Morita's κ relations

Below are notes of my discussions in the past week with Oscar Randal-Williams about his further development of Morita's method.

A. Morita's relations

Morita's geometric idea involves the θ divisor on the universal Picard variety over M_g. In order to avoid Artin stack issues (which Morita and Randal-Williams confront directly), I will move the discussion to $M_{g,1}$.

Let Pic^0_g denote the universal moduli space of degree 0 line bundles over $M_{g,1}$,

$$\nu : \mathrm{Pic}^0_g \to M_{g,1} \ .$$

By using the marked point p on the curve, the universal Picard varieties of different degrees are all canonically isomorphic. On the universal curve (pulled-back from $M_{g,1}$),

$$\pi : \mathrm{C}_g \to \mathrm{Pic}^0_g \ ,$$

there exists a universal line bundle L with the properties:

(i) over the moduli point $[C, L] \in \mathrm{Pic}^0_g$, the universal line bundle L is isomorphic to L,

(ii) after pulling-back via the section determined by the marking

$$s_p : \mathrm{Pic}^0_g \to \mathrm{C}_g \ ,$$

the universal bundle is trivial

$$s_p^*(\mathrm{L}) \cong \mathcal{O} \ .$$

Properties (i) and (ii) uniquely specify L.

Morita's basic identity, after the results of Ebert and Randal-Williams [2, 5], takes the form

$$\pi_*(c_1(\mathrm{L})^2)^{g+1} = 0 \ \in H^{2g+2}(\mathrm{Pic}^0_g) \ . \tag{1}$$

My translation of the ideas of Morita, Ebert, and Randal-Williams is as follows. From the Leray spectral sequence, there is a part of $H^2(\mathrm{Pic}^0_g)$ which comes from the monodromy invariant part of H^2 of the fibers of ν. The intersection form of the curve is such a monodromy invariant element and therefore defines $\theta \in H^2(\mathrm{Pic}^0_g)$. Alternatively, $\theta \in H^2(\mathrm{Pic}^0_g)$ is characterized by

(i) the restriction of θ to the fibers of ν is the intersection form,

(ii) after pulling-back via the section determined by the 0 in the fiberwise Picard group

$$z : M_{g,1} \to \mathrm{Pic}^0_g,$$

we have $z^*(\theta) = 0 \in H^2(M_{g,1})$.

The defining properties of universal bundle L imply the relation

$$\pi_*(c_1(\mathrm{L})^2) = -2\theta \in H^2(\mathrm{Pic}^0_g) \ .$$

The restriction of $\pi_*(c_1(\mathrm{L})^2)$ to the fibers of ν recovers the negative of twice the intersection form by a standard GRR calculation, see §VIII.2 of [1]. The vanishing conditions (ii) match.

Locally on Pic^0_g over an open set of $M_{g,1}$, we can use the Gauss-Manin flat structure on H^1 of the universal curve to write canonically the 2-form representing θ in the usual way. Since only the fiber coordinates with respect to ν appear in θ, the vanishing after raising to the $g+1$ power is immediate.

Whether relation (1) holds at the level of algebraic cycles in Chow is not immediately clear.[1] The relation relies fundamentally on the flat structure of the local system of H^1 over the moduli of curves. The Mumford relation for the vanishing of the even Chern characters of the Hodge bundle relies upon the same flatness (and is algebraic by a different argument).

A useful extension of (1) is the following statement. If

$$\pi : \mathrm{C} \to M$$

is any family of nonsingular genus g curves over a base M with a line bundle $\mathcal{L}$ on C of π-relative degree 0, then

$$\pi_*(c_1(\mathcal{L})^2)^{g+1} = 0 \ \in H^{2g+2}(M) \ . \tag{2}$$

The above form of the vanishing appears in [2, 5].

[1] After I lectured in the April 2012 moduli workshop at KTH Stockholm, Farkas reminded me of Van der Geer's calculation in the Dutch intercity volume which precisely implies $\theta^{g+1} = 0$ in Chow (tensor $\mathbb{Q}$). So all the relations discussed here hold in Chow.

The derivation of (2) from (1) is simple. It is enough to prove (2) after pull-back to C,

$$\pi^*\pi_*(c_1(\mathcal{L})^2)^{g+1} = 0 \ \in H^{2g+2}(\mathrm{C}) \ . \tag{3}$$

There is a map to the moduli space of curves

$$\phi : \mathrm{C} \to M_{g,1}$$

obtained from the universal curve

$$\pi_1 : \mathrm{C}^2_M \to \mathrm{C}$$

with canonical section s. The line bundle $\mathcal{L}$ pulls-back to C^2_M, but may not be trivial along the section. The modification

$$\mathcal{L} \otimes \pi_1^* s^*(\mathcal{L}^{-1})$$

is degree 0 on the fibers of π_1 and is trivial on the section s. Thus,

$$\phi^*(\mathrm{L}) \cong \mathcal{L} \otimes \pi_1^* s^*(\mathcal{L}^{-1})$$

and the ϕ pull-back of (1) easily implies (3) using the degree 0 hypothesis.

B. How to construct relations

Randal-Williams [5] proposes the following application of (1). Let C^n_g be the n^{th} fiber product of the universal curve over M_g, and let

$$\pi : \mathrm{C}^{n+1}_g \to \mathrm{C}^n_g$$

be the universal curve. Let $\mathcal{L}$ be any line bundle on C^{n+1}_g of degree 0 on the fibers of π. We may take

$$\mathcal{L} = \beta K_{n+1} + \sum_{i=1}^{n} \alpha_i [D_{i,n+1}]$$

where and K_{n+1} is the contangent line, $D_{i,n+1}$ are the diagonals, and

$$\beta(2g-2) + \sum_{i=1} \alpha_i = 0 \ .$$

Then, using maps to Pic^0_g and the main relation (1), we conclude

$$\pi_*(c_1(\mathcal{L})^2)^{g+1} = 0 \ \in H^{2g+2}(\mathrm{C}^n_g) \ .$$

We can cut further and push-down to M_g to obtain κ relations.

B. Example in Wick form

Randal-Williams and I studied some particular cases rather carefully. By using Wick form and further tricks, computations in genus 24 were possible. Recall, relations in $R^{12}(M_{24})$ are particularly interesting since the known methods provide, by Carel Faber's calculations, only 40 independent relations (while 41 are necessary for the Gorenstein condition to hold).

For g, we consider $g-1$ marked points. On the universal curve C_g^g over C_g^{g-1}, take the line bundle

$$\mathcal{L} = K_g - 2\sum_{i=1}^{g-1}[D_{i,g}] \ .$$

The push-down of the square is

$$\pi_*(c_1(\mathcal{L})^2) = \kappa_1 + 8\sum_{i<j}[D_{ij}] - 8\sum_{i=1}^{g-1} K_i$$

on C_g^{g-1}. Our choice was made so D_{ij} and $-K_i$ have the same coefficient.

A codimension 12 relation in genus 24 is obtained by

$$\mu_*\left(\left(\kappa_1 + 8\sum_{i<j}[D_{ij}] - 8\sum_{i=1}^{g-1} K_i\right)^{35}\right) = 0 \tag{4}$$

where $\mu : \mathrm{C}_{24}^{23} \to M_{24}$ is the forgetful map. The main point is calculating the push-forward

$$\mu_*\left(\left(\sum_{i<j}[D_{ij}] - \sum_{i=1}^{g-1} K_i\right)^{r}\right) = 0 \tag{5}$$

for all r. Since κ_1 is pulled-back from M_{24}, the relation (4) is determined by (5) for $23 \le r \le 35$.

The Wick formalism exactly executes the latter computations. The answer is developed for all genus simultaneously. Let

$$D = \sum_{i<j}[D_{ij}] - \sum_{i=1}^{n} K_i$$

be the divisor on C_g^n. We will calculate the class

$$Z(r,n) = \mu_*(D^r) \ \in R^{r-n}(M_g)$$

where $\mu : \mathrm{C}_g^n \to M_g$.

The first step is to consider the generating series

$$F = \sum_{n>0} \sum_{r>=0} Z(r,n) \frac{t^r}{r!} \frac{x^n}{n!} \ .$$

We will extract the connected graph contribution. In the expansion of D^r, if a monominal containing the D_{ij} connects all the n factors, then such a term contributes exactly $(-1)^{r-n+1}\kappa_{r-n}$ to the class $Z(r,n)$.

Following the notation of my article [3] on κ classes for the moduli of curves of compact type (where the Wick formalism is explained in more detail), let $C(r,n)$ be the summand of D^r on C_g^n corresponding to connected monomial (after the specialization $K_i = -1$ and $D_{ij} = 1$). We see

$$\sum_{n>0} \sum_{r>=0} C(r,n) \frac{t^r}{r!} \frac{x^n}{n!}$$

$$= \log \left(1 + \sum_{n>0} \sum_{r>=0} (n(n-1)/2 + n)^r \frac{t^r}{r!} \frac{x^n}{n!} \right)$$

$$= \log \left(1 + \sum_{n>0} \sum_{r>=0} \left(\frac{n(n+1)}{2} \right)^r \frac{t^r}{r!} \frac{x^n}{n!} \right) \ .$$

To calculate the original F, we can exponentiate

$$F = \exp \left(\sum_{n>0} \sum_{r>=0} (-1)^{r-n+1} \kappa_{r-n} C(r,n) \frac{t^r}{r!} \frac{x^n}{n!} \right) \ .$$

We have calculated explicitly the following three nontrivial relations in $R^{12}(M_{24})$ pushed-down from C_{24}^{23}:

$$\mu_* \left(\left(\kappa_1 + 8 \sum_{i<j} [D_{ij}] - 8 \sum_{i=1}^{g-1} K_i \right)^{35} \right) \ = \ 0 \ ,$$

$$\mu_* \left(\left(\kappa_1 + 8 \sum_{i<j} [D_{ij}] - 8 \sum_{i=1}^{g-1} K_i \right)^{34} \cdot [D_{12}] \right) \ = \ 0 \ ,$$

$$\mu_* \left(\left(\kappa_1 + 8 \sum_{i<j} [D_{ij}] - 8 \sum_{i=1}^{g-1} K_i \right)^{33} \cdot [D_{12}]^2 \right) \ = \ 0 \ .$$

We have sent all three relations to Carel Faber. He has checked that all three lie in the span of the 40 known κ relations in $R^{12}(M_{24})$.

Based on the evidence above, it is reasonable to guess the entire output of the method discussed here lies in the span of the FZ relations (see [4]). I would further guess the result, in fact, equals the span of FZ but that prediction is probably premature.

Rahul 8/10/2011

References

[1] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves*, Springer 1984.

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[3] R. Pandharipande, *The κ ring of the moduli of curves of compact type*, Acta Math. (to appear).

[4] R. Pandharipande and A. Pixton, *Relations in the tautological ring*, arXiv:1101.2236.

[5] O. Randal-Williams, *Relations among tautological classes revisited*, arXiv:1012.1430.