MFO Report / Rahul Pandharipande (Log workshop, May 2024)

Title: Hodge integrals, Abelian varieties, and the Hilbert scheme of points of the plane

The Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ is by far the most studied moduli space of varieties in algebraic geometry. The Hilbert scheme $\mathsf{Hilb}(\mathbb{C}^2, d)$ of d points of the plane $\mathbb{C}^2$ is arguably the moduli of space of sheaves with the richest known structure. Gromov-Witten theory, via the virtual class of the moduli space of stable maps, provides a system of correspondences between these moduli spaces of varieties and sheaves:

$$\overline{\mathcal{M}}_{g,n} \ \leftarrow \ [\overline{\mathcal{M}}_{g,n}(\mathsf{Hilb}(\mathbb{C}^2, d), \beta)]^{\mathrm{vir}} \ \rightarrow \mathsf{Hilb}(\mathbb{C}^2, d)^n\,.$$

The data of all these correspondences as the genus g, the marking number n, and the curve class $\beta \in H_2(\mathsf{Hilb}(\mathbb{C}^2, d))$ vary constitutes the CohFT associated to $\mathsf{Hilb}(\mathbb{C}^2, d)$ called $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2, d))$.

The study of the genus 0 part of $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2, d))$ was undertaken 20 years ago by myself and A. Okounkov [OP1]. We found that the entire genus 0 part (in other words, the quantum cohomology of $\mathsf{Hilb}(\mathbb{C}^2, d)$) is controlled by the operator of quantum multiplication by the (unique up to scale) divisor class of $\mathsf{Hilb}(\mathbb{C}^2, d)$. The main result of [OP1] is the calculation of this operator in the Fock space description by Nakajima [N] and Grojnowski [G] of the cohomology of $\mathsf{Hilb}(\mathbb{C}^2, d)$. The genus 0 study played an important role in the investigation of the GW/DT correspondence of [MNOP] for local curves [BP,OP2]

The main goal of my lecture was to show the richness of the higher genus geometry of $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2, d))$. In the past year, the picture in genus 1 has become clearer. There are several approaches to $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2, d))$ in higher genus (see [PT]) including Hodge integrals for the families Gromov-Witten theory of the universal curve over $\overline{\mathcal{M}}_{g,n}$. In the case of the genus 1 series for $\mathsf{Hilb}(\mathbb{C}^2, d)$ corresponding to a single insertion of the divisor class (parallel to the fundamental genus 0 calculation discussed above), a complete solution is obtained via the Hodge integral study. Moreover, the result is connected in an essential way to the Noether-Lefschetz theory of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g as computed recently by A. Iribar López.

The formula of the basic genus 1 series (a result in 2024 of myself with A. Iribar López and H.-H. Tseng) and is:

$$-\big\langle (2) \big\rangle_1^{\mathsf{Hilb}(\mathbb{C}^2,d)} = -\frac{1}{24}\frac{(t_1+t_2)^2}{t_1 t_2}\Big(\mathsf{Tr}_d + \sum_{k=2}^{d-1} \frac{\sigma(d-k)}{d-k}\mathsf{Tr}_k\Big)\,.$$

Here, (2) denotes the divisor class of $\mathsf{Hilb}(\mathbb{C}^2, d)$ viewed as an element in Fock space. The variables t_1 and t_2 are the standard equivariant parameters for the scaling actions on the components of $\mathbb{C}^2$. The trace of the operator of quantum multiplication by the class $-(2)$ on the cohomology of $\mathsf{Hilb}(\mathbb{C}^2, d)$ is defined to be $(t_1 + t_2) \cdot \mathsf{Tr}_d$. The function $\sigma(m)$ is the sum of the divisors of the integer m. We note that

$$\big\langle (2) \big\rangle_1^{\mathsf{Hilb}(\mathbb{C}^2,d)}, \mathsf{Tr}_2 \ldots, \mathsf{Tr}_d$$

are all q series, where q is the Novikov parameter associated to the curve classes of $\mathsf{Hilb}(\mathbb{C}^2, d)$. In fact, these are all rational functions in q.

The study of the above genus 1 formula relies also upon related results of S. Canning, F. Greer, C. Lian. D. Oprea, S. Molcho, and A. Pixton.

Bibliography

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