{\mathcal{M}}_g$.

Question D. *Find a polynomial formula in logarithmic boundary divisors for* λ_g *in log Chow (without using Pixton's formula).*

The larger bChow ring of $\overline{\mathcal{M}}_g$ is defined as a limit over *all* blow-ups:

$$\mathsf{CH}^\star(\overline{\mathcal{M}}_g) \subset \mathsf{logCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \subset \mathsf{bCH}^\star(\overline{\mathcal{M}}_g)\,.$$

The bChow ring is by far the largest of the three Chow constructions. In Section 7, we show the main questions of the paper become trivial in bChow. In fact, for every nonsingular variety X, we have

$$\mathsf{divbCH}^\star(X) = \mathsf{bCH}^\star(X)\,.$$

The logarithmic geometry of $\overline{\mathcal{M}}_g$ is therefore the natural place to study Question D for λ_g.

1.6 Acknowledgments

D. Holmes, D. Ranganathan, and J. Wise have suggested that the Θ-formula (2) should extend over the moduli of curves in some form in log geometry (based on their understanding of the logarithmic Picard stack [41]). Our initial motivation here was to study geometric obstructions to such an extension. While the simplest form is excluded, Theorem 6 supports the idea of the existence of some perturbed extension of (2) in log Chow. Our development of the logarithmic tautological ring of (X, D) emerged from the proof of Theorem 6. We are very grateful to Holmes, Ranganathan, and Wise for extensive discussions of these topics.

We have also benefitted from related conversations with Y. Bae, C. Faber, T. Graber, S. Grushevsky, M. Olsson, A. Pixton, R. Vakil, and D. Zakharov. The results of the paper were presented in the *Algebraic Geometry Seminar* at Stanford in the fall of 2020 (with a lively and very helpful discussion afterwards).

S.M. was supported by ERC-2017-AdG-786580-MACI. R.P. was supported by SNF-200020-182181, ERC-2017-AdG-786580-MACI, and SwissMAP. J.S. was supported by the SNF Early Postdoc Mobility grant 184245 and thanks the Max Planck Institute for Mathematics in Bonn for its hospitality. This project has received funding from the European Research Council (ERC) under the European Union Horizon 2020 research and innovation program (grant agreement No 786580).

2 λ_g in the Chow ring

2.1 Proof of Theorem 3

The tautological subring $\mathsf{RH}^\star(\overline{\mathcal{M}}_{g,n})$ is defined as the image of the cycle map

$$\mathsf{R}^\star(\overline{\mathcal{M}}_{g,n}) \twoheadrightarrow \mathsf{RH}^\star(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{H}^{2\star}(\overline{\mathcal{M}}_{g,n})\,.$$

We will use the complex degree grading for $\mathsf{RH}^\star$ and the real degree grading (as usual) for $\mathsf{H}^\star$. Let

$$\mathsf{divRH}^\star(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{RH}^\star(\overline{\mathcal{M}}_{g,n}) \quad \text{and} \quad \mathsf{divH}^\star(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{H}^{2\star}(\overline{\mathcal{M}}_{g,n})$$

be the subrings generated respectively by $\mathsf{RH}^1(\overline{\mathcal{M}}_{g,n})$ and $\mathsf{H}^2(\overline{\mathcal{M}}_{g,n})$. Since

$$\mathsf{RH}^1(\overline{\mathcal{M}}_{g,n}) = \mathsf{H}^2(\overline{\mathcal{M}}_{g,n})\,,$$

we have

$$\mathsf{divRH}^\star(\overline{\mathcal{M}}_{g,n}) = \mathsf{divH}^{2\star}(\overline{\mathcal{M}}_{g,n})\,. \tag{4}$$

We will use the complex degree grading for both $\mathsf{divRH}^\star$ and $\mathsf{divH}^\star$. Since

$$\mathsf{CH}^1(\overline{\mathcal{M}}_{g,n}) \stackrel{\simeq}{=} \mathsf{H}^2(\overline{\mathcal{M}}_{g,n})$$

via the cycle class map, we obtain a surjection

$$\mathsf{divCH}^\star(\overline{\mathcal{M}}_{g,n}) \twoheadrightarrow \mathsf{divH}^\star(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{H}^{2\star}(\overline{\mathcal{M}}_{g,n})\,.$$

The following stronger result implies Theorem 3.

Theorem 3/Cohomology. For all $g \geq 3$, we have $\lambda_g \notin \mathsf{divH}^\star(\overline{\mathcal{M}}_g)$.

Proof. For $g = 3$, we have complete control of the tautological rings in Chow and cohomology (since the intersection pairing to $\mathsf{R}_0(\overline{\mathcal{M}}_g) \stackrel{\sim}{=} \mathbb{Q}$ is nondegenerate for tautological classes). In particular,

$$\mathsf{R}^\star(\overline{\mathcal{M}}_3) \stackrel{\sim}{=} \mathsf{RH}^\star(\overline{\mathcal{M}}_3)\,.$$

In degree 3,

$$\mathsf{divRH}^3(\overline{\mathcal{M}}_3) \subset \mathsf{RH}^3(\overline{\mathcal{M}}_3)$$

is a 9-dimensional subspace of a 10-dimensional space. Explicit calculations with the Sage program *admcycles* [12] show $\lambda_3 \notin \mathsf{divRH}^3(\overline{\mathcal{M}}_3)$. We conclude $\lambda_3 \notin \mathsf{divH}^\star(\overline{\mathcal{M}}_3)$ by (4).

Since duality holds, we also understand $\mathsf{RH}^*(\overline{\mathcal{M}}_{3,1})$ completely:

$$\mathsf{divRH}^3(\overline{\mathcal{M}}_{3,1}) \subset \mathsf{RH}^3(\overline{\mathcal{M}}_{3,1})$$

is a 28-dimensional subspace of a 29-dimensional space. But remarkably, a calculation by *admcycles* shows

$$\lambda_3 \in \mathsf{divRH}^3(\overline{\mathcal{M}}_{3,1})\ !$$

The containment appears miraculous. Is there a geometric explanation?

The tautological ring $\mathsf{RH}^*(\overline{\mathcal{M}}_{4,1})$ is also completely under control in codimension 4:

$$\mathsf{divRH}^4(\overline{\mathcal{M}}_{4,1}) \subset \mathsf{RH}^4(\overline{\mathcal{M}}_{4,1})$$

is a 103-dimensional subspace of a 191-dimensional space. An *admcycles* calculation shows

$$\lambda_4 \notin \mathsf{divRH}^4(\overline{\mathcal{M}}_{4,1})\,. \tag{5}$$

The result (5) implies $\lambda_4 \notin \mathsf{divRH}^4(\overline{\mathcal{M}}_4)$ by a pull-back argument and

$$\lambda_4 \notin \mathsf{divH}^\star(\overline{\mathcal{M}}_4)$$

since divisor classes are tautological.

For $g \geq 5$, a boundary restriction argument is pursued. Suppose, for contradiction,

$$\lambda_g \in \mathsf{divH}^g(\overline{\mathcal{M}}_g)\,. \tag{6}$$

Then, by pull-back, we have

$$\lambda_g \in \mathsf{divH}^g(\overline{\mathcal{M}}_{g,1})\,. \tag{7}$$

Consider the standard boundary inclusion

$$\delta : \overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{g,1}\,.$$

As usual, we have

$$\delta^*(\lambda_g) = \lambda_{g-1} \otimes \lambda_1 \, .$$

Then (7) implies

$$\lambda_{g-1} \otimes \lambda_1 \in \mathsf{divH}^g(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2}) \, . \tag{8}$$

Since $\mathsf{H}^1(\overline{\mathcal{M}}_{g-1,1})$ and $\mathsf{H}^1(\overline{\mathcal{M}}_{1,2})$ both vanish,

$$\mathsf{divH}^\star(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2}) = \mathsf{divH}^\star(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{divH}^*(\overline{\mathcal{M}}_{1,2}) \, .$$

We therefore can write $\mathsf{divH}^g(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2})$ as

$$\begin{aligned} & \mathsf{divH}^g(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{divH}^0(\overline{\mathcal{M}}_{1,2}) \\ \oplus \quad & \mathsf{divH}^{g-1}(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{divH}^1(\overline{\mathcal{M}}_{1,2}) \\ \oplus \quad & \mathsf{divH}^{g-2}(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{divH}^2(\overline{\mathcal{M}}_{1,2}) \, . \end{aligned} \tag{9}$$

After multiplying with ψ_1 (corresponding to the original marking of $\overline{\mathcal{M}}_{g,1}$) and pushing both (8) and (9) to the factor $\overline{\mathcal{M}}_{g-1,1}$, we conclude

$$\lambda_{g-1} \in \mathsf{divH}^{g-1}(\overline{\mathcal{M}}_{g-1,1}) \, .$$

By descending induction, we contradict (5). Therefore (7) and hence also (6) must be false. $\diamondsuit$

2.2 With marked points

The proof of Theorem 3 in cohomology shows

$$\lambda_g \notin \mathsf{divH}^g(\overline{\mathcal{M}}_{g,1}) \tag{10}$$

for $g \geq 4$. By using (10) as a starting point, we can study

$$\lambda_g \in \mathsf{divH}^g(\overline{\mathcal{M}}_{g,n})$$

for $g \geq 4$ and $n \geq 2$ using the boundary restrictions

$$\widehat{\delta} : \overline{\mathcal{M}}_{g,n-1} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n} \, .$$

The argument used in the proof then easily yields the following statement with markings.

Theorem 3/Markings. For all $g \geq 4$ and $n \geq 0$, we have

$$\lambda_g \notin \mathsf{divH}^\star(\overline{\mathcal{M}}_{g,n}) \, .$$

2.3 Proof of Theorem 4

Define the subalgebra of tautological classes

$$\mathsf{RH}^\star_{\leq k}(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{RH}^\star(\overline{\mathcal{M}}_{g,n})$$

generated by classes of complex degrees less than or equal to k. Since all divisors are tautological,

$$\mathsf{divRH}^\star(\overline{\mathcal{M}}_{g,n}) = \mathsf{RH}^\star_{\leq 1}(\overline{\mathcal{M}}_{g,n})\,.$$

The arguments in Sections 2.1 and 2.2 naturally generalize to address the following question: *when is*

$$\lambda_{g-r} \in \mathsf{RH}^{g-r}_{\leq k}(\overline{\mathcal{M}}_{g,n})\,?$$

A crucial case of the question (from the point of view of boundary restriction arguments) is for $n = 1$. Let $\mathsf{Q}_g(r,k)$ be the statement

$$\lambda_{g-r} \notin \mathsf{RH}^{g-r}_{\leq k}(\overline{\mathcal{M}}_{g,1})$$

which may be true or false.

For example, $\mathsf{Q}_g(r, g-r)$ is false essentially by definition. In fact,

$$\mathsf{Q}_g(s, g-r) \ \text{ is false for all } s \geq r$$

for the same reason. By Mumford's formula for the Chern character of the Hodge bundle (and the vanishing of even Chern characters),

$$\mathsf{Q}_g(r-1, g-r) \ \text{ is false whenever } g-r \text{ is odd.}$$

The boundary arguments used in Sections 2.1 and 2.2 yield the following two results.

Proposition 8 *If $\mathsf{Q}_g(r,k)$ is true, then $\mathsf{Q}_{g+1}(r,k)$ and $\mathsf{Q}_{g+1}(r+1,k)$ are true.*

Proposition 9 *If $\mathsf{Q}_g(r,k)$ is true, then*

$$\lambda_{g-r} \notin \mathsf{RH}^{g-r}_{\leq k}(\overline{\mathcal{M}}_{g,n})$$

for all $n \geq 0$.

Since the $k = 1$ case has already been analyzed, we consider now $k = 2$. The first relevant *admcycles* calculation is

$$\lambda_3 \notin \mathsf{RH}^3_{\leq 2}(\overline{\mathcal{M}}_{4,1})\,,$$

so $\mathsf{Q}_4(1,2)$ is true. The corresponding subspace here is of dimension 91 inside a 93 dimensional space. As a consequence of Propositions 8 and 9 , we obtain the following result.

Proposition 10 *For all $g \geq 4$ and $n \geq 0$, we have*

$$\lambda_{g-1} \notin \mathsf{RH}^{g-1}_{\leq 2}(\overline{\mathcal{M}}_{g,n}).$$

A much more complicated *admcycles* calculation shows

$$\lambda_5 \notin \mathsf{RH}^5_{\leq 2}(\overline{\mathcal{M}}_{5,1}),$$

so $\mathsf{Q}_5(0,2)$ is true. The corresponding subspace here is of dimension 1314 inside a 1371 dimensional space. As a consequence of Propositions 8 and 9, we find

$$\lambda_g \notin \mathsf{RH}^g_{\leq 2}(\overline{\mathcal{M}}_{g,n}) \tag{11}$$

for all $g \geq 5$ and $n \geq 0$. For $g \geq 7$, the equality

$$\mathsf{RH}^2(\overline{\mathcal{M}}_g) = \mathsf{H}^4(\overline{\mathcal{M}}_g)$$

is shown by combining results of Edidin [14] and Boldsen [9]. We provide a summary of the argument in Appendix A. For $g \geq 7$, the cycle map

$$\mathsf{CH}^\star_{\leq 2}(\overline{\mathcal{M}}_g) \to \mathsf{H}^{2\star}(\overline{\mathcal{M}}_g)$$

therefore factors through $\mathsf{RH}^\star_{\leq 2}(\overline{\mathcal{M}}_g)$. Then, the non-containment (11) completes the proof of Theorem 4. ◊

2.4 Cases of Pixton's conjecture (Proposition 5)

For the proofs of Theorem 3 and 4, dimensions and bases of the following graded parts of tautological rings are required:

$$\mathsf{RH}^4(\overline{\mathcal{M}}_{4,1}), \qquad \dim_{\mathbb{Q}} = 191,$$
$$\mathsf{RH}^5(\overline{\mathcal{M}}_{5,1}), \qquad \dim_{\mathbb{Q}} = 1314.$$

These cases are possible to analyze (via *admcycles*) since the dual pairings are found to have kernels exactly spanned by Pixton's relations. A discussion of the *admcycles* calculation is presented in Appendix B.

Pixton has conjectured that his relations always provide all tautological relations. Dual pairings are known to be insufficient to prove Pixton's conjecture in all cases, see [48, 49] for a more complete discussion.

3 The log Chow ring

3.1 Definitions

Let (X, D) be a nonsingular variety[8] X with a normal crossings divisor

$$D = D_1 \cup \ldots \cup D_\ell \subset X$$

[8]For a nonsingular Deligne-Mumford stack X and a normal crossings divisor $D \subset X$, the definitions are the same.

with ℓ irreducible components. The divisor $D \subset X$ is called the *logarithmic boundary*. An *open stratum*

$$S \subset X$$

is an irreducible quasiprojective subvariety satisfying two properties:

(i) S is étale locally the transverse intersections of the branches of the D_i which meet S.

(ii) S is maximal with respect to (i).

The set $U = X \setminus D$ is an open stratum. Every open stratum is nonsingular. A *closed stratum* is the closure of an open stratum.

If all D_i are nonsingular and all intersections

$$D_{i_1} \cap \ldots \cap D_{i_k}$$

are irreducible and nonempty, then there are exactly 2^ℓ open strata.

Our main interest will be the case $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ where the normal crossings divisors have self-intersections. The open strata defined above for $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ are the same as the usual open strata of the moduli space of stable curves.

An open stratum $S \subset X$ is *simple* if the closure

$$\overline{S} \subset X$$

is nonsingular. A *simple blow-up* of (X, D) is a blow-up of X along the closure $\overline{S} \subset X$ of a simple stratum. Let

$$\widetilde{X} \to X \tag{12}$$

be a simple blow-up along $\overline{S}$. Let

$$\widetilde{D} = \widetilde{D}_1 \cup \ldots \widetilde{D}_\ell \cup E \subset \widetilde{X}$$

be the union of the strict transforms $\widetilde{D}_i$ of D_i along with the exceptional divisor E of the blow-up (12). Then, $(\widetilde{X}, \widetilde{D})$ is also a nonsingular variety with a normal crossings divisor. An *iterated blow-up*

$$(\widehat{X}, \widehat{D}) \to (X, D)$$

is a finite sequence of simple blow-ups of varieties with normal crossings divisors.[9]

The log Chow group of (X, D) is defined a limit over all iterated blow-ups along special varieties,

$$\mathsf{logCH}^*(X, D) = \varinjlim_{Y \in \mathsf{logB}(X,D)} \mathsf{CH}^*(Y)\,.$$

[9] An iterated blow-up is a special type of log blow-up. Since we are taking a limit, we do not have to consider all log blow-ups.

Here, $\mathsf{logB}(X, D)$ is the category of iterated blow-ups of (X, D): objects in $\mathsf{logB}(X)$ are iterated blow-ups of (X, D) and morphisms in $\mathsf{logB}(X)$ are iterated blow-ups.

Since (X, D) is the trivial iterated blow-up of itself, there is canonical algebra homomorphism

$$\mathsf{CH}^\star(X) \to \mathsf{logCH}^\star(X, D)$$

which is injective (since an inverse map of $\mathbb{Q}$-vectors spaces is obtain by proper push-forward). We therefore view $\mathsf{CH}^\star(X)$ as a subalgebra of $\mathsf{logCH}^\star(X, D)$. Every Chow class on X canonically determines a log Chow class for (X, D).

3.2 Calculation in genus 2

We will prove Proposition 7: *there does not exist a class* $\mathsf{T} \in \mathsf{logCH}^1(\overline{\mathcal{M}}_2)$ *satisfying*

$$\mathsf{T}|_{\mathcal{M}_2^{\mathrm{ct}}} = 0 \quad \textit{and} \quad \lambda_2 = \frac{\mathsf{T}^2}{2!} \in \mathsf{logCH}^2(\overline{\mathcal{M}}_2)\,.$$

Proof. Denote by $\pi_* : \mathsf{logCH}^\star(\overline{\mathcal{M}}_2) \to \mathsf{CH}^\star(\overline{\mathcal{M}}_2)$ the push-forward from log Chow to ordinary Chow. We will prove a stronger claim: *there does not exist a class* $\mathsf{T} \in \mathsf{logCH}^1(\overline{\mathcal{M}}_2)$ *satisfying*

$$\mathsf{T}|_{\mathcal{M}_2^{\mathrm{ct}}} = 0 \quad \textit{and} \quad \pi_*\left(\lambda_2 - \frac{\mathsf{T}^2}{2!}\right) = 0 \in \mathsf{CH}^2(\overline{\mathcal{M}}_2)\,. \tag{13}$$

Denote by $U_2 \subseteq \overline{\mathcal{M}}_2$ the open subset obtained by removing all closed strata of codimension at least 3. By the excision exact sequence of Chow groups, we have

$$\mathsf{CH}^2(U_2) \cong \mathsf{CH}^2(\overline{\mathcal{M}}_2)$$

and thus we can verify the stronger claim by working over U_2.

The open set U_2 has open strata of codimension 1 and 2. Since blow-ups along codimension 1 strata do not change U_2, the only simple blow-ups

$$U_2' \to U_2$$

are along codimension 2 open strata (all of which are special in U_2). Since the codimension 2 open strata of U_2 do not intersect (nor self-intersect), we obtain a $\mathbb{P}^1$-bundle as an exceptional divisor which contains 0 and ∞ sections[10] which are codimension 2 strata of U_2'. The iterated blow-ups

$$\widehat{U}_2 \to U_2$$

are then simply towers of blow-ups of these codimension 2 toric strata in successive exceptional divisors.

[10]Depending upon monodromy, there are either two distinct sections or a single double-section.

Assume $\mathsf{T} \in \mathsf{logCH}^1(U_2)$ satisfies the conditions (13). Since T restricts to zero over the compact type locus, T can be represented as

$$\mathsf{T} \in \mathsf{CH}^1(\widehat{U}_2)$$

on an iterated blow-up

$$\widehat{U}_2 \to U_2$$

with all blow-up centers living over strata in the complement of the compact type locus.

There is a single codimension 1 stratum $\Delta_0 \subset U_2$ and two codimension 2 strata $B, C \subset U_2$ contained in the complement of the compact type locus (see Figure 1).

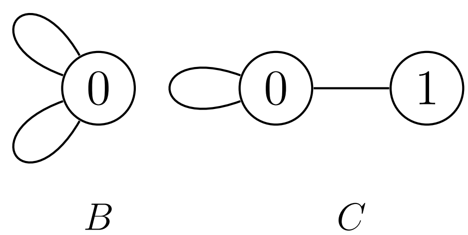


Figure 1: The stable graphs associated to the codimension 2 boundary strata B, C contained in U_2

Denote by $E_B^1, \ldots, E_B^\ell$ and $E_C^1, \ldots, E_C^m$ the exceptional divisors of blow-ups with centers lying over B, C. Then T has a representation[11]

$$\mathsf{T} = a \cdot [\Delta_0] + \sum_{i=1}^{\ell} b_i [E_B^i] + \sum_{j=1}^{m} c_j [E_C^j] \,.$$

After taking the square and pushing forward, we claim

$$\pi_* \left(\mathsf{T}^2\right) = x \cdot [\Delta_0]^2 + y \cdot [B] + z \cdot [C] \,, \tag{14}$$

with $x, y, z \in \mathbb{Q}$ satisfying

$$x = a^2 \geq 0 \quad \text{and} \quad z \leq 0 \,.$$

The claim follows from the following observations:

- In T^2, all mixed terms $[\Delta_0] \cdot [E_B^i]$ and $[\Delta_0] \cdot [E_C^j]$ vanish after pushforward to U_2, since

$$\pi_*([\Delta_0] \cdot [E_B^i]) = [\Delta_0] \cdot \pi_* [E_B^i] = [\Delta_0] \cdot 0 = 0 \,.$$

[11] Here, $[\Delta_0]$ is defined via pull-back (not strict transformation).

- Similarly, since $B \cap C = \emptyset$ in U_2 (as we have removed the codimension 3 stratum of $\overline{\mathcal{M}}_2$), we have $[E_B^i] \cdot [E_C^j] = 0$.
- Denote by $\mathbf{M} \in \mathsf{Mat}_{\mathbb{Q},m\times m}$ the matrix defined by

$$\pi_*\left([E_C^{j_1}] \cdot [E_C^{j_2}]\right) = \mathbf{M}_{j_1,j_2}[C]\,.$$

  A basic fact is that $\mathbf{M}$ is negative semidefinite. Therefore, for $\mathbf{b} = (b_i)_{i=1}^{\ell}$, we have

$$\pi_*\left(\sum_{j=1}^{m} b_j[E_C^j]\right)^2 = \underbrace{(\mathbf{b}^\top \mathbf{M}\mathbf{b})}_{=z\leq 0}[C]\,.$$

- The pushforward

$$\pi_*\left(\sum_{i=1}^{\ell} b_i[E_B^i]\right)^2$$

  is supported on B and thus is a multiple $y \cdot [B]$ of the fundamental class of B.

After substituting (14) in the second condition of (13), we conclude the existence of $x, y, z \in \mathbb{Q}$ with $x \geq 0$ and $z \leq 0$ satisfying

$$x \cdot [\Delta_0]^2 + y \cdot [B] + z \cdot [C] = 2\lambda_2 \in \mathsf{CH}^2(U_2)\,. \tag{15}$$

Using *admcycles* (see Appendix B.3), we can explicitly identify all classes in (15) in

$$\mathsf{CH}^2(U_2) \cong \mathbb{Q}^2\,.$$

The corresponding affine linear equation has the solution space

$$x = z - \frac{1}{120}, \quad y = -\frac{5}{24} \cdot z + \frac{11}{2880}\,.$$

But for $z \leq 0$, we have

$$z - 1/120 < 0\,,$$

which contradicts the assumption $x \geq 0$. Therefore, there can not exist a class

$$\mathsf{T} \in \mathsf{logCH}^1(U_2)$$

satisfying conditions (13). ◊

4 Relationship with logarithmic geometry

4.1 Overview

The definitions of Section 3 are natural from the perspective of logarithmic geometry. The choice of the divisor D on X can be seen as the choice of a log structure on X. We briefly recall the relevant definitions and constructions of logarithmic geometry.

4.2 Definitions

A log structure on a scheme X is a sheaf of monoids M_X on the étale site of X together with a homomorphism[12]

$$\exp : M_X \to \mathcal{O}_X$$

which induces an isomorphism $\exp^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ on units.

- Morphisms of log schemes $(X, M_X) \to (Y, M_Y)$ are morphisms of schemes

  $$f : X \to Y$$

  together with homomorphisms of sheaves of monoids $f^{-1}M_Y \to M_X$ which are compatible with the structure map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ in the obvious sense.

- Log structures can be pulled back. Given a morphism of schemes

  $$f : X \to Y$$

  and a log structure M_Y on Y, there is an induced log structure f^*M_Y on X, generated by $f^{-1}M_Y$ and the units $\mathcal{O}_X^*$.

The basics of log schemes can be found in Kato's original article on the subject [Ka].

The category of log schemes is, in practice, too large for geometric study. It is therefore common to work in smaller categories by requiring additional properties to hold. For our purposes, we will work only with in the category of fine and saturated log schemes, usually termed *f.s. log schemes.* The prototype of such a log scheme is

$$A_P = \mathrm{Spec}(k[P])\,,$$

the spectrum of the algebra generated by a *fine and saturated monoid* P: a finitely generated monoid P which injects into its Grothendieck group P^{gp} and which is saturated there,

$$nx \in P \ \text{ for } \ n \in \mathbb{N}\,,\ x \in P^{\mathrm{gp}} \quad \Longrightarrow \quad x \in P\,.$$

The sheaf M_{A_P} here is the subsheaf of $\mathcal{O}_{A_P}$ generated by P and the units of $\mathcal{O}_{A_P}$.

All of the log schemes which arise for us will be comparable to A_P on the level of log structures. More precisely we require our log schemes X to admit the following local charts: for each $x \in X$, there must be an étale neighborhood

$$i : U \to X\,,$$

an f.s. monoid P, and a map $g : U \to A_P$ such that

$$i^*M_X = g^*M_{A_P}\,.$$

[12] $\mathcal{O}_X$ here is sheaf of monoids under multiplication.

Since we are always working with f.s. log schemes, the chart P at x can in fact always be chosen to be isomorphic to the characteristic monoid[13]

$$\overline{M}_{X,\overline{x}} = M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$$

at x.

4.3 Normal crossings pairs

Let us now return to the situation of interest for the paper: a pair (X, D) of a nonsingular scheme (or Deligne-Mumford stack) with a normal crossings divisor $D \subset X$. The pair (X, D) determines a sheaf M_X on the étale site of X by setting

$$M_X(p : U \to X) = \{f \in \mathcal{O}_U : f \text{ is a unit on } p^{-1}(X - D)\}$$

for each étale map $p : U \to X$. The sheaf of units $\mathcal{O}^*_X$ is a subsheaf of M_X. We write

$$\overline{M}_X = M_X/\mathcal{O}^*_X$$

for the *characteristic monoid* of X. Normal crossings pairs (X, D), with the log structure described above, are precisely the log schemes which are log smooth over the base field Spec k with trivial log structure.

When the irreducible components of D do not have self intersections, the log structure M_X of (X, D) can be defined on the Zariski topology of X. The result is a technically simpler theory. The pair (X, D) is then called a *toroidal embedding (without self intersection)* in [33]. However, for a general pair (X, D), M_X can only be defined on the étale site of X. The general étale case differs from the Zariski case in two key aspects: the irreducible components of D can self-intersect, and the log structure M_X, while locally constant on a stratum, can globally acquire monodromy.

The characteristic monoid $\overline{M}_X$ is a constructible sheaf on X. The connected components of the loci on which $\overline{M}_X$ is constant define a stratification of X, which is precisely the stratification of Section 3.1. Indeed, for a geometric point $x \in X$,

$$\overline{M}_{X,\overline{x}} = \mathbb{N}^r$$

where r is the number of branches (in the étale topology) of D that contain x.

A combinatorial space can be built from the information contained in $\overline{M}_X$. There are two basic approaches. The first, which is more geometric and more evidently combinatorial, is to build the *cone complex* $C(X, D)$ of (X, D). We briefly outline the construction (details can be found in [11] and [1]).

We begin with the case where M_X is defined Zariski locally on X (when the irreducible components of D do not have self-intersections). Then, $C(X, D)$ is a rational polyhedral cone complex, see [33]:

[13]Since we are working with étale sheaves, the stalk is computed in the étale topology; $\overline{x}$ denotes the étale stalk.

- For each point $x \in X$, the characteristic monoid $\overline{M}_{X,\overline{x}}$ determines a rational polyhedral cone

$$\sigma_{X,x} = \mathrm{Hom}_{\mathrm{Monoids}}(\overline{M}_{X,\overline{x}}, \mathbb{R}_{\geq 0})$$

  together with an integral structure

$$N_{X,x} = \mathrm{Hom}(\overline{M}^{\mathrm{gp}}_{X,\overline{x}}, \mathbb{Z})$$

- When x belongs to a stratum $S \subset X$ and y belongs to the closure $\overline{S} \subset X$, there are canonical inclusions

$$\sigma_{X,x} \subset \sigma_{X,y}\,, \quad N_{X,x} \subset N_{X,y}\,.$$

- We glue the cones $\sigma_{X,x}$ together with their integral structures to form the complex

$$C(X,D) = \varinjlim_{x \in X} (\sigma_{X,x}, \sigma_{X,x} \cap N_{X,x})\,.$$

- More effectively, instead of working with all points $x \in X$, we can take the finite set $\{x_S\}$ of the generic points of the strata of (X, D). Then,

$$C(X,D) = \varinjlim_{x_S} (\sigma_{X,x_S}, \sigma_{X,x_S} \cap N_{X,x_S})\,.$$

  In other words, $C(X, D)$ is the dual intersection complex of (X, D).

When M_X is defined only on the étale site, we build the cone complex $C(X, D)$ by descent.

- We find an étale (but not necessarily proper), strict ($f^*M_X = M_Y$) cover $f : Y \to X$ which is *as fine as possible* (called atomic or small in the literature): the log structure on Y is defined on the Zariski site of Y, and each connected component of Y has a unique closed stratum. Taking a further such cover V of the fiber product $Y \times_X Y$ if necessary, we find a groupoid presentation

$$V \rightrightarrows Y \to X\,.$$

- We define

$$C(X,D) = \varinjlim [C(V) \rightrightarrows C(Y)]$$

The coequalizer $C(X, D)$ does not depend on the groupoid presentation of X. Moreover, $C(X, D)$ is a complex of cones, but no longer a rational polyhedral cone complex. For each point $x \in X$, there is a canonical map

$$\sigma_{X,x} \to C(X,D)\,,$$

but the map may no longer be injective. As the étale local branches of the divisor D may be connected globally on X, the faces of the cones $\sigma_{X,x}$ may be glued to each other in $C(X, D)$, and they may naturally acquire automorphisms coming from the monodromy of the branches of D.

4.4 Artin fans

An equivalent combinatorial space is the Artin fan $\mathcal{A}_X$ of (X, D). The Artin fan is defined by gluing, instead of the dual cones $\sigma_{X,x}$ of $\overline{M}_{X,\overline{x}}$, the quotient stacks

$$\mathcal{A}_{\overline{M}_{X,\overline{x}}} = \left[\operatorname{Spec}(k[\overline{M}_{X,\overline{x}}]) \,/\, \operatorname{Spec}(k[\overline{M}^{\mathrm{gp}}_{X,\overline{x}}]) \right] .$$

The gluing is exactly the same as for $C(X, D)$ as explained above. When M_X is defined on the Zariski site of X,

$$\mathcal{A}_X = \varinjlim_{x\in X} \mathcal{A}_{\overline{M}_{X,\overline{x}}} = \varinjlim_{x_S} \mathcal{A}_{\overline{M}_{X,\overline{x_S}}} ,$$

and when M_X is defined only on the étale site of X,

$$\mathcal{A}_X = \varinjlim [\mathcal{A}_V \rightrightarrows \mathcal{A}_Y] ,$$

for an atomic presentation $\varinjlim[V \rightrightarrows Y] = X$ as before.

The Artin fan $\mathcal{A}_X$ captures exactly the same combinatorial information as the cone complex $C(X, D)$, but is geometrically less intuitive. Nevertheless, the Artin fan has the advantage of coming with a *smooth* morphism of stacks

$$\alpha : X \to \mathcal{A}_X .$$

4.5 Logarithmic modifications

The cone complex $C(X, D)$ encodes an important operation: *logarithmic modification* of X. Logarithmic modifications correspond to subdivisions of $C(X, D)$. A subdivision of $C(X, D)$ is, by definition, a compatible subdivision of all the cones $\sigma_{X,x}$ compatible with the gluing relations. Each subdivision $\sigma'_{X,x} \to \sigma_{X,x}$ determines dually a map $\overline{M}_{X,\overline{x}} \to \overline{M}'_{X,\overline{x}}$, and so a map

$$\left[\operatorname{Spec}(k[\overline{M}'_{X,\overline{x}})] \,/\, \operatorname{Spec}(k[\overline{M}^{\mathrm{gp}}_{X,\overline{x}}]) \right] \to \left[\operatorname{Spec}(k[\overline{M}_{X,\overline{x}}]) \,/\, \operatorname{Spec}(k[\overline{M}^{\mathrm{gp}}_{X,\overline{x}}]) \right] .$$

The compatibility of the subdivisions with respect to the gluing relations in $C(X, D)$ implies that these maps glue to a *proper* and *birational* map

$$\mathcal{A}'_X \to \mathcal{A}_X .$$

Then, we define

$$X' = X \times_{\mathcal{A}_X} \mathcal{A}'_X \to X$$

which is proper and birational over X. Moreover, X' is a log scheme, and we have

$$\mathcal{A}_{X'} = \mathcal{A}'_X .$$

Geometrically, subdivisions come in three levels of generality:

- General subdivisions simply produce proper birational maps $X' \to X$, which are isomorphisms over $X - D$. Such maps are called *logarithmic modifications*

- Log blow-ups are a special kind of subdivision. They are the subdivisions of $C(X,D)$ into the domains of linearity of a piecewise linear function on $C(X,D)$, and they correspond to a sheaf of monomial ideals,
$$I \subset M_X \, .$$
The map $X' \to X$ is then projective and is the normalization of the blow-up of X along the sheaf of ideals $\exp(I) \subset \mathcal{O}_X$.
- Star subdivisions along simple strata S correspond to the most basic logarithmic modifications. The strata of X are, by construction, in bijection with the cones of $C(X,D)$. We obtain a subdivision by subdividing σ_{X,x_S} along its barycenter. A simple blow-up along $\overline{S}$ corresponds precisely to the star subdivision of the cone σ_{X,x_S}. Further applications of the star subdivision operation are discussed in section 5.3.

Although star subdivisions are the simplest and most basic subdivisions, we need not consider more general subdivisions for our purposes. We are only concerned with statements that are valid over some arbitrarily fine subdivision, and the star subdivisions along simple strata are cofinal in this setting: for each subdivision
$$C(X,D)' \to C(X,D)$$
there is a further subdivision $C(X,D)'' \to C(X,D)'$ such that the composition $C(X,D)'' \to C(X,D)$ is the composition of star subdivisions along simple strata. So the reader can restrict attention to simple blow-ups without any loss of generality.

We define a category $\mathsf{logM}(X,D)$ whose objects are log modifications
$$X' \to X$$
obtained via subdivisions of $C(X,D)$. There is a unique morphism $X'' \to X'$ if and only if X'' is a log modification of X'. Following [7], we then define
$$\mathsf{logCH}^\star(X,D) = \varinjlim_{X' \in \mathsf{logM}(X)} \mathsf{CH}^\star(X') \, .$$
As simple blowups are cofinal among log modifications, we have, equivalently,
$$\mathsf{logCH}^\star(X,D) = \varinjlim_{X' \in \mathsf{logB}(X,D)} \mathsf{CH}^\star(X')$$
as defined in Section 3.1.

5 The divisor subalgebra of log Chow

5.1 Definitions

Let (X,D) be a nonsingular variety X with a normal crossings divisor
$$D = D_1 \cup \ldots \cup D_\ell \subset X$$

with ℓ irreducible components. Let

$$\mathsf{divlogCH}^\star(X,D) \subset \mathsf{logCH}^\star(X,D)$$

be the subalgebra generated by the classes of all the components of the associated normal crossings divisors of all iterated blow-ups of X.

Let $S \subset X$ be an open stratum of codimension s, let $\overline{S} \subset X$ be the closure, and let

$$\epsilon : \widetilde{S} \to X$$

be the normalization of $\overline{S}$ equipped with a canonical map ϵ to X. The normalization $\widetilde{S}$ is nonsingular and separates the branches of the self-intersections of $\overline{S}$. The map ϵ is an immersion and therefore has a well-defined normal bundle

$$\mathsf{N}_\epsilon = \epsilon^* T_X / T_{\widetilde{S}}$$

of rank s.

An open stratum $S \subset X$ of codimension s is étale locally cut out by s branches of the full divisor D. These s branches are partitioned by monodromy orbits over S. Each monodromy orbit determines a summand of N_ϵ. We obtain a canonical splitting of N_ϵ corresponding to monodromy orbits

$$\mathsf{N}_\epsilon = \oplus_{\gamma \in \mathsf{Orb}(S)} \mathsf{N}_\epsilon^\gamma \,, \qquad \mathsf{rank}(\mathsf{N}_\epsilon^\gamma) = |\gamma| \,,$$

where $\mathsf{Orb}(S)$ is the set of monodromy orbits of the branches of D cutting out S, and $|\gamma|$ is the number of branches in the orbit γ. For polynomials P_γ in the Chern classes of $\mathsf{N}_\epsilon^\gamma$, we define

$$[S, \{P_\gamma\}_{\gamma \in \mathsf{Orb}(S)}] = \epsilon_* \left(\prod_{\gamma \in \mathsf{Orb}(S)} P_\gamma(\mathsf{N}_\epsilon^\gamma) \right) \in \mathsf{CH}^\star(X) \,. \tag{16}$$

We define *normally decorated classes* by the following more general construction. Let G be the monodromy group of the s branches of D which cut out S. Over $\widetilde{S}$, there is a principal G-bundle

$$\mu : \widetilde{P} \to \widetilde{S}$$

over which the s branches determine s line bundles

$$N_1, \ldots, N_s \,. \tag{17}$$

The G-action on $\widetilde{P}$ permutes the line bundles (17) via the original monodromy representation. Let P_G be any G-invariant polynomial in the Chern classes $c_1(N_i)$. Since $P_G(c_1(N_1), \ldots, c_1(N_s))$ is G-invariant,

$$P_G(c_1(N_1), \ldots, c_1(N_s)) \in \mathsf{CH}^\star(\widetilde{S}) \,.$$

We define a *normally decorated strata class* by

$$[S, P_G] = \epsilon_*(P_G(c_1(N_1), \ldots, c_1(N_s))) \in \mathsf{CH}^\star(X) \,.$$

Construction (16) is a special case of a normally decorated strata class.

A fundamental result about the log Chow ring of (X, D) is the following inclusion.

Theorem 11 *Let (X, D) be a nonsingular variety with a normal crossings divisor. Let $S \subset X$ be an open stratum. Every normally decorated class associated to S lies in $\mathsf{divlogCH}^\star(X, D)$.*

5.2 Proof of Theorem 11

Theorem 11 is almost trivial if every irreducible component D_i of D is nonsingular. The complexity of the argument occurs only in case there are irreducible components with self-intersections.

Proof. Let $S \subset X$ be an open stratum of codimension s. The first case to consider is when S is simple. Then, the closure

$$\overline{S} \subset X$$

in nonsingular and no normalization is needed,

$$\epsilon : \overline{S} \to X \,.$$

Let G be the monodromy of the s branches of D which cut out S. We must prove

$$[S, P_G] = \epsilon_*(P_G(c_1(N_1), \ldots, c_1(N_s))) \in \mathsf{divlogCH}^\star(X)$$

for every G-invariant polynomial P_G.

We argue by induction on the degree of P_G. The base case is when P_G is of degree 0. We can take $P_G = 1$, and we must prove

$$[S, 1] = \epsilon_*[S] \in \mathsf{divlogCH}^\star(X, D)\,. \tag{18}$$

Our argument requires a blow-up construction which we term an explosion.

The *explosion* of (X, D) along a simple stratum S,

$$e : \mathsf{E}_S(X, D) \to X\,, \tag{19}$$

is defined by a sequence of blow-ups of X. To describe the blow-ups locally[14] near a point $p \in S$, let

$$B_1, \ldots, B_s$$

be the branches of D cutting out S near p.

- At the 0^{th} stage, we blow-up S, the intersection of all s branches $B_1, \ldots, B_s$.

[14]Throughout the proof of Theorem 11, the terms local, near, and open refer to the Euclidian topology since we must separate branches.

Consider next the strict transform of the intersection of $s-1$ branches. For each choice of $s-1$ branches, the strict transform of the intersection is nonsingular of codimension $s-1$ over an open set of $p \in X$. Moreover, the strict transforms of the intersections of different sets of $s-1$ branches are disjoint over an open set of $p \in X$.

- At the 1^{st} stage, we blow-up all s of these strict transforms of intersections of $s-1$ branches.

Then, the strict transforms of the intersections of $s-2$ branches among $B_1, \ldots, B_s$ are nonsingular of codimension $s-2$ and disjoint over an open set of $p \in X$.

- At the 2^{nd} stage, we blow-up all $\binom{s}{2}$ of these strict transforms of intersections of $s-2$ branches.

We proceed in the above pattern until we have completed $s-1$ stages.

- At the j^{th} stage, we blow-up all $\binom{s}{j}$ strict transforms of intersections of $s-j$ branches.

The explosion (19) is the result after stage[15] $j = s-1$. Since the above blow-ups are defined symmetrically with respect to the branches B_i, the definition is well-defined globally on X.

Near S, all the prescribed blow-ups are of simple loci, but non-simplicity may occur away from S. In order for the explosion to be a sequence of simple blow-ups, some extra blow-ups may be required far from S. Since we will only be interested in the geometry near S, the blow-ups related to non-simplicity away from S are not important for our argument (and are not included in our notation).

A local study shows the following properties of the explosion

$$e : \mathsf{E}_S(X, D) \to X \,,$$

near S:

(i) The inverse image $e^{-1}(S) \subset \mathsf{E}_S(X, D)$ is a nonsingular irreducible subvariety which we denote by $\mathsf{E}_S(S)$ and call the *exceptional divisor* of the explosion. We denote the inclusion by

$$\iota : \mathsf{E}_S(S) \to \mathsf{E}_S(X, D) \,.$$

(ii) Let N_S be the rank s normal bundle of S in X. The fibers of the projective normal bundle

$$\mathsf{P}(\mathsf{N}_S) \to S \tag{20}$$

have a canonical (unordered) set of s coordinate hyperplanes determined by the s local branches of D cutting out S. In the fibers of (20), these relative hyperplanes determine s coordinate points, $\binom{s}{2}$ coordinate lines, $\binom{s}{3}$ coordinate planes, and so on.

[15] At stage $j = s-1$, we are blowing-up divisors, so no change occurs in the space. But we include the $j = s-1$ stage to uniformize our later notation for exceptional divisors.

(iii) The restriction of the explosion morphism to the exceptional divisor

$$e_S : \mathsf{E}_S(S) \to S$$

is obtained from $\mathsf{P}(\mathsf{N}_S) \to S$ by first blowing-up the coordinate points, and then blowing-up the strict transforms of the coordinate lines, and so on. For

$$1 \leq j \leq s-1\,,$$

the j^{th} stage of the construction of the explosion restricts to the blow-up of the strict transform of the $(j-1)$-dimensional coordinate linear spaces of the fibers of (20).

(iv) On $\mathsf{E}_S(S)$, we have a distinguished set of divisors

$$E_0, E_1, \ldots, E_s \in \mathsf{CH}^1(\mathsf{E}_S(S))\,.$$

Here, E_0 is the pull-back to $\mathsf{E}_S(S)$ of

$$\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1) \to \mathsf{P}(\mathsf{N}_S)$$

determined by the 0^{th} stage of the construction of the explosion. Then, $E_j \in \mathsf{CH}^1(\mathsf{E}_S(S))$ is the pull-back to $\mathsf{E}_S(S))$ of the exceptional divisor obtained from the blow-up of strict transform of the $(j-1)$-dimensional coordinate linear spaces in the fibers of (20).

(v) Every class of the form

$$[\mathsf{E}_S(S)] \cdot \mathsf{F}(E_0, \ldots, E_{s-1}) \in \mathsf{CH}^*(\mathsf{E}_S(X,D))$$

where F is a polynomial, lies in the divisor ring of log Chow,

$$[\mathsf{E}_S(S)] \cdot \mathsf{F}(E_0, \ldots, E_{s-1}) \in \mathsf{divlogCH}^*(X,D)\,.$$

The claim follows the geometric construction of the explosion. To start, $\mathsf{E}_S(S)$ is a component of the associated normal crossings divisor of $\mathsf{E}_S(X,D)$. For each $0 \leq j \leq s_1$, E_j comes from the pull-back of a divisor stratum of the blow-up at the j^{th} stage.

To the explosion geometry, we can apply Fulton's excess intersection formula. We start with the 0^{th} stage:

$$e_0 : X_0 \to X$$

is the blow-up along S, and

$$e_0^*[S] = [\mathsf{P}(\mathsf{N}_S)] \cdot c_{s-1}\left(\frac{\mathsf{N}_S}{\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)}\right)\,.$$

When we pull-back $e_0^*[S]$ all the way to $\mathsf{E}_S(X,D)$, we obtain[16]

$$e^*[S] = [\mathsf{E}_S(S)] \cdot c_{s-1}\left(\frac{\mathsf{N}_S}{\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)}\right)\,.$$

[16] We have omitted the pull-backs in the notation inside the argument of c_{s-1}.

By definition, we have
$$c(\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)) = 1 + E_0 \, .$$
By property (v) above for the explosion geometry, to prove
$$\epsilon_*[S] \in \mathsf{divlogCH}^\star(X, D) \, , \tag{21}$$
we need only show
$$c_k(\mathsf{N}_S) = \mathsf{F}_k(E_0, \ldots, E_{s-1}) \in \mathsf{CH}^k(\mathsf{E}_S(S)) \tag{22}$$
for polynomials F_k for $1 \leq k \leq s-1$.

The claim (22) is established directly by the following basic formula of the explosion geometry. For $0 \leq j \leq s-1$, let
$$\mathsf{L_j} = \sum_{i=0}^{j} E_i \, .$$
Let σ_k be the k^{th} elementary symmetric polynomial. Then,
$$c_k(\mathsf{N}_S) = \sigma_k(\mathsf{L}_0, \ldots, \mathsf{L}_{s-1}) \in \mathsf{CH}^k(\mathsf{E}_S(S)) \, . \tag{23}$$
The proof of (21), which establishes the base case $P_G = 1$ of the induction, follows from formula (23).

Let $\mathsf{T} = (\mathbb{C}^*)^s$. Let t_i denote the weight of the standard representation of the i^{th} factor of T. To prove formula (23), we consider the universal T-equivariant model where $S \subset X$ is
$$\mathbf{0} \in \mathbb{C}^s$$
and the logarithmic boundary $H \subset \mathbb{C}^s$ is the union of the s coordinate hyperplanes. Then, the T-action on
$$e_{\mathbf{0}} : \mathsf{E}_{\mathbf{0}}(\mathbb{C}^s, H) \to \mathbf{0}$$
has $s!$ isolated T-fixed points naturally indexed by elements of the symmetric group Σ_s. The weights of the divisors
$$\mathsf{L}_0, \ldots, \mathsf{L}_{s-1}$$
with their canonical T-equivariant lifts at the T-fixed point $\gamma \in \Sigma_s$ are
$$t_{\gamma(1)}, t_{\gamma(2)}, t_{\gamma(3)}, \ldots, t_{\gamma(s)}$$
respectively. Formula (23) then follows immediately for the T-equivariant model. The general case of (23) is a formal consequence.

We now will establish the induction step. Let $S \subset X$ be a simple stratum of codimension s with monodromy group[17] G of the branches of D cutting out S. We must prove
$$[S, P_G] = \epsilon_*(P_G(c_1(N_1), \ldots, c_1(N_s))) \in \mathsf{divlogCH}^\star(X, D)$$

[17] The geometry involved in the proof of the base case of the induction was fully symmetric with respect to the branches, so the group G did not play a role.

for every G-invariant polynomial P_G. By induction, we assume the truth of the statement for polynomials of lower degree.

Let P_G be a G-equivariant polynomial in $c_1(N_1), \ldots, c_1(N_s)$ of degree $d > 0$. We will prove a stronger property for the induction argument:

$$\epsilon_*(P_G(c_1(N_1), \ldots, c_1(N_s))) \in \mathsf{divlogCH}^\star(X, D)$$

can be expressed as a linear combination of terms of the form

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d$$

where the $\widehat{D}_i$ *are components of the logarithmic boundary of an iterated blow-up of the explosion* $\mathsf{E}_S(X, D)$ *and* $\widehat{D}_1$ *lies over*

$$\mathsf{E}_S(S) \subset \mathsf{E}_S(X, D) .$$

Our proof of the base of the induction establishes the stronger property.

We can assume P_G is the summation[18] M_G of the G-orbit of a degree d monomial M,

$$M_G = \frac{1}{|\mathsf{Stab}(M)|} \sum_{g \in G} g(M) .$$

We will study the geometry of the the exceptional divisor of the explosion

$$e_S : \mathsf{E}_S(S) \to S$$

locally over an analytic open set $U_p \subset S$ of $p \in S$.

Over small enough U_p, we can separate all the branches $B_1, \ldots, B_s$ of D which cut out S, and we can write

$$M = c_1(N_1)^{m_1} \cdots c_1(N_s)^{m_s} = B_1^{m_1} \cdots B_s^{m_s} . \tag{24}$$

Over U_p, we can separate all the exceptional divisors of all the blow-ups in the construction of

$$\mathsf{E}_S(S) \to \mathsf{P}(\mathsf{N}_S)$$

explained in (iii) above. There are $2^s - 2$ such exceptional divisor in bijective correspondence to all the proper coordinate linear spaces of the fiber $\mathsf{N}_S|_p$ of N_S at p. We denote these $2^s - 2$ exceptional divisors by E_Λ where

$$\Lambda \subset \mathsf{N}_S|_p$$

is a proper coordinate linear space. As before, we denote the pull-back of $\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)$ to $\mathsf{E}_S(S)$ by E_0.

Via the pull-back formula for B_i, we have

$$e^*(N_i) = E_0 + \sum_{\Lambda \subset H_i} E_\Lambda \ \in \mathsf{CH}^1(e^{-1}(U_p)) , \tag{25}$$

[18] The stabilizer factor occurs to correct for overcounting.

where $H_i \subset \mathsf{N}_S|_p$ is the hyperplane associated to B_i. We now substitute formula (25) into (24) to find

$$M \in \mathbb{Q}[E_0, \{E_\Lambda\}_\Lambda] \,.$$

Of course, M has degree d in the divisors E_0 and $\{E_\Lambda\}_\Lambda$.

Let M^E be a monomial of degree d in the divisors

$$E_0 \quad \text{and} \quad \{E_\Lambda\}_\Lambda \,. \tag{26}$$

The monodromy group G acts[19] canonically on the set (26) leaving E_0 fixed. Let

$$M^E_G = \frac{1}{|\mathsf{Stab}(M^E)|} \sum_{g\in G} g(M^E)$$

be the summation over the G-orbit of M^E. Since M^E_G is G-invariant, M^E_G is a well-defined class

$$M^E_G \in \mathsf{CH}^d(\mathsf{E}_S(S)) \,.$$

To prove the stronger induction step, we need only prove[20]

$$\iota_*\left(M^E_G \cdot c_{s-1}\left(\frac{\mathsf{N}_S}{\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)}\right)\right) \in \mathsf{divlogCH}^*(X,D) \tag{27}$$

can be expressed as a linear combination of terms of the form

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d$$

where the $\widehat{D}_i$ are components of the logarithmic boundary of an iterated blow-up of the explosion $\mathsf{E}_S(X,D)$ and $\widehat{D}_1$ lies over $\mathsf{E}_S(S)$. To see why the claim for (27) is enough, we write

$$\begin{aligned} e^*[S, M_G] &= \sum_{M^E_G} e^*[S]\cdot M^E_G \\ &= \sum_{M^E_G} [\mathsf{E}_S(S)]\cdot c_{s-1}\left(\frac{\mathsf{N}_S}{\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)}\right)\cdot M^E_G \\ &= \sum_{M^E_G} \iota_*\left(M^E_G \cdot c_{s-1}\left(\frac{\mathsf{N}_S}{\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)}\right)\right) . \end{aligned}$$

The first equality is written with the understanding that $e^*[S]$ is supported on $\mathsf{E}_S(S)$.

To study M^E_G, we take a geometric approach. If M^E is just E_0^d, then (27) is already of the claimed form by our analysis in the base case. Otherwise, M^E

[19] The G-action on $\{E_\Lambda\}_\Lambda$ preserves the dimension of Λ. Moreover, for a group element $g \in G$, if $g(E_\Lambda) \neq E_\Lambda$, then

$$g(E_\Lambda) \cap E_\Lambda = \emptyset \,.$$

[20] Recall, ι is the inclusion $\iota : \mathsf{E}_S(S) \to \mathsf{E}_S(X,D)$.

has at least one factor E_Λ. Since $\{E_\Lambda\}_\Lambda$ is a set of simple normal crossings divisors on $\mathsf{E}_S(S)$, we can write M^E (if nonzero) as

$$M^E = E_{\Lambda_1} \cdots E_{\Lambda_t} \cdot \widetilde{M}^E ,$$

where $\Lambda_1, \ldots, \Lambda_t$ are distinct divisors with a nonempty transverse intersection

$$I_{U_p} = \Lambda_1 \cap \ldots \cap \Lambda_t \quad \text{over} \quad U_p .$$

Moreover, we can assume *every* divisor of the monomial $\widetilde{M}^E$ contains I_{U_p}. When the monodromy invariant M^E_G is considered, we obtain a nonsingular subvariety of $\mathsf{E}_S(S)$ of codimension t,

$$V \subset \mathsf{E}_S(S)$$

which is a simple stratum of $\mathsf{E}_S(X, D)$,

$$\epsilon^V : V \to \mathsf{E}_S(X, D) .$$

Over U_p, the subvariety V restricts to the union[21] of the distinct G-translates of I_{U_p}. The crucial geometric observation is

$$\iota_*(M^E_G) = \epsilon^V_*(\widetilde{P}) \in \mathsf{CH}^\star(\mathsf{E}_S(X, D) ,$$

where $\widetilde{P}$ is defined by $\widetilde{M}^E$ and is of degree at most $d-1$.

We can apply the strong induction property: the class

$$\epsilon^V_*(\widetilde{P}) \in \mathsf{divlogCH}^\star(X, D)$$

can be expressed as a linear combination of terms of the form

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d$$

where the $\widehat{D}_i$ are components of the logarithmic boundary of an iterated blow-up of the explosion of V in $\mathsf{E}_S(X, D)$ and $\widehat{D}_1$ lies over

$$\mathsf{E}_V(V) \subset \mathsf{E}_S(X, D) .$$

Then, the claim

$$\iota_* \left(M^E_G \cdot c_{s-1} \left(\frac{\mathsf{N}_S}{\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)} \right) \right) \in \mathsf{divlogCH}^*(X, D) \tag{28}$$

holds by the analysis of

$$c_{s-1} \left(\frac{\mathsf{N}_S}{\mathcal{O}_{\mathsf{P}(\mathsf{N}_S)}(-1)} \right)$$

on $\mathsf{E}_S(S)$ in the base case of the induction. Since each monomial

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d$$

[21]The distinct G-translates of I_{U_p} are disjoint, see Footnote 19.

of $\epsilon_*^V(\widetilde{P})$ lies over $\mathsf{E}_V(V)$ which, in turn, lies over $\mathsf{E}_S(S)$, the analysis of the base case yields the desired result (28).

The induction argument is complete, so we have proven Theorem 11 in case S is a simple stratum of (X, D). The general case follows by repeated application of the result for a simple stratum.

Let $S \subset X$ be a stratum with a singular closure

$$\overline{S} \subset X\,.$$

The first step is to blow-up simple strata in $\overline{S}$,

$$\widehat{X} \to X\,,$$

until the strict transform of $\overline{S}$,

$$\widehat{S} \subset \widehat{X}\,,$$

is nonsingular. Since S is simple stratum of the blow-up $\widehat{X}$, we can apply Theorem 11 to $S \subset \widehat{X}$.

Via the blow-down map, we have

$$\widehat{S} \to \overline{S}\,.$$

There are two discrepancies to handle before deducing Theorem 11 for normally decorated classes associated to $S \subset X$ from the result for normally decorated classes associated to $S \subset \widehat{X}$:

(i) The fundamental class $[\widehat{S}] \in \mathsf{CH}^\star(\widehat{X})$ is not the pull-back of $[\overline{S}] \in \mathsf{CH}^\star(X)$.

(ii) The normal directions of $\widehat{S} \subset \widehat{X}$ differ from the pull-backs of the normal directions of $\overline{S} \subset X$.

However, both discrepancies are corrected by applying the simple stratum result to the lower dimensional strata occurring in $\widehat{S} \setminus S$. $\Diamond$

5.3 Explosion geometry and barycentric subdivision

The explosion operation $E(X, D)$ along a stratum simple $S \subset X$, which appeared in the proof 5.2, is an essentially combinatorial operation that has a natural interpretation in terms of the geometry of the cone complex $C(X, D)$.

Consider first a cone σ of dimension n in a lattice N, and let A_σ be the associated toric variety. Let $\mathcal{A}_\sigma$ be the associated Artin fan, which is simply the stack quotient of A_σ by the corresponding dense torus T_σ. The logarithmic stratification of A_σ is precisely the stratification defined by the orbits of T_σ, and there is a bijective dimension reversing correspondence between faces of σ and strata. We write $\sigma(k)$ for the k-dimensional faces of σ and thus the codimension k strata of A_σ.

For each face τ of σ, the barycenter b_τ of τ is the sum

$$b_\tau = \sum_{v_i \in \tau \cap \sigma(1)} v_i$$

of the primitive vectors along the extremal rays of τ. For any flag

$$\tau_0 \subset \tau_1 \subset \cdots \subset \tau_k$$

of faces of σ, the barycenters $b_{\tau_0}, \cdots, b_{\tau_k}$ span a cone. The set of all such cones, for all flags in σ, forms a subdivision of σ, which we call the *barycentric subdivision* $\widetilde{\sigma}$ of σ.

Alternatively, we can build the barycentric subdivision inductively: at step 1, we start with the star subdivision over the barycenter of faces in $\sigma(n)$ (where σ has dimension n), then take the start subdivision over faces in $\sigma(n-1)$, and so on, terminating after $n-1$ steps with $\sigma(2)$, after which the operation no longer has effect. We thus produce a sequence of $n-1$ subdivisions

$$\widetilde{\sigma} = \sigma_{n-1} \to \sigma_{n-1} \cdots \to \sigma_1 \to \sigma_0 = \sigma$$

When $\sigma = \mathbb{R}^n_{\geq 0}$, which is our main case of interest, the barycentric subdivision has $n!$ maximal cones.

The barycentric subdivision of σ produces a log modification

$$\widetilde{A}_\sigma \to A_\sigma\,,$$

which is in fact a log blow-up. More precisely, we have constructed the subdivision $\widetilde{A}_\sigma \to A_\sigma$ as a sequence

$$\widetilde{A}_\sigma = A_{n-1} \to A_{n-2} \to \cdots A_1 \to A_0 = A_\sigma$$

and the map $A_k \to A_{k-1}$ is determined by the subdivision $\sigma_k \to \sigma_{k-1}$, which is the subdivision corresponding to the domains of linearity of a piecewise linear function – see [33] for the construction. In the case of interest,

$$\sigma = \mathbb{R}^n_{\geq 0}\,,$$

the map $A_1 \to A_0$ is the blowup of $\mathbb{A}^n$ at the origin, $A_2 \to A_1$ is the blowup along the strict transforms of the coordinate lines, and in general $A_k \to A_{k-1}$ is the blowup along the strict transforms of the dimension $k-1$ hyperplanes of $\mathbb{A}^n$ in A_{k-1}. Thus, the barycentric subdivision of $\mathbb{A}^n$ is precisely the explosion of $\mathbb{A}^n$ along the origin.

The barycentric subdivision construction is clearly equivariant and therefore descends to the Artin fan $\mathcal{A}_\sigma$ of A_σ. Furthermore, the subdivision is the same on isomorphic faces of σ and invariant with respect to automorphisms of σ. Consequently, given any cone complex C, the barycentric subdivisions of individual cones glue to a global subdivision of C, and that is true even if faces of C are identified or if there is monodromy in C. Thus, for a normal crossings pair (X, D), we can define the barycentric subdivision $\widetilde{C}(X, D)$ of the cone complex $C(X, D)$, and equivalently, a log blow-up

$$\widetilde{\mathcal{A}}_X \to \mathcal{A}_X$$

of the Artin fan. We also obtain *globally* a log blow-up

$$(\widetilde{X}, \widetilde{D}) = X \times_{\mathcal{A}_X} \widetilde{\mathcal{A}}_X \to (X, D)$$

with Artin fan is $\mathcal{A}_{\widetilde{X}} = \widetilde{\mathcal{A}}_X$.

The explosion of Section 5.2 is a local operation, centered along a simple stratum S. A quasi-projective stratum S (not necessarily simple) of a normal crossings pair (X, D) corresponds to a cone σ of $C(X, D)$. More precisely, the quasi-projective stratum S corresponds to the interior of σ, and the whole of σ corresponds to a canonical open set U in X that contains S as its minimal stratum. The open set U consists of the interiors of the intersections of all divisors that contain S, including $X - D$, regarded as the intersection of no divisors. The explosion $\mathsf{E}_S(U, D|_U))$ is well-defined.

The cone σ has a cover by $\mathbb{R}^n_{\geq 0}$, and, in general, is obtained from $\mathbb{R}^n_{\geq 0}$ by potentially identifying faces and taking a quotient by a group G. The group G is precisely the monodromy group of the divisors D that cut out S considered in Section 5. Similarly, the Artin fan $\mathcal{U}$ of U is the analogous groupoid quotient of $[\mathbb{A}^n/\mathbb{G}_m^n]$, with S corresponding to the minimal stratum

$$B(G_m^n \rtimes G) \subset \mathcal{U}\,.$$

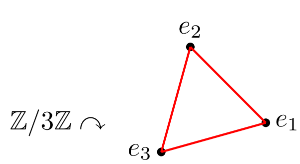
From the discussion of the barycentric subdivision of $\mathbb{A}^n$, we see that $\mathsf{E}_S(U, D_U)$ is precisely the barycentric subdivision $\widetilde{X} \to X$ restricted to U. We may thus view the barycentric subdivision as globalizing the explosion geometry.

If the stratum S is simple, the explosion of Section 5.2 is defined over a neighborhood of $\overline{S}$. However, the extension no longer coincides with the barycentric subdivision. The barycentric subdivision performs additional blowups, first blowing up all minimal strata in the closure of S (and also strata around $\overline{S}$ whose closure does not necessarily meet S).

We illustrate the concepts discussed above through an example. Let (X, D) be a log scheme whose cone complex is the cone over an equilateral triangle, with all edges identified and with monodromy $\mathbb{Z}/3\mathbb{Z}$. For example, we can construct (X, D) by taking

$$X \to B$$

over a nonsingular base B with a non-trivial loop in $\pi_1(B)$ to be a family with fiber $\mathbb{A}^3$ in which the non-trivial loop cyclically permutes the coordinate hyperplanes of $\mathbb{A}^3$. The divisor $D \subset X$ is then the union of these coordinate hyperplanes over B.



A cross section of the cone complex $C(X, D)$

The log scheme (X, D) has four strata: the open set $X - D$, corresponding to the empty face of the triangle (or, equivalently, the vertex of the cone over the triangle), the interior of the divisor D corresponding to the vertex

$$e_1 = e_2 = e_3\,,$$

the locus which is étale locally the intersection of exactly two irreducible components of D corresponding to edge

$$\overline{e_1 e_2} = \overline{e_1 e_3} = \overline{e_2 e_3}\,,$$

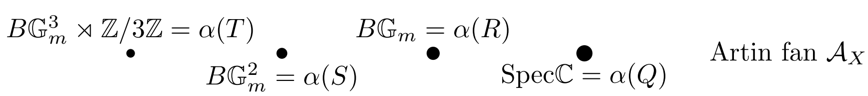
and the triple point singularity corresponding to the whole triangle. We name the strata Q, R, S, T respectively. While T is simple, S is not, since

$$\overline{S} = S \cup T$$

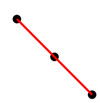
is not normal. The strata are taken bijectively to points of the Artin fan via the map

$$\alpha : X \to \mathcal{A}_X$$

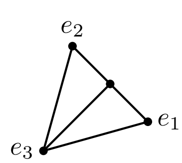
We depict the Artin fan as four points, each isomorphic to $B\mathbb{G}_m^k \rtimes G$ as indicated, with points drawn increasingly bigger to describe the topology (the closure contains all smaller points).



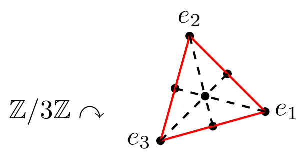
Consider the explosion of the quasi-projective stratum S depicted by the open line segment $\overline{e_1 e_2}$. The open set U over which the explosion is defined is $Q \cup R \cup S$. The explosion of S is the barycentric subdivision of $\overline{e_1 e_2}$:



However, the above explosion does not extend away from U. The blowup of $\overline{S}$, over an étale cover of X is depicted as



But the blow-up does not descend to X as it does not respect the face identifications/automorphisms of $C(X, D)$. The barycentric subdivision is depicted as



The corresponding log blow-up restricts to the explosion over U. Over X, the log blow-up is not the blow-up of $\overline{S}$, but the explosion of T.

5.4 Tautological classes

Let (X, D) be a nonsingular variety with a normal crossings divisor. We define the *logarithmic tautological ring*

$$\mathsf{R}^\star(X, D) \subset \mathsf{CH}^\star(X)$$

to be the $\mathbb{Q}$-linear subspace spanned by all normally decorated strata classes (which is easily seen to be closed under the intersection product). Theorem 11 can then be written as

$$\mathsf{R}^\star(X, D) \subset \mathsf{divlogCH}^\star(X, D)\,.$$

The logarithmic tautological ring of (X, D) depends strongly on the divisor D. For example, if X is irreducible and $D = \emptyset$, then there is only one stratum and

$$\mathsf{R}^\star(X, \emptyset) = \mathbb{Q}\,.$$

For the moduli space of curves, the inclusion

$$\mathsf{R}^\star(\overline{\mathcal{M}}_g, \Delta_0) \subset \mathsf{R}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)\,,$$

is proper for $g \geq 2$. Furthermore, the inclusion

$$\mathsf{R}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \subset \mathsf{R}^\star(\overline{\mathcal{M}}_g)$$

in the standard tautological ring[22] is proper for $g \geq 3$ since $\mathsf{R}^\star(\overline{\mathcal{M}}_g)$ contains κ and ψ classes which do not appear in the logarithmic constructions.

Let (X, D) be a nonsingular variety with a normal crossings divisor. Let

$$\pi : \widetilde{X} \to X$$

be a simple blow-up of (X, D). Let $\widetilde{D} \subset \widetilde{X}$ be the associated normal crossings divisor. We will prove the following two basic properties of logarithmic tautological rings.

[22] $\mathsf{R}^\star(\overline{\mathcal{M}}_g)$ is definitely not equal to $\mathsf{R}^\star(\overline{\mathcal{M}}_g, \emptyset)$!

Theorem 12 *The pull-back*

$$\pi^* : \mathsf{R}^\star(X,D) \to \mathsf{CH}^\star(\widetilde{X})$$

has image in $\mathsf{R}^\star(\widetilde{X},\widetilde{D})$.

Theorem 13 *The push-forward*

$$\pi_* : \mathsf{R}^\star(\widetilde{X},\widetilde{D}) \to \mathsf{CH}^\star(X)$$

has image in $\mathsf{R}^\star(X,D)$.

By Theorems 12 and 13, we can simply write

$$\pi^* : \mathsf{R}^\star(X,D) \to \mathsf{R}^\star(\widetilde{X},\widetilde{D})\,, \quad \pi_* : \mathsf{R}^\star(\widetilde{X},\widetilde{D}) \to \mathsf{R}^\star(X,D)\,.$$

Theorems 12 and 13 will proven in Section 5.6 via the geometry of the Artin fan. As a consequence, we will present a more conceptual (but less constructive) proof of Theorem 11.

5.5 The Chow ring of the Artin fan

Let (X,D) be a nonsingular variety with a normal crossings divisor. We relate here the normally decorated strata classes of (X,D) to Chow classes on the Artin fan $\mathcal{A}_X$ of (X,D). Here, since $\mathcal{A}_X$ is a smooth, finite type algebraic stack stratified by quotient stacks, it has well-defined Chow groups $\mathsf{CH}^\star(\mathcal{A}_X)$ with an intersection product as defined in [35]. As we explain in Section 4.4, there is a smooth morphism to the Artin fan,

$$\alpha : X \to \mathcal{A}_X\,.$$

Theorem 14 *There is a canonical isomorphism*

$$\mathsf{CH}^\star(\mathcal{A}_X) \cong \mathrm{PP}^\star(C(X,D))$$

between the Chow ring of $\mathcal{A}_X$ *and the algebra of piecewise polynomial functions on the cone complex* $C(X,D)$.

Proof. By construction, the Artin fan $\mathcal{A}_X$ has a presentation as a colimit

$$\mathcal{A}_X = \varinjlim_{x\in\mathcal{S}} \mathcal{A}_x\,,$$

where $\mathcal{S}$ is a finite diagram, each map $\mathcal{A}_x$ is a stack of the form $[\mathbb{A}^n/\mathbb{G}_m^n]$, and all maps in the diagram are étale. First, we note that for the individual stacks $\mathcal{A}_x = [\mathbb{A}^n/\mathbb{G}_m^n]$ we have

$$\mathsf{CH}^\star([\mathbb{A}^n/\mathbb{G}_m^n]) \cong \mathsf{CH}^\star([\mathsf{Spec}(\mathbb{C})/\mathbb{G}_m^n]) \cong \mathbb{Q}[x_1,\ldots,x_n]\,. \tag{29}$$

The first equality is because

$$[\mathbb{A}^n/\mathbb{G}_m^n] \to [\mathsf{Spec}(\mathbb{C})/\mathbb{G}_m^n]$$

is a vector bundle and induces an isomorphism of Chow groups by [35, Theorem 2.1.12 (vi)]. The second equality is because the equivariant Chow ring of a product of tori is a polynomial algebra [15, Section 3.2], which can be identified with polynomials on the cone $\sigma_{X,x}$ associated to $\mathcal{A}_x$ (appearing in the colimit presentation of $C(X, D)$).

For the entire Artin fan $\mathcal{A}_X$, we claim

$$\mathsf{CH}^\star \mathcal{A}_X = \varprojlim_{x \in \mathcal{S}} \mathsf{CH}^\star \mathcal{A}_x \,. \tag{30}$$

If we can show equality (30), then Theorem 14 will follow since the result holds for each term on the right hand side by (29). Piecewise polynomial functions on $C(X, D)$ are defined by the corresponding limit presentation.

All the stacks appearing in (30) are very special: they are nonsingular and have a stratification with strata isomorphic to

$$B(\mathbb{G}_m^n \rtimes G)$$

with G a finite group. For the argument below, it will be more convenient to index Chow groups by the dimension of the cycles (instead of the codimension) and prove[23]

$$\mathsf{CH}_\star(\mathcal{A}_X) = \varprojlim_{x \in \mathcal{S}} \mathsf{CH}_\star(\mathcal{A}_x) \,. \tag{31}$$

Let $\mathcal{C}$ denote the full 2-subcategory of the 2-category of algebraic stacks with $\mathrm{Ob}(\mathcal{C})$ given by algebraic stacks $\mathcal{A}$ with a stratification by stacks of the form $B(\mathbb{G}_m^n \rtimes G)$, with G a finite group. Similarly, let $\mathcal{C}^\circ$ be the full 2-subcategory of $\mathcal{C}$ with objects given by stacks of the form $B\mathbb{G}_m^n$. We start with a stack[24] $\mathcal{A}_X \in \mathcal{C}$ with a colimit presentation

$$\mathcal{A}_X = \varinjlim_{x \in \mathcal{S}} \mathcal{A}_x = \mathcal{A}_X$$

where $\mathcal{A}_x \in \mathcal{C}^\circ$ and all maps in the diagram are étale. We will prove (31) by induction on the number of strata of $\mathcal{A}_X$.

Assume first that there is a unique stratum,

$$\mathcal{A}_X = B(\mathbb{G}_m^n \rtimes G) \,,$$

and all maps in the diagram $\mathcal{S}$ are isomorphisms. Then the groupoid $\varinjlim_{x \in \mathcal{S}} \mathcal{A}_x$ is equivalent to the quotient $B\mathbb{G}_m^n/G$, and the statement is equivalent to

$$\mathsf{CH}_*(B(\mathbb{G}_m^n \rtimes G)) = \mathsf{CH}_*(B\mathbb{G}_m^n)^G \,,$$

[23] A similar formula and computation for the Chow groups of the stack of expanded pairs appears in [43].

[24] The case of interest is the Artin fan $\mathcal{A}_X$ of X, but in the argument we allow $\mathcal{A}_X$ to be arbitrary in $\mathcal{C}$ in order to run the induction.

which is true (see [6, Lemma 3.20]). In general, we pick an open stratum , $U \in \mathcal{A}_X$, with preimage $U_x \in \mathcal{A}_x$. Then, by [35, Proposition 4.2.1] we have an exact sequence

$$\mathsf{CH}(U,1) \longrightarrow \mathsf{CH}(Z) \longrightarrow \mathsf{CH}(\mathcal{A}_X) \longrightarrow \mathsf{CH}(U) \longrightarrow 0$$

with $Z = \mathcal{A}_X - U$. Since U is of the form $U = B(\mathbb{G}_m^n \rtimes G)$, we can use [6, Proposition 3.14, Remark 3.21] to see that

$$\mathsf{CH}(U,1) = \mathsf{CH}(U) \otimes_{\mathbb{Q}} \mathsf{CH}(\mathrm{Spec}(\mathbb{C}),1)$$

Then by [6, Remark 3.18], the connecting homomorphism $\mathsf{CH}(U,1) \to \mathsf{CH}(Z)$ vanishes. So we obtain an exact sequence

$$0 \longrightarrow \mathsf{CH}(Z) \longrightarrow \mathsf{CH}(\mathcal{A}_X) \longrightarrow \mathsf{CH}(U) \longrightarrow 0\,,$$

and the same sequence holds with $\mathcal{A}_X$ replaced by $\mathcal{A}_x$, U by U_x, and Z by $Z_x = \mathcal{A}_x - U_x$. As projective limits are left exact, we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{CH}(Z) & \longrightarrow & \mathsf{CH}(\mathcal{A}_X) & \longrightarrow & \mathsf{CH}(U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_{x\in\mathcal{S}} \mathsf{CH}(Z_x) & \longrightarrow & \varprojlim_{x\in\mathcal{S}} \mathsf{CH}(\mathcal{A}_x) & \longrightarrow & \varprojlim_{x\in\mathcal{S}} \mathsf{CH}(U_x) & & \end{array}$$

By induction, the left and right vertical arrows are isomorphisms. But the bottom row is exact as well: the composed map

$$\mathsf{CH}(\mathcal{A}_X) \to \mathsf{CH}(\mathcal{U}) \cong \varprojlim_{x\in\mathcal{S}} \mathsf{CH}(U_x)$$

is surjective and factors through $\varprojlim_{x\in\mathcal{S}} \mathsf{CH}(\mathcal{A}_x)$. Thus the map

$$\mathsf{CH}(\mathcal{A}_X) \to \varprojlim_{x\in\mathcal{S}} \mathsf{CH}(\mathcal{A}_x)$$

is an isomorphism as well. ◊

Theorem 15 *The logarithmic tautological ring*

$$\mathsf{R}^\star(X,D) \subset \mathsf{CH}^\star(X)$$

coincides with the image $\alpha^*\mathsf{CH}^\star(\mathcal{A}_X) \subset \mathsf{CH}^\star(X)$

Proof. Fix a stratum $S \subset X$ with closure $\overline{S} \subset X$, and normalization

$$\epsilon : \widetilde{S} \to \overline{S} \subset X\,.$$

Consider the cone complex $C(X,D)$ and the Artin fan $\mathcal{A}_X$ of (X,D) with

$$\alpha : X \to \mathcal{A}_X\,.$$

Let $\widetilde{P}$ be the total space of the principal G-bundle over the normalization $\widetilde{S}$ defined by the branches of D in Section 5.1,

$$\mu : \widetilde{P} \to \widetilde{S}\,, \quad \mu_X = \epsilon \circ \mu : \widetilde{P} \to X\,.$$

We observe that all the relevant geometry is pulled back from the from the Artin fan $\mathcal{A}_X$: the stratum S corresponds to the stratum

$$\alpha(S) = \mathcal{S} \subset \mathcal{A}_X$$

with closure $\overline{\mathcal{S}} = \alpha(\overline{S})$. Let $\widetilde{\mathcal{S}}$ be the normalization of $\overline{\mathcal{S}}$, and let

$$\mu : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{S}}\,, \quad \mu_{\mathcal{A}} = \widetilde{\mathcal{P}} \to \mathcal{A}_X$$

be the total space of the principal G-bundle over $\widetilde{\mathcal{S}}$. Then,

$$S = \mathcal{S} \times_{\mathcal{A}_X} X\,, \quad \overline{S} = \overline{\mathcal{S}} \times_{\mathcal{A}_X} X\,, \quad \widetilde{S} = \widetilde{\mathcal{S}} \times_{\mathcal{A}_X} X\,, \quad \widetilde{P} = \widetilde{\mathcal{P}} \times_{\mathcal{A}_X} X\,.$$

Furthermore, since the map α is smooth, we find that $N_{\widetilde{S}/X}$ is the pullback of $N_{\widetilde{\mathcal{S}}/\mathcal{A}_X}$, and the splitting of $N_{\widetilde{S}/X}$ on $\widetilde{P}$ into line bundles is pulled back from the splitting of $N_{\widetilde{\mathcal{S}}/\mathcal{A}_X}$ on $\widetilde{\mathcal{P}}$. In other words, we have a Cartesian diagram:

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\alpha_P} & \widetilde{\mathcal{P}} \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_{\mathcal{A}}} \\ X & \xrightarrow{\alpha} & \mathcal{A}_X\,. \end{array}$$

Normally decorated strata classes on $\overline{S}$ have the form $\mu_{X*}\alpha_P^*(\gamma)$ for $\gamma \in \mathsf{CH}^\star(\widetilde{\mathcal{P}})$. As α, α_P are smooth, $\mu_{X*}\alpha_P^* = \alpha^*\mu_{\mathcal{A}*}$. Therefore,

$$R(X,D) \subset \alpha^*\mathsf{CH}^\star(\mathcal{A}_X)\,.$$

To show the reverse inclusion, we perform a barycentric subdivision on $C(X,D)$ which, in turn, determines a map

$$\mathcal{A}_{X'} \to \mathcal{A}_X$$

of Artin fans, a log blow-up

$$X' = \mathcal{A}_{X'} \times_{\mathcal{A}_X} X \to X\,,$$

and a normal crossings pair (X', D').

Over the original quasi-projective stratum $S \subset X$, a stratum $T \subset X'$ can be found which is the total space of the principal G-bundle

$$\epsilon : T = P \to S\,,$$

where G is the monodromy group of the branches of D which cut out S. The claim easily follows from the analysis of the explosion in Section 5.2. However, $\overline{T}$ may not be a complete intersection of distinct components of the logarithmic boundary of $\widetilde{X}$. To achieve the complete intersection property for strata, we must iterate the subdivision.

We now take the barycentric subdivision of $C(X', D')$ to obtain $C(\widehat{X}, \widehat{D})$. We obtain log blow-ups

$$f : \widehat{X} = \mathcal{A}_{\widehat{X}} \times_{\mathcal{A}_X} X \to X\,, \quad g : \mathcal{A}_{\widehat{X}} \to \mathcal{A}_X\,,$$

and we also have

$$\widehat{\alpha} : \widehat{X} \to \mathcal{A}_{\widehat{X}}\,.$$

The stratum $T \subset X'$ lifts to

$$T \subset \widehat{X}$$

and is a complete intersection of logarithmic boundary divisors in $\widehat{X}$. The same is true on the level of Artin fans: the total space of the principal G-bundle

$$\mathcal{P} \to \mathcal{S} = \alpha(S) \subset \mathcal{A}_X$$

is a stratum $\mathcal{T} \subset \mathcal{A}_{\widehat{X}}$ with

$$T = \mathcal{T} \times_{\mathcal{A}_{\widehat{X}}} \widehat{X}\,.$$

The advantage of working with $\widehat{X}$ is that the cone complex $C(\widehat{X}, \widehat{D})$ is an honest rational polyhedral cone complex. As a consequence, the Chow ring

$$\mathsf{CH}^\star(\mathcal{A}_{\widehat{X}}) \cong \mathsf{PP}(C(\widehat{X}))$$

is particularly simple:

- $\mathsf{CH}^\star(\mathcal{A}_{\widehat{X}})$ is generated by $\mathsf{CH}^1(\mathcal{A}_{\widehat{X}}) \cong \mathrm{Pic}(\mathcal{A}_{\widehat{X}})$,
- $\mathrm{Pic}(\mathcal{A}_{\widehat{X}})$ is isomorphic to the group of piecewise linear functions $\mathrm{PL}(C(\widehat{X}, \widehat{D}))$,
- For all strata, $\overline{\mathcal{Z}} \subset \mathcal{A}_{\widehat{X}}$, $\mathsf{CH}^\star(\overline{\mathcal{Z}})$ is generated by the image of

$$\mathrm{Pic}(\mathcal{A}_{\widehat{X}}) \to \mathsf{CH}^1(\overline{\mathcal{Z}})\,.$$

Let $\gamma \in \alpha^* \mathsf{CH}^\star(\mathcal{A}_X)$. We must show

$$\gamma \in R(X, D)\,.$$

We may assume that γ is supported on $\overline{S}$ for some stratum $S \subset X$. Suppose, by induction, we have shown that every such class supported on a stratum $\overline{S}'$ with

$$\dim S' < \dim S$$

is in $R(X, D)$. Suppose further that we can find a class $\delta \in R(X, D)$ such that γ equals δ on S. Then,

$$\gamma - \delta \in \alpha^* \mathsf{CH}^\star(\mathcal{A}_X)$$

supported on lower dimensional strata and therefore in $R(X, D)$. Therefore, we have $\gamma \in R(X, D)$ as well.

By the induction argument, we can assume that the stratum $S \subset X$ is closed. Consider the Cartesian diagrams

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\alpha}} & \mathcal{A}_{\widehat{X}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\alpha} & \mathcal{A}_X \end{array}$$

which restrict over $\mathcal{S}$ to

$$\begin{array}{ccc} T & \xrightarrow{\widehat{\alpha}} & \mathcal{T} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{\alpha} & \mathcal{S}\,. \end{array}$$

Since $\gamma \in \alpha^* \mathsf{CH}^\star(\mathcal{A}_X)$, we have

$$f^*(\gamma) = f^* \alpha^*(\beta) = \widehat{\alpha}^* g^*(\beta)\,,$$

for some $\beta \in \mathsf{CH}^\star(\mathcal{A}_X)$. Since $\mathcal{T}$ is a G-bundle over $\mathcal{S}$, we have $g^*(\beta)$ is a G-invariant class on $\mathcal{T}$. Suppose the codimension of S in X is k, and let

$$f^* N_{S/X} = \oplus_1^k N_i$$

be the splitting of $N_{S/X}$. Then, we also have

$$g^* N_{\mathcal{S}/\mathcal{X}} = \oplus_1^k \mathcal{N}_i\,, \quad \widehat{\alpha}^* \mathcal{N}_i = N_i\,.$$

Since $\mathcal{T}$ is the complete intersection of k divisors in $\mathcal{A}_{\widehat{X}}$, we have

$$\mathcal{T} \cong B\mathbb{G}_m^k \quad \text{and} \quad \mathsf{CH}^\star(\mathcal{T}) = \mathbb{Q}[x_1, \cdots, x_k]\,.$$

The basis elements x_i of $\mathsf{CH}^\star(\mathcal{T})$ are precisely the Chern classes of the line bundles $\mathcal{N}_i$. Therefore $g^*(\beta)$ is a G-invariant polynomial in the Chern classes of $\mathcal{N}_i$ and

$$f^*(\gamma) = \widehat{\alpha}^* g^*(\beta)\,,$$

so γ is precisely a normally decorated stratum class as desired. $\Diamond$

Theorem 15 immediately implies that $\mathsf{R}^\star(X, D) \subset \mathsf{CH}^\star(X)$ is closed under the intersection product (a claim which was left to the reader in Section 5.4). We can also provide a second proof of Theorem 11 based on the above study of the Artin fan.

Corollary 16 *We have* $\mathsf{R}^\star(X,D) \subset \mathsf{divlogCH}^\star(X,D)$.

Proof. Let $(\widehat{X},\widehat{D})$ be the log blow-up corresponding to the double barycentric subdivision,

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\alpha}} & \mathcal{A}_{\widehat{X}} \\ \big\downarrow f & & \big\downarrow g \\ X & \xrightarrow{\alpha} & \mathcal{A}_X \,. \end{array}$$

Let $\gamma \in \mathsf{R}^\star(X,D)$. By Theorem 15, $\gamma \in \alpha^*\mathsf{CH}^\star(\mathcal{A}_X)$ and therefore

$$f^*(\gamma) \in \widehat{\alpha}^*\mathsf{CH}^\star(\mathcal{A}_{\widehat{X}})\,.$$

Since $\mathsf{CH}^\star(\mathcal{A}_{\widehat{X}})$ is generated by divisors, we have $f^*(\gamma) \in \mathsf{divCH}^\star(\widehat{X})$. ♢

The proof of Theorem 15 immediately yields a finer statement: $\mathsf{R}^\star(X,D)$ lies in the subalgebra generated by logarithmic divisors of the log blow-up associated to the second barycentric subdivision of the Artin fan of (X,D). In fact, very little of the double barycentric subdivision is needed. A closer analysis of the proof of Theorem 15 yields an even finer statement: $\mathsf{R}^\star(X,D)$ lies in the subalgebra generated by logarithmic divisors of any log modification $(\widetilde{X},\widetilde{D})$ of (X,D) for which $\widetilde{X}$ is nonsingular and the cone complex $C(\widetilde{X},\widetilde{D})$ is the cone over a simplicial complex: each cone in $C(\widetilde{X},\widetilde{D})$ is the cone over a simplex, and the intersection of any two cones in $C(\widetilde{X},\widetilde{D})$ is a face of each[25]. Geometrically, the simplicial condition implies that the irreducible components of $\widetilde{D}$ are nonsingular and that the closures of strata of $(\widetilde{X},\widetilde{D})$ are complete intersections of irreducible components of $\widetilde{D}$. The barycentric subdivision of any normal crossings pair (X,D) is always a cone over a simplicial complex, but for any given example, a much more efficient choice $(\widetilde{X},\widetilde{D})$ may be available.

5.6 Proofs of Theorems 12 and 13

Fix a normal crossings pair (X,D) with Artin fan $\mathcal{A}_X$ and map

$$\alpha : X \to \mathcal{A}_X\,.$$

Consider an arbitrary smooth log modification

$$f : \widetilde{X} \to X$$

necessarily of the form $(\widetilde{X},\widetilde{D})$ with an associated map

$$\widetilde{\alpha} : \widetilde{X} \to \mathcal{A}_{\widetilde{X}}\,.$$

[25] Equivalently, $C(\widetilde{X},\widetilde{D})$ can be piecewise-linearly embedded in a vector space.

As in the proof of Theorem 15, we have a Cartesian diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\alpha}} & \mathcal{A}_{\widetilde{X}} \\ \Big\downarrow f & & \Big\downarrow g \\ X & \xrightarrow{\alpha} & \mathcal{A}_X \,. \end{array}$$

By Theorem 15,

$$\mathsf{R}^\star(X,D) = \alpha^* \mathsf{CH}^\star(\mathcal{A}_X) \quad \text{and} \quad \mathsf{R}^\star(\widetilde{X},\widetilde{D}) = \widetilde{\alpha}^* \mathsf{CH}^\star(\mathcal{A}_{\widetilde{X}})\,.$$

Since $f_* \widetilde{\alpha}^*(\widetilde{\delta}) = \alpha^* g_*(\widetilde{\delta})$, we have[26]

$$f_* \mathsf{R}^\star(\widetilde{X},\widetilde{D}) = \mathsf{R}^\star(X,D)\,.$$

Similarly, since and $f^*\alpha^*(\delta) = \widetilde{\alpha}^* g^*(\delta)$, we have $f^*\mathsf{R}^\star(X,D) \subset \mathsf{R}^\star(\widetilde{X},\widetilde{D})$. ◊

6 Pixton's formula for $\lambda_g \in \mathsf{CH}^\star(\overline{\mathcal{M}}_g)$

6.1 Strata

Pixton's formula for the double ramification cycle $\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$ is expressed as a sum over strata of $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ indexed by the set $\mathsf{G}_{g,n}$ of stable graphs. We present here Pixton's formula with an emphasis on the special case

$$\mathsf{DR}_{g,\emptyset} = (-1)^g \lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_g)\,.$$

We refer the reader to [31, 46] for a more detailed discussion about double ramification cycles, stable graphs, Pixton's formula, and the relation to classical Abel-Jacobi theory.

6.2 Weightings

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ satisfy $\sum_{i=1}^n a_i = 0$. Let

$$\Gamma \in \mathsf{G}_{g,n}$$

be a stable graph[27] of genus g with n legs. A *weighting* of Γ is a function on the set of half-edges,

$$w : \mathrm{H}(\Gamma) \to \mathbb{Z},$$

which satisfies the following three properties:

(i) $\forall h_i \in \mathrm{L}(\Gamma)$, corresponding to the marking $i \in \{1, \ldots, n\}$,

$$w(h_i) = a_i \ ,$$

[26] We conclude equality instead of inclusion since g_* is surjective.

[27] Here and in Pixton's formula in Section 6.3, we follow the notation of [31, Sections 0.3 and 0.4]. The factors of 2 are treated equivalently but slightly differently in [5, 32].

(ii) $\forall e \in \mathrm{E}(\Gamma)$, corresponding to two half-edges $h, h' \in \mathrm{H}(\Gamma)$,

$$w(h) + w(h') = 0\,,$$

(iii) $\forall v \in \mathrm{V}(\Gamma)$,

$$\sum_{v(h)=v} w(h) = 0\,,$$

where the sum is taken over *all* $n(v)$ half-edges incident to v.

In the case $A = \emptyset$, the set of half-edges $\mathrm{H}(\Gamma)$ has no legs ($n = 0$).

Let r be a positive integer. A *weighting mod* r of Γ is a function,

$$w : \mathrm{H}(\Gamma) \to \{0, \ldots, r-1\},$$

which satisfies exactly properties (i-iii) above, but with the equalities replaced, in each case, by the condition of *congruence mod* r. The set $\mathsf{W}_{\Gamma,r}$ is finite, with cardinality $r^{h^1(\Gamma)}$.

6.3 Formula for double ramification cycles

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ satisfy $\sum_{i=1}^n a_i = 0$. Let r be a positive integer. We denote by

$$\mathsf{P}_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$$

the degree d component of the tautological class

$$\sum_{\Gamma \in \mathsf{G}_{g,n}} \sum_{w \in \mathsf{W}_{\Gamma,r}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma *} \Bigg[\prod_{i=1}^n \exp(a_i^2 \psi_{h_i}) \cdot \prod_{e=(h,h') \in \mathrm{E}(\Gamma)} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \Bigg]. \quad (32)$$

in $R^*(\overline{\mathcal{M}}_{g,n})$.

The following fundamental polynomiality property of $\mathsf{P}_g^{d,r}(A)$ has been proven by Pixton, see [31, Appendix].

Proposition 17 (Pixton) *For fixed g, A, and d, the class*

$$\mathsf{P}_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$$

is polynomial in r (for all sufficiently large r).

We denote by $\mathsf{P}_g^d(A)$ the value at $r = 0$ of the polynomial associated to $\mathsf{P}_g^{d,r}(A)$ by Proposition 17. In other words, $\mathsf{P}_g^d(A)$ is the *constant* term of the associated polynomial in r. Pixton's formula for double ramification cycles is

$$\mathsf{DR}_{g,A} = 2^{-g}\, \mathsf{P}_g^g(A) \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,.$$

6.4 Examples in the $A = \emptyset$ case

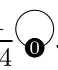
For the reader's convenience, we present here the first few examples[28] of Pixton's formula for λ_g obtained by calculating $(-1)^g \mathsf{DR}_{g,\emptyset}$.

Each labeled graph Γ describes a moduli space $\overline{\mathcal{M}}_\Gamma$ (a product of moduli spaces associated with the vertices of Γ), a tautological class $\alpha \in R^*(\overline{\mathcal{M}}_\Gamma)$, and a natural map

$$\xi : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_g \,.$$

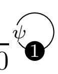
Our convention in the formulas below is that the graph Γ represents the cycle class $\xi_* \alpha$. For instance, assume the graph carries no ψ-classes and the class α equals 1. Since the map ξ is of degree $|\mathrm{Aut}(\Gamma)|$ onto its image, the cycle class represented by Γ is then $|\mathrm{Aut}(\Gamma)|$ times the class of the image of ξ.

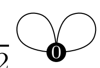
Genus 1.

$\lambda_1 = \frac{1}{24}$ 

.

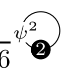
Genus 2.

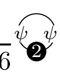
$\lambda_2 = \frac{1}{240}$ 

$+ \frac{1}{1152}$ 

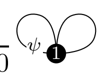
.

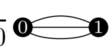
Genus 3.

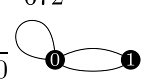
$\lambda_3 = \frac{1}{2016}$ 

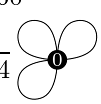
$+ \frac{1}{2016}$ 

$- \frac{1}{672}$ 

$+ \frac{1}{5760}$ 

$- \frac{13}{30240}$ 

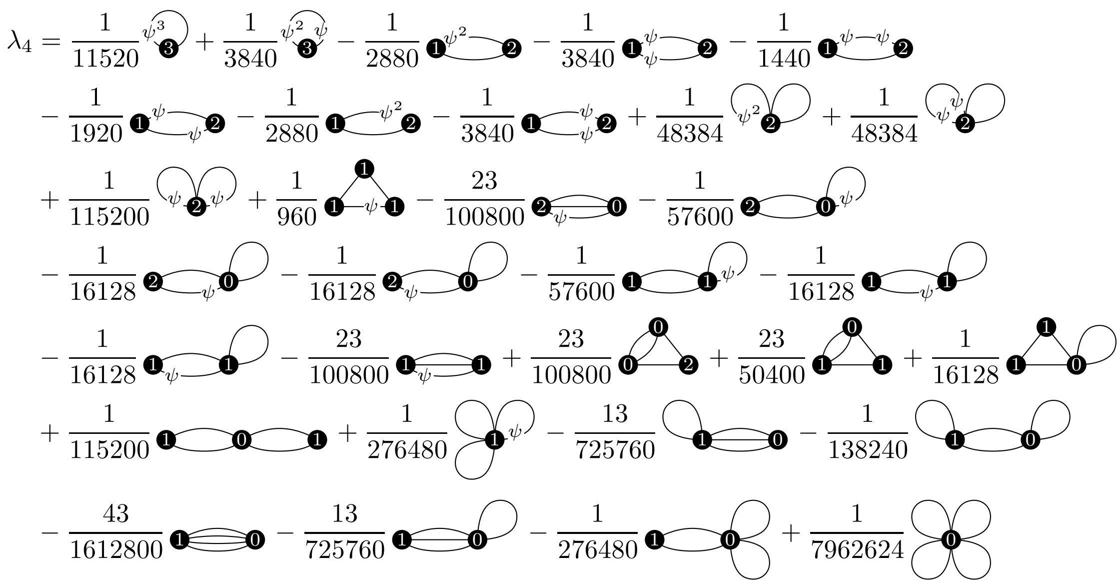
$- \frac{1}{5760}$ 

$+ \frac{1}{82944}$ 

.

[28] The graphics are by F. Janda.

Genus 4.



6.5 Proof of Theorem 6

We analyze Pixton's formula in the $A = \emptyset$ case,

$$\lambda_g = (-1)^g \mathsf{DR}_{g,\emptyset} \in \mathsf{CH}^g(\overline{\mathcal{M}}_g)\,.$$

Since $A = \emptyset$, the sum (32) is over stable graphs $\Gamma \in \mathsf{G}_g$ corresponding to strata of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$.

• By the definition of a *weighting mod* r, the weights

$$w(h)\,,\ w(h')$$

on the two halves of *every* separating edge e of Γ must both be 0. The factor in Pixton's formula for e,

$$\frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}}\,,$$

then vanishes and kills the contribution of Γ to $\mathsf{P}_g^g(\emptyset)$. Therefore, nonvanishing terms in the sum (32) must correspond to graphs with *no* separating edges.

• Since $A = \emptyset$, the term

$$\prod_{i=1}^{n} \exp(a_i^2 \psi_{h_i})$$

drops out of (32).

• The classes which do appear in (32) are the normal bundle terms $\psi_h + \psi_{h'}$ at each edge of Γ.

Since the formula (32) respects the automorphisms of the stable graph Γ, we obtain the following result.

Proposition 18 *The class $\lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_g)$ is a sum of normally decorated classes associated to strata of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$ corresponding to stable graphs $\Gamma \in \mathsf{G}_g$ with no separating edges.*

Theorem 6 is then an immediate consequence of Proposition 18 and Theorem 11. Proposition 18 reflects a very special property of λ_g obtained from Pixton's formula. ♢

Since every edge of every stable graph $\Gamma \in \mathsf{G}_g$ which appears in Pixton's formula for λ_g is non-separating, we actually have

$$\lambda_g \in \mathsf{R}^\star(\overline{\mathcal{M}}_g, \Delta_0)\,.$$

Theorem 11 then implies a refinement of Theorem 6,

$$\lambda_g \in \mathsf{divlogCH}^\star(\overline{\mathcal{M}}_g, \Delta_0)\,.$$

By applying Pixton's formula for the double ramification cycle

$$\mathsf{DR}_{g,(0,\ldots,0)} = (-1)^g \lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,,$$

an identical argument yields

$$\lambda_g \in \mathsf{divlogCH}^\star(\overline{\mathcal{M}}_{g,n}, \Delta_0)$$

for $2g-2+n > 0$.

6.6 More general DR cycles

Let $A = (a_1, \ldots, a_n)$ be a vector of integers satisfying $\sum_{i=1}^n a_i = 0$. Pixton's formula for the double ramification cycle

$$\mathsf{DR}_{g,A} \in \mathsf{R}^\star(\overline{\mathcal{M}}_{g,n})$$

together with Theorem 11 yields the following result (the proof is exactly the same as the proof of Theorem 6).

Theorem 19 *We have $\mathsf{DR}_{g,A} \in \underline{\mathsf{div}}\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n})$ where*

$$\underline{\mathsf{div}}\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n})$$

is the subalgebra generated by logarithmic boundary divisors together with the cotangent line classes $\psi_1, \ldots, \psi_n$.

Theorem 19 provides half of the proof of Conjecture C concerning the lifted double ramification cycle $\widetilde{\mathsf{DR}}_{g,A}$. The second half requires a study of the difference between $\mathsf{DR}_{g,A}$ and $\widetilde{\mathsf{DR}}_{g,A}$ as developed in [40].

The special case $A = (0,\ldots,0)$ related to the class λ_g is simpler since no cotangent line classes appear at the markings in Pixton's formula. Moreover, there is no change in the lift for $A = (0,\ldots,0)$:

$$\mathsf{DR}_{g,(0,\ldots,0)} = \widetilde{\mathsf{DR}}_{g,(0,\ldots,0)} \in \underline{\mathsf{div}}\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n})\,.$$

The ω^k-twisted double ramification cycle [27] is also governed by Pixton's formula [5],

$$\mathsf{DR}^k_{g,A} \in \mathsf{R}^\star(\overline{\mathcal{M}}_{g,n})\,, \quad \sum_{i=1}^n a_i = k(2g-2)\,.$$

The analogue of Theorem 19 can be proven for the ω^k-twisted double ramification cycle, but the divisor subalgebra of $\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n})$ must include κ_1 together with the cotangent line classes ψ_i and the logarithmic boundary divisors. Conjecture C can then also be promoted to a statement for the lifted ω^k-twisted double ramification cycle (again including κ_1 in the subalgebra).

6.7 Pixton's generalized boundary strata classes

In [52], Pixton has defined a subalgebra of the tautological ring $\mathsf{R}^*(\overline{\mathcal{M}}_{g,n})$ spanned by *generalized boundary strata classes*: tautological classes $[\Gamma]$ associated to prestable graphs Γ of genus g with n legs.

If Γ is a semistable graph (every genus 0 vertex is incident to at least two legs or half-edges), then Pixton's definition takes a simple form. Let Γ' be the stabilization of Γ. The class $[\Gamma]$ is defined as a push-forward under the gluing map $\xi_{\Gamma'}$ of products of classes $\psi_1,\ldots,\psi_n$ and classes $\psi_h + \psi_{h'}$ for half-edges (h,h') forming an edge of Γ'. The analysis of Section 6.5 then implies

$$[\Gamma] \in \underline{\mathsf{div}}\mathsf{logCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

in the semistable case.

Pixton's boundary class for more general unstable graphs has κ classes and will likely not lie in any version of $\mathsf{divlogCH}^\star(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$.

7 The bChow ring

Let X be a nonsingular variety. Given the additional data of a normal crossings divisor $D \subset X$ we defined the log Chow ring of the pair (X,D). This is a variant of a much larger ring, the *bChow ring* of X. We define

$$\mathsf{bCH}^\star(X) = \varinjlim_{Y\in\mathsf{B}(X)} \mathsf{CH}^\star(Y)\,,$$

where $\mathsf{B}(X)$ is the category of nonsingular blow-ups of X: objects in $\mathsf{B}(X)$ are proper birational maps

$$Y \to X$$

with Y nonsingular and morphisms in $\mathsf{B}(X)$ are proper birational maps over X. For a longer introduction to the bChow ring, see [28]. Some of the ideas involved go back to papers of Shokurov [54, 55]. See also Aluffi [4] for similar constructions.

Let $[Z \to X]$ and $[Y \to X]$ be objects of $\mathsf{B}(X)$. If $Z \to X$ factors as

$$Z \to Y \to X \,,$$

then there is a unique morphism from $[Z \to X]$ to $[Y \to X]$ in $\mathsf{B}(X)$, and we call $Z \to X$ a *refinement* of $Y \to X$. The transition maps in the above colimit are given by pullbacks

$$f^\star : \mathsf{CH}^*(Y) \to \mathsf{CH}^\star(Z)$$

for refinements $Z \xrightarrow{f} Y \to X$.

Unlike, $\mathsf{logCH}^\star(X)$, the bChow ring does *not* depend upon the choice of a normal crossings divisor $D \subset X$. However, given such a choice there is always a tower of natural inclusions

$$\mathsf{CH}^\star(X) \subset \mathsf{logCH}^\star(X) \subset \mathsf{bCH}^\star(X)\,.$$

Since the centers of the blow-up are so restricted in the definition of $\mathsf{logCH}^\star(X)$, we view $\mathsf{CH}^\star(X)$ and $\mathsf{logCH}^\star(X)$ as relatively close in size. On the other hand, $\mathsf{bCH}^\star(X)$ is very much larger.

Let $\mathsf{divbCH}^\star(X)$ be the subalgebra of $\mathsf{bCH}^\star(X)$ generated by divisors. More precisely,

$$\mathsf{divbCH}^\star(X) = \varinjlim_{Y \in \mathsf{B}(X)} \mathsf{divCH}^\star(Y)\,.$$

While the proof of the claim

$$\lambda_g \in \mathsf{divlogCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

depended upon special properties of λ_g, the parallel bChow statement

$$\lambda_g \in \mathsf{divbCH}^*(\overline{\mathcal{M}}_g)$$

immediately follows from a general result.

Theorem 20 *For every nonsingular quasi-projective variety*[29] X, *bChow is generated by divisor classes,*

$$\mathsf{divbCH}^*(X) = \mathsf{bCH}^\star(X)\,.$$

[29]The statement holds verbatim for nonsingular Deligne-Mumford stacks which admit finite resolutions of sheaves by vector bundles.

Proof. Let $\alpha \in \mathsf{CH}^\star(Y)$ for an object $[Y \to X]$ in $\mathsf{B}(X)$. We will find a refinement $Z \to Y$ for which

$$f^* a \in \mathsf{divCH}(Z)\,.$$

Since Y is nonsingular and quasi-projective, the Chern classes of vector bundles generate $\mathsf{CH}^\star(Y)$. We can assume $\alpha = c_i(E)$ for a vector bundle E on Y. By [26, Corollary 2], there is a blow-up

$$g : W \to Y$$

where W is nonsingular and g^*E contains a subline bundle L,

$$0 \to L \to g^*E \to g^*E/L \to 0\,.$$

Applying the same argument to the quotient bundle g^*E/L, we find inductively a nonsingular blow-up

$$f : Z \to Y$$

for which f^*E has a filtration with line bundles as quotients. Therefore,

$$f^*c_i(E) = c_i(f^*E)$$

is in $\mathsf{divCH}^\star(Z)$. ♢

The quasi-projective hypothesis is used only for vector bundle resolutions. In fact, the hypothesis is not necessary. Theorem 20 can be proven locally near any cycle

$$S \subset X$$

by successive blow-ups along nonsingular centers to resolve S and appropriately modify the Chern classes of the normal bundle of S. We leave the details for the interested reader.

A The fourth cohomology group of $\overline{\mathcal{M}}_g$

In the proof of Theorem 4, we require the equality[30]

$$\mathsf{H}^4(\overline{\mathcal{M}}_g) = \mathsf{RH}^2(\overline{\mathcal{M}}_g)\,. \tag{33}$$

for sufficiently large g, In other words, the fourth cohomology group of $\overline{\mathcal{M}}_g$ is spanned by tautological classes for sufficiently high g.

Equality (33) was first proven by Edidin [14] for $g \geq 12$. Edidin bounded the Betti number $\mathsf{h}^4(\overline{\mathcal{M}}_g)$ from above and then showed by intersection calculations

[30] We use, as before, the complex grading on $\mathsf{RH}^\star$.

that the span of the tautological classes[31] in codimension 2 achieves the required rank. Edidin used the interior result

$$\mathsf{H}^4(\mathcal{M}_g) = \mathsf{RH}^2(\mathcal{M}_g) \tag{34}$$

proven by Harer [25] for $g \geq 12$. The interior statement (34) was later proven for $g \geq 9$ by Ivanov [29] and strengthened further to $g \geq 7$ by Boldsen [9] which improved Edidin's bound.

Theorem 21 ([14], [29], [9]) *We have* $\mathsf{H}^4(\overline{\mathcal{M}}_g) = \mathsf{RH}^2(\overline{\mathcal{M}}_g)$ *for* $g \geq 7$.

B Computations in *admcycles*

B.1 Verification of Pixton's conjecture

In [51], Pixton proposed a set of relations between tautological classes on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves. These were proven to hold in cohomology [49] and in Chow [30]. Pixton furthermore conjectured that his relations span the *complete* set of relations among tautological classes. The relations were implemented by Pixton in the mathematical software SageMath [53] and later incorporated in the SageMath package *admcycles*. Assuming Pixton's conjecture, the software computes a basis of the $\mathbb{Q}$-vector spaces $\mathsf{R}^d(\overline{\mathcal{M}}_{g,n})$ and express tautological classes in the basis.

In Proposition 5, we state that Pixton's conjecture holds for the spaces

$$\mathsf{R}^4(\overline{\mathcal{M}}_{4,1}) \text{ and } \mathsf{R}^5(\overline{\mathcal{M}}_{5,1})\,.$$

Assuming the conjecture, *admcycles* computes the rank of these two spaces to be 191 and 1371 respectively. If the conjecture was false, the rank of one (or both) of the groups would have to be strictly smaller. However, using *admcycles*, we verify that the ranks of the intersection pairings

$$\mathsf{R}^4(\overline{\mathcal{M}}_{4,1}) \otimes \mathsf{R}^6(\overline{\mathcal{M}}_{4,1}) \to \mathbb{Q} \quad \text{and} \quad \mathsf{R}^5(\overline{\mathcal{M}}_{5,1}) \otimes \mathsf{R}^8(\overline{\mathcal{M}}_{5,1}) \to \mathbb{Q}$$

are bounded from below by 191 and 1371 respectively. The rank bounds are obtained by taking generating sets of $\mathsf{R}^4(\overline{\mathcal{M}}_{4,1})$ and $\mathsf{R}^5(\overline{\mathcal{M}}_{5,1})$ and computing the matrix of pairings with generators in $\mathsf{R}^6(\overline{\mathcal{M}}_{4,1})$ and $\mathsf{R}^8(\overline{\mathcal{M}}_{5,1})$ respectively. For the rank bounds of pairing, we do *not* assume anything about the relations between the above generators, though we are allowed to use the known relations [49] to reduce the size of the generating sets.

The computations were performed on a server of the Max-Planck Institute for Mathematics in Bonn[32], taking two days in the case of $\overline{\mathcal{M}}_{4,1}$ and 31 days for

[31] Edidin does not use the language of tautological classes as we now do, but all of his generators are in fact tautological: they are given by the classes κ_2, κ_1^2, pushforwards of λ- and ψ-classes under boundary divisor gluing maps, and fundamental classes of strata of codimension 2.

[32] The program ran on a single thread of the available CPU (Intel Xeon Prozessor E5-2667 v2) taking about 60 GB of RAM due to the large amounts of intermediate data to store (such as the list of Pixton's relations, sets of tautological generators, etc).

$\overline{\mathcal{M}}_{5,1}$. Without substantial improvements of the algorithm, it is thus unlikely that Pixton's conjecture can be verified in this way for significantly larger g, n, and d. We warmly thank the Max-Planck Institute for providing the computer infrastructure for our computations.

B.2 Computations in proofs of Theorems 3 and 4

Once we have verified Pixton's conjecture (as above[33]). for $\mathsf{RH}^d(\overline{\mathcal{M}}_{g,n})$, we can explicitly check whether

$$\lambda_d \in \mathsf{RH}^d_{\leq k}(\overline{\mathcal{M}}_{g,n})\,.$$

Several such checks used in the proofs of Theorems 3 and 4 were made using *admcycles*.

We provide below an example of the computation showing that the class λ_3 is not contained in the space

$$\mathsf{divRH}^3(\overline{\mathcal{M}}_3) \subset \mathsf{RH}^3(\overline{\mathcal{M}}_3)\,,$$

which is a 9-dimensional subspace of a 10-dimensional space. We first create the list `divcl` of divisor classes on $\overline{\mathcal{M}}_3$, compute the set of triple products of such classes, and then take the span `divR` of the vectors representing them in a basis of $\mathsf{RH}^3(\overline{\mathcal{M}}_3)$. We verify that `divR` is 9-dimensional inside the 10-dimensional ambient space $\mathsf{RH}^3(\overline{\mathcal{M}}_3)$. Finally, we compute the class λ_3 and verify that the associated vector `Lv` is not contained in `divR`.

```
sage: from admcycles import *
sage: divcl = tautgens(3,0,1)
sage: divp = [a*b*c for a in divcl for b in divcl for c in divcl]
sage: divR = span(u.toTautbasis() for u in divp)
sage: (divR.rank(), divR.degree())
(9, 10)
sage: L = lambdaclass(3,3,0)
sage: Lv = L.toTautbasis()
sage: Lv in divR
False
```

B.3 Proof of Proposition 7

We record below the computation in *admcycles* used in the proof of Proposition 7. We create the classes $\lambda_2, [\Delta_0], [B]$ and $[C]$ and represent the class defined by

$$2\lambda_2 - x \cdot [\Delta_0]^2 - y \cdot [B] - z \cdot [C]$$

in the vector `diff` with respect to a basis of $\mathsf{CH}^2(\overline{\mathcal{M}}_2) = \mathsf{R}^2(\overline{\mathcal{M}}_2)$. We then solve the equation `diff=0` to find the formula for x and y in terms of the variable z used in the proof.

[33]Our verification method also then shows $\mathsf{R}^d(\overline{\mathcal{M}}_{g,n}) = \mathsf{RH}^d(\overline{\mathcal{M}}_{g,n})$

We remark that in the definition of the class Delta0 we need to divide by 2 since this is the degree of the gluing morphism parameterizing the boundary divisor Δ_0.

```
sage: from admcycles import *
sage: lambda2 = lambdaclass(2,2,0)
sage: Delta0 = 1/2 * irrbdiv(2,0)
sage: gammaB = StableGraph([0],[[1,2,3,4]],[(1,2),(3,4)])
sage: B = gammaB.boundary_pushforward()
sage: gammaC = StableGraph([0,1],[[1,2,3],[4]],[(1,2),(3,4)])
sage: C = gammaC.boundary_pushforward()
sage: x, y, z = var('x, y, z')
sage: diff = (2*lambda2 - x*Delta0^2 - y*B - z*C).toTautbasis()
sage: diff
(476*x + 1824*y - 96*z - 3, -144*x - 576*y + 24*z + 1)
sage: solve([diff[i]==0 for i in (0,1)], x,y,z)
[[x == r1 - 1/120, y == -5/24*r1 + 11/2880, z == r1]]
```

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