I would like to start by
[d]iscussing some basic aspects of the
[...]M stack $\overline{\mathcal{M}}_{1,1}$. Everything I
[s]ay will be elementary — and there
[wi]ll be no formal development
[of] the theory of Stacks as in the
[p]revious talks. Some of the Constructions
[I] will discuss helped me begin to
[un]derstand the geometry of stacks when
[I] started learning about them
(which was not such a long time ago).

What is $\overline{M}_{1,1}$ ?

Start: genus 1 stable curve with 1 marked point is either

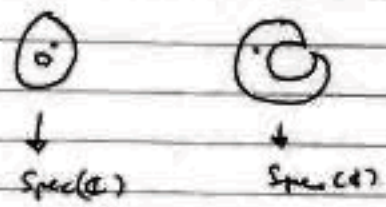
$\overline{M}_{1,1}$ category

$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ S \arrow[u, bend left, "s"] \end{tikzcd}$$

"flat family of 1-pointed genus 1 stable curves"

$\overline{M}_{1,1}$ category fibered in groupoids over Sch.

If I consider the objects over Spec(ℂ) I find.




If I have an object over Spec ℂ of a DM stack, there is an intrinsic aut group given by self arrows in the category.

DM → finite aut group.

Here Aut [ 

$\downarrow$ )$^1$ ]

Spec($\mathbb{C}$)

=

Alg Aut 

Why : Morphism.

 $\xrightarrow{f}$ 

$\downarrow$          $\downarrow$

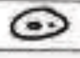
Spec($\mathbb{C}$) $\longrightarrow$ Spec($\mathbb{C}$)

f is any aut of 

which preserves the

marking.

In particular,

$$\mathrm{Aut}\left( \right.$$  $$\left. \right) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \\ \text{ord } 4, 6 & j = 0, 1728 \end{cases}$$

$$\mathrm{Aut}\left( \right.$$  $$\left. \right) = \mathbb{Z}/2\mathbb{Z}$$

Next I would like to explain how to realize $\bar{M}_{1,1}$ as a quotient stack in concrete terms.

(Bill already briefly indicated a path to $M_{1,1}$ by étale charts, we take here an alternate global approach).

First idea (motivation)

Elliptic Curve $\longleftrightarrow$ plane curve $C \cdot \mathbb{CP}^2 = \mathbb{P}(V)$

look at the locus of ns or 1-nodal plane cubics in $\mathbb{P}(V)$.

$$\mathcal{U} \subset \mathbb{P}(\mathrm{Sym}^3 V^*)$$
$\{$ ns or 1-nodal plane curves

$\mathcal{U}$ is a 9 dim open ns variety.

Of course, if we just care about the intrinsic geom of the elliptic curve, we care not about the proj embedding

$PGL(V)$ acts on $\mathcal{U}$,

Consider the quotient space.

$\left[ \mathcal{U} / PGL(V) \right]$ DM quotient stack.
dim $9-8=1$!

$$\tilde{\mathcal{M}}_{1,1} \overset{?}{\rightsquigarrow} \left[ \mathcal{U} / PGL(V) \right]$$

Take an element of $\left[ \mathcal{U} / PGL(V) \right]$

over $\mathrm{Spec}(S)$:

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{U} \\ \downarrow & & \\ \mathrm{Spec}(S) & & \end{array}$$

$$\begin{tikzcd} \mathcal{C}_P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ P \arrow[r] \arrow[d] & \mathcal{U} \\ \operatorname{Spec}(k) & \end{tikzcd}$$

Th $\quad \begin{matrix} \mathcal{C}_P/_{PGL(V)} \\ \downarrow \\ P/_{PGL(V)} \end{matrix} \implies \begin{matrix} \mathcal{C} \\ \downarrow \\ \operatorname{Spec}(k) \end{matrix}$

but no point!

We haven't proven there is no clever way to get a point, but we rule out the possibility of an isom between $\widetilde{\mathcal{M}}_{1,1}$ and $[U/_{PGL(V)}]$ by check Auts.

What is het $\left[\begin{array}{c} P \longrightarrow \mathcal{U} \\ \downarrow \\ \mathrm{Spec}(k) \end{array}\right]$ ?

Well def:

$$\begin{tikzcd} & & \mathcal{U} \\ P \arrow[urr] \arrow[r, "f"] \arrow[d] & P \arrow[ur] \arrow[d] & \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(k) & \end{tikzcd}$$

Unravell:

Let $[E] \in \mathcal{U}$ whose orbit correspond to an object.

And $\left[\begin{array}{c} P \to \mathcal{U} \\ \downarrow \\ \mathrm{Spec}\, k \end{array}\right]$ is isom to $\mathrm{Proj}\ \mathrm{Aut}(E \subset \mathbb{P}(V))$

Question What are the proj aut $C \subset \mathbb{P}(V)$

$C$ is non deg, so

Proj Aut $(C \subset \mathbb{P}(V)) \subset$ Aut $C$

$$\text{Proj Aut } (C \subset \mathbb{P}(V)) = \left\{ \gamma \in \text{Aut}(C) \;\middle|\; \begin{array}{c} \gamma^* \mathcal{O}_C(1) \\ \cong \\ \mathcal{O}_C(1) \end{array} \right\}$$

$C = \bigcirc$ $\quad \mathcal{L} = \mathcal{O}_C(1)$

might as well take $\quad \mathcal{L} = \mathcal{O}(3p)$.

Proj Aut gen $\mathbb{Z}/3\mathbb{Z}$. 3 torsion

gen a group of ord $\boxed{18}$

Too big for $\mu_3$.

Next idea is to somehow add the marking explicitly.

Instead of $\mathcal{U} \subset \mathbb{P}(\mathrm{Sym}^3 V^*)$

we could take $\begin{matrix} \mathcal{E} \\ \downarrow \\ \mathcal{U} \end{matrix}$ universal curve

Try $\left[ \mathcal{E} / \mathbb{P}GL(V) \right]$ a very six

$\dim = 10 - 8 = 2$.

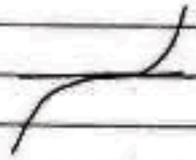
What we need is a 9-fold

Cover of $\left[ \mathcal{U} / \mathbb{P}GL(V) \right]$ involving

a point on the curve.

$\mathcal{W} = \{ (C, p) \in \mathcal{U} \times \mathbb{P}(V) \mid p$ is a flex of $C. \}$

Recall flex



$C \subset \mathbb{P}(V)$ has 9 flexes.

Claim: $\left[ \mathcal{W} / \mathbb{P}GL(V) \right] \cong \widetilde{M}_{1,1}$

Fact:

$$\begin{array}{ccc} P \to \mathcal{W} & \rightsquigarrow & (C, p) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & & \mathrm{Spec}\, k. \end{array}$$

Further back.

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ S \arrow[u, bend left, "s"] \end{tikzcd}$$

Let $\mathcal{L} = \mathcal{O}_C[3s]$

Let $\mathcal{B} = \pi_* \mathcal{L}$

rank 3 vector bundle on $S$.

Let $\begin{tikzcd} \mathcal{P} \arrow[d] \\ S \end{tikzcd}$ bundle of proj frames of $\mathcal{B}$.

$\{ \underbrace{B_s \cong V^*}_{\text{mod scalars}} \mid s \in S \}$

Consider $\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{W} \\ S \end{tikzcd}$

It now easy to define various structure on $\bar{\mathcal{M}}_{1,1}$:

A sheaf on $\bar{\mathcal{M}}_{1,1}$ is given by an equivariant coherent sheaf on $\mathcal{W}$.

The most important one is the line bundle

$$\begin{array}{c}\mathcal{L}_1\\ \downarrow\\ \bar{\mathcal{M}}_{1,1}\end{array}$$

the cotangent line.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] \\ W \arrow[u, bend left, "s"] \end{tikzcd}$$ Sctn

$$s^* \omega_\pi = \begin{array}{c} \mathcal{L}_1 \\ \downarrow \\ W \end{array}$$ $PGL(V)$ - equivt.

Ask: integral theory

$[\bar{\mathcal{M}}_{1,1}] \in A_1$

$c_1(\mathcal{L}_1) = \psi_1$ ← first Ch Cl.

$$\int_{\bar{\mathcal{M}}_{1,1}} c_1(\mathcal{L}_1) = \int_{\bar{\mathcal{M}}_{1,1}} \psi_1 = \langle \tau_1 \rangle$$

$$= ?$$

$$\begin{array}{c}\overline{\mathcal{M}}_{3,1}\\ \epsilon\downarrow\\ \mathrm{Spec}(\mathbb{C})\end{array} \qquad \text{proper, not rep.}$$

$$\int_{\overline{\mathcal{M}}_{3,1}} \psi = \epsilon_*\left([\overline{\mathcal{M}}_{3,1}] \cap c_1(\mathbb{L}_1)\right) \in A_0(\mathrm{Spec}\,\mathbb{C}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

There is now a real point.

How to calculate $\langle \tau_1 \rangle$ ?

Idea (i) find an object

of $\overline{M}_{1,1}$ over a

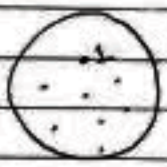
complete curve $S$

$$\lambda \left( \begin{array}{c} E \\ \downarrow \\ S \end{array} \right)$$

(ii) by 2-Y $\quad S \to \overline{M}_{1,1}$

by push-pull

$$\underbrace{\int_S f^*\psi_1}_{\text{Schur Calc}} = \deg(f) \cdot \int_{\overline{M}_{1,1}} \psi_1$$



$\mathbb{P}^2$ 8 pt.

Let $s \left( \begin{array}{c} C \\ \downarrow \\ \mathbb{P}^1 \end{array} \right.$ pencil of cubics thru 8 pts.

(1) $$\int_{\mathbb{P}^1} f^* \psi_1 = \int_{\mathbb{P}^1} s^* \omega_\pi$$

$$= -\int_C [s]^2$$

$$= +1.$$

We need to find the factor

deg(f).

idea: Let $[\gamma] \in \overline{M}_{1,1}$

be the substratum.

$$\underbrace{\int_S f^* [\gamma]} = \deg(f) \cdot \int_{\overline{M}_{1,1}} [\gamma]$$

\# nodal issues
in the pencil

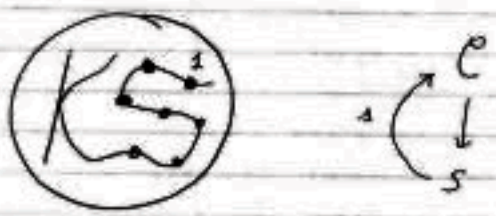
$$\boxed{12} = \deg(f) \cdot \boxed{\tfrac{1}{2}}$$

$$\boxed{24} = \deg(f)$$

Find: $\langle \tau_1 \rangle = \frac{1}{24}$.

You may ask what happens when we chose a different family.
If you believe what you've been told, the answer $\langle \tau_1 \rangle = 1/24$ will always be obtained.

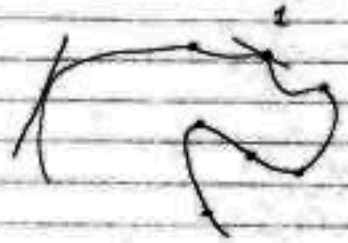
Here is another family




Claim there 7 fixed pts tangent to a fixed line

(i) Calculate $\int_S S^1 \gamma$

Ans



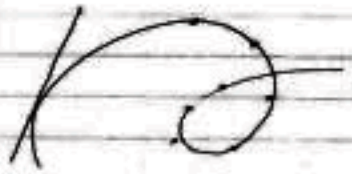

# of sol = $\boxed{4}$

(ii) Calculate def (f)

As before def (f) =

$$2 \cdot \int_S S^1 [\gamma].$$

How to find $\int_S f^*[\gamma]$ ?

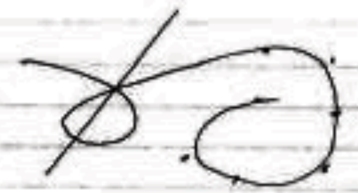


\# nodal
curves tangent to line at the 7 pt

$\boxed{36}$.

So for $4 = 2 \cdot 36 \langle \tau_1 \rangle$ doesn't work!

In fact there are some elements
of the family with nodes.



Turns out there are $\boxed{6}$ of them.

Then calculate

$$4 = 2\,(36 + 6)\ \langle \tau_1 \rangle \qquad \left[\begin{array}{c}\text{Still}\\ \text{Fails}\end{array}\right]$$

$$S \longrightarrow \overline{M}_{1,1}$$

Etale at

type I $\gamma$ $\qquad 4 = 2\,(36 + 2\cdot 6)\langle \tau_1 \rangle$

ramifies at $\qquad \frac{1}{24} = \langle \tau_1 \rangle$

type II $\gamma$

Stop!

We note we can learn about

$Pic(\mathcal{M}_{1,1})$ by $\langle\tau_1\rangle = \frac{1}{24}$ calculation.

$\mathcal{L}_1$ restr to cotangent li $\mathcal{M}_{1,1}$

Claim order of $\mathcal{L}_1$ in $Pic(\mathcal{M}_{1,1})$

is at least 12.

proof: Suppose $\mathcal{L}_1^{\otimes k} \cong \mathcal{O}$ on $\mathcal{M}_{1,1}$

then $\mathcal{L}_1^{\otimes k}$ has a ration section

with zeros or pole on $[\delta]$

$$\int_{\overline{M}_{1,1}} \psi_1^{k} = \frac{n}{2}$$

$$\langle\tau_1\rangle = \frac{n}{k \cdot 2} = \frac{1}{24} \qquad k \geq 12.$$

Further thing we can see

let

$$\begin{array}{c} N \\ \downarrow \\ \bar{M}_{1,1} \end{array}$$

be

any line bundle, we can see

$$\int_{\bar{M}_{1,1}} c_1(N) \text{ has denom at most } 24.$$

$\bar{M}_{1,1} \sim \{ j = 0, 1728 \}$

"

$$\left[ \frac{P^2 - 2pt1}{\mathbb{Z}/_2} \right]$$

$N^{\otimes 2}$ has a section on $\bar{M}_{1,1} \setminus \{ j = 0, 1728 \}$

$$\int_{\bar{M}_{1,1}} 2c_1(N) = \frac{n}{4} + \frac{m}{6}$$

I would consider $\bar{M}_{1,1}$

an orbifold, not by Billi def

Orbifold Euler Char.

(i) Stratify by ext group $G$

(ii) $\chi = \sum_{Y} \chi(Y_G)/|G|$

$$\chi(\bar{M}_{1,1}) = \frac{\chi(\mathbb{P}^1 - 2\,pt)}{2} + \frac{1}{4} + \frac{1}{6}$$

$$= -\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$$

$$= \boxed{-\frac{1}{12}}.$$