Maps to a moving elliptic curve

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Moduli spaces of genus g Hurwitz covers of CP^1 yield cycles on the moduli space of genus g curves. The study of these classes is a very well-developed subject. Much less studied is the moduli space of Hurwitz covers of elliptic curves. I presented results and open questions related to ramified covers of a moving elliptic curve (connected mainly to tautological classes on moduli spaces (of curves, Abelian varieties, and K3 surfaces), but also to quantum cohomology, Hilbert schemes, and Hodge integrals).

More specifically, there are 8 variations of the moduli problem which I wish to consider given by the following 3 binary choices:

(i) The target elliptic curve may be fixed or moving.
(ii) The moduli space may be of stable maps or of admissible covers.
(iii) The class in the moduli space of domain curves may be taken in cohomology or Chow.

Part I: Modularity for a stable maps to a fixed elliptic curve E in cohomology.

The central result here is due to Oberdieck-Pixton: the quasi-modularity of the cohomological cycle,

$$\sum_{d=0}^{\infty} q^d [\overline{M}_{g,n}(E,d)]^{vir} \ \in \ H^*(\overline{M}_{g,n}) \otimes \mathsf{QMod}\,.$$

In the above result, the virtual class defines (via push-forward) a cohomology class on the moduli space of stable curves. The proof uses the magical polynomiality of the double ramification cycle (a property proven from Pixton's formula). The algebra of quasi-modular forms, QMod, is generated by the Eisenstein series $E_2(q), E_4(q), E_6(q)$.

A fundamental open question concerns the parallel claim for a moving elliptic target. Let

$$\pi : \mathcal{E} \to \overline{M}_{1,1}$$

be the universal family of stable curves of genus 1 with 1 marked point.

Question 1: Does quasi-modularity hold in the moving case:

$$\sum_{d=0}^{\infty} q^d [\overline{M}_{g,n}(\pi,d)]^{vir} \ \in \ H^*(\overline{M}_{g,n}) \otimes \mathsf{QMod}\ ?$$

Calculations by Carl Lian in genus 2 and 3 appear to suggest that an affirmative answer is not impossible.

Part II: Quantum cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ and Hodge integrals.

The GW/DT and Crepant Resolution correspondences connect a web of mathematical theories (by the work of many authors). The genus 0 quantum cohomology of $\mathsf{Hilb}(\mathsf{C}^2)$ was calculated by Okounkov and myself almost 20 years ago. The genus 1 quantum cohomology of $\mathsf{Hilb}(\mathsf{C}^2)$ precisely yields results about Hodge integrals over the moduli spaces of maps to a moving elliptic curve. A simple nontrivial example (from more recent joint work with H.-H.Tseng) is

$$\sum_{n=1}^{\infty} \frac{u^{2n-1}}{(2n-1)!} \int_{\mathsf{Adm}_1^{n+1}((2)^{2n})_{d=2}} \lambda_{n+1}\lambda_{n-1} \ = \ \frac{i}{24} \cdot \frac{1-e^{iu}}{1+e^{iu}}\,.$$

Here, $\mathsf{Adm}_1^{n+1}((2)^{2n})_{d=2}$ is the moduli of admissible covers of degree 2 of a genus 1 curve with $2n$ simple branch points (so the domain has genus $n+1$).

Question 2: Can a closed form be found for the parallel integral over $\mathsf{Adm}_1^{n+1}((2)^{2n})_d$ for higher d? The answer is known to be a rational function in $q = -e^{iu}$, but which rational function?

Part III: The cycle theory of the Torelli map.

Consider the Torelli map from the moduli space of curves of compact type to the moduli space of principally polarized abelian varieties:

$$\mathsf{Tor} : M_g^{ct} \to A_g \,.$$

The Jacobians of curves which admit a d-cover of an elliptic curve define a Noether-Lefschetz locus

$$\mathsf{NL}_d \subset A_g$$

of codimension $g-1$. In joint work with Canning and Oprea in 2023, we studied the class

$$\Delta_g = \mathsf{Tor}^*[\mathsf{NL}_1] - \frac{(-1)^{g+1}g}{6B_{2g}}\lambda_{g-1} \,.$$

For all g, Δ_g lies in the Gorenstein kernel of the tautological ring $\mathsf{R}^\star(M_g^{ct})$. The study is related to the recent verification of Pixton's conjecture for M_6^{ct} by Canning-Larson-Schmitt, which shows $\Delta_6 \neq 0$. To control the higher d cases, $\mathsf{Tor}^*[\mathsf{NL}_d]$, the answers to Questions 1 and 2 are required.

Question 3: Can we explain all of the Gorenstein kernel of $\mathsf{R}^\star(M_g^{ct})$ using the cycles $\mathsf{Tor}^*[\mathsf{NL}_d]$?

While Questions 1 and 2 are likely solvable in some way using ideas not so far ahead of where we are in the field, Question 3 would require a larger leap.

The moduli spaces of bi-elliptic curves determine non-tautological cycles on the moduli spaces of curves and Abelian varieties. I ran out of time, but I also wanted to propose a construction of non-tautological classes on the projective bundle of linear sections over the moduli of quasi-polarized $K3$ surfaces: the locus of sections which yield bi-elliptic curves. However, I do not know how to prove these cycles are non-tautological for any moduli space of $K3$ surfaces.