Hilbert scheme of infinite affine space

Thanks, Rahul!

j.w. Marc, Joachim, Denis, Burt

Grothendieck: Hilb parametrizes points of a given scheme.

S-base, X - S-scheme, $\mathrm{Hilb}(X/S)(T) = \left\{ \begin{array}{c} Z \hookrightarrow X_T \\ p \searrow \ \swarrow \\ T \end{array} \ p \text{ finite flat (fin. presented)} \right\}$

This functor of pts is represented by a scheme, at least when X qproj.

5 $\mathrm{Hilb}(X) = \coprod_{d \geq 0} \mathrm{Hilb}_d(X)$, d = degree of p.

Usually: X smooth surface $\rightsquigarrow$ rich subject (Göttsche, Nakajima)

many invariants are computed

- $\mathrm{Hilb}(X)$ is smooth
- $\mathrm{Hilb}_d(X) \to \mathrm{Sym}_d(X) = X^d/\Sigma_d$ is resolution of singularities.
- X K3-surface $\Rightarrow \mathrm{Hilb}_d(X)$ is a hyperkähler manifold.

We're interested in $\mathrm{Hilb}(\mathbb{A}^n)$.

- $\mathrm{Hilb}_d(\mathbb{A}^2)$, $\mathrm{Hilb}_3(\mathbb{A}^n)$ are smooth schemes with nice stratifications, induced by a generic $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ and its "attraction sets".

  reembed 3 pts into $\mathbb{A}^2$

5 • But $d, n \uparrow \rightsquigarrow$ things quickly become a disaster

Joachim: $\mathrm{Hilb}(\mathbb{A}^{16})$ satisfies Murphy's law, i.e. has ALL types of singularities (up to retraction).

Since geometry is so complicated, ask about hom. type?

Let $\mathrm{Hilb}(\mathbb{A}^\infty) := \mathrm{colim}_n \mathrm{Hilb}(\mathbb{A}^n)$ along $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$.

ind-scheme

concentrated in deg 0

Thm A ($d\geq 1$) $H^*(\mathrm{Hilb}_d(\mathbb{A}^\infty_{\mathbb{C}}), \mathbb{Z}) \simeq \mathbb{Z}[c_1,\dots,c_{d-1}]$, $|c_i|=2i$.

5 More generally, A^* – ℓ-adic coh. / KH / MGL / de Rham in char 0 / ...,
then $A^{(*)}(\mathrm{Hilb}_d(\mathbb{A}^\infty_S)) \simeq A^*(S)[[c_1,\dots,c_{d-1}]]$. (can apply $\mathrm{Re}_{\mathbb{C}}$)
$(2*,*)$ formal power series

These A^* are cohomology theories on S-schemes, which are determined by values on affines (more generally, satisfy descent ①), and, crucially, are $\mathbb{A}^1$-invariant ②, i.e. $\forall\, Y\in \mathrm{Sch}_S$ $A^*(Y) \xrightarrow[\sim]{p^*} A^*(\mathbb{A}^1\times Y)$, and are oriented ③ (have Thom isoms), and RHS computes $A^*(\mathrm{Gr}_{d-1}(\mathbb{A}^\infty))$.

Motivic htpy thy (homotopy thy for schemes) studies cohomological invariants of schemes
10 that satisfy ① and ②. universal ways to impose ① and ②: motivic homotopy types

$$\begin{tikzcd}
\mathrm{Sch}_S \arrow[r, hook, "\text{Yoneda}"] & P(\mathrm{Sm}_S) \arrow[r, "\text{sheaf.}"] \arrow[d, shift left, "L_{\mathbb{A}^1}"] \arrow[rr] & \mathrm{Shv}(\mathrm{Sm}_S) \arrow[r] & \mathrm{Shv}_{\mathbb{A}^1}(\mathrm{Sm}_S) \arrow[ll, bend right, "L_{mot}"'] \\
& P_{\mathbb{A}^1}(\mathrm{Sm}_S) \arrow[u, shift left] \arrow[urr] & &
\end{tikzcd}$$

complicated

$P(\mathrm{Sm}_S) \ni F$

$$L_{\mathbb{A}^1}F(Y) = \operatorname{colim}_{n\in\Delta^{op}} F(Y\times\Delta^n) \qquad \Delta^n = \mathrm{Spec}\, k[t_0,\dots,t_n]/_{\sum t_i - 1} \simeq \mathbb{A}^n$$

$$\pi_0 L_{\mathbb{A}^1}F(Y) = \mathrm{coeq}\left(F(Y\times\mathbb{A}^1) \rightrightarrows F(Y)\right)$$

Two maps $F \overset{\varphi}{\underset{\psi}{\rightrightarrows}} G$ is an $\mathbb{A}^1$-homotopic if $L_{\mathbb{A}^1}\varphi \simeq L_{\mathbb{A}^1}\psi$.

Example: $H: \mathbb{A}^1\times F \to G$ s.t. $H|_0 = \varphi$, $H|_1 = \psi$ gives an $\mathbb{A}^1$-homotopy

$\varphi: F\to G$ is an $\mathbb{A}^1$-equivalence if $\exists\, F \underset{\psi}{\overset{\varphi}{\rightleftarrows}} G$ st. $\varphi\psi \simeq_{\mathbb{A}^1} id_G$, $\psi\varphi \simeq_{\mathbb{A}^1} id_F$.

So, by the properties of A^* our main thm A follows from:

Thm B S – any base. ($d\geq 1$) Then the map of ind-schemes

$$\mathrm{Gr}_{d-1}(\mathbb{A}^\infty_S) \to \mathrm{Hilb}_d(\mathbb{A}^\infty_S),$$

5 sending a subspace of $\mathbb{A}^n$ to pt at 0 with such tangent space,

aka $[\mathcal{O}^{n} \twoheadrightarrow \mathcal{E}] \mapsto [\mathrm{Sym}_{\mathcal{O}}(\mathcal{O}^{n}) \twoheadrightarrow \mathcal{O} \oplus \mathcal{E}]$
is an $\mathbb{A}^1$-equivalence (on affines) square zero ext. of $\mathcal{O}$ by $\mathcal{E}$
avoid writing! in the future

Rem. $\underset{\text{ind-scheme}}{\mathrm{Hilb}_d(\mathbb{A}^\infty_S)} \subset \underset{\text{scheme}}{\mathrm{Hilb}_d(\mathbb{A}^N_S)}$ is an $\mathbb{A}^1$-equiv.

5 **Rem.** $\mathrm{Sym}_d(\mathbb{A}^\infty)$ is $\mathbb{A}^1$-contractible ($\mathbb{A}^1$-equiv. to $*$), because $\mathbb{A}^1 \times \mathrm{Sym}_d(\mathbb{A}^\infty) \to \mathrm{Sym}_d(\mathbb{A}^\infty)$ contracts everything to 0, but $\mathbb{A}^1 \times \mathrm{Hilb}_d(\mathbb{A}^\infty) \dashrightarrow \mathrm{Hilb}_d(\mathbb{A}^\infty)$ is not well-defined.

40-45

Proof of Thm. B Consider moduli stacks FFlat_d and Vect_d (presheaves of groupoids) $\in P(\mathrm{Sm}_S)$.

- The forgetful maps $\mathrm{Hilb}_d(\mathbb{A}^\infty) \to \mathrm{FFlat}_d$ and $\mathrm{Gr}_d(\mathbb{A}^\infty) \to \mathrm{Vect}_d$ are $\mathbb{A}^1$-equivalences on affines:
5 the spaces of choices of embeddings into $\mathbb{A}^\infty$ and sections of a v.b. are $\mathbb{A}^1$-contractible, when evaluated on affine schemes. because we can extend along $\partial\Delta^i \hookrightarrow \Delta^i$

Hence suffices: $\mathrm{Vect}_{d-1} \xrightarrow{} \mathrm{FFlat}_d$ is an $\mathbb{A}^1$-equivalence
square zero extension $\mathcal{E} \mapsto \mathcal{O} \oplus \mathcal{E}$

- The $\mathbb{A}^1$-inverse $\mathrm{FFlat}_d \to \mathrm{Vect}_{d-1}$ is $A \mapsto A/\mathcal{O}$.

Need to show: $\mathrm{FFlat}_d \to \mathrm{Vect}_{d-1} \to \mathrm{FFlat}_d$ is $\mathbb{A}^1$-equiv. to $\mathrm{id}_{\mathrm{FFlat}_d}$.
$A \longmapsto A/\mathcal{O} \oplus \mathcal{O}$

We'll define an explicit homotopy along $\mathbb{A}^1$.

Rees algebra. k-ring, A - k-algebra with increasing filtration $0 \subset A_0 \subset A_1 \subset \dots$
$1 \in A_0$, $A_i \cdot A_j \subset A_{i+j}$, $A = \bigcup A_i$. Then
10 $R(A) := \bigoplus \underbrace{A_i \cdot t^i}_{\deg i} \subset A[t]$ is graded $k[t]$-algebra with (grading comes from $A[t]$, $\deg t$)

$R(A)/(t) \simeq gr(A)$ and $R(A)/(t-1) \simeq A$. deg 1

If all $gr_i A := A_i/A_{i-1}$ are flat over k, so is $R(A)$ over $k[t]$.

We apply this to the canonical filtration $\underset{A_0}{k\cdot 1} \subset \underset{A_1}{A}$ of any $A \in FFlat_d(k)$, $d \geq 1$.

We get $R(A)/(t) \simeq k \oplus \underbrace{A/k}_{\text{zero product here}}$; $R(A)/(t-1) \simeq A$.

So the map $A \mapsto R(A)$ is natural and extends to

$$\mathbb{A}^1 \times FFlat_d \to FFlat_d$$

(aka $FFlat_d(k) \to FFlat_d(\mathbb{A}^1_k)$) $\mathrm{Maps}(\mathbb{A}^1, FFlat_d)(k)$

which provides the $\mathbb{A}^1$-htpy between $A \mapsto A/\!\!/_{\mathcal{O}} \oplus \mathcal{O}$ and id_A. ∎

~1 hr

Application to K-thy.

X qproj. S-scheme $\Rightarrow$ $K(X) \simeq Vect(X)^{gp}$ ← group completion wrt $\oplus$ of v.b., make π_0 a group

in general, $K \simeq L_{zar} Vect^{gp}$

$Vect^{gp} = Vect[-\mathcal{O}] \underset{\mathbb{A}^1\text{-equiv. on affines}}{\simeq} \mathbb{Z} \times Vect_\infty$, where $Vect_\infty = \mathrm{colim}(Vect_0 \xrightarrow{+\mathcal{O}} Vect_1 \to \dots)$

5 +construction invisible ⇝

This is one of the features of motivic htpy thy: explicit constructions for K-thy as a motivic space.

Similarly, $FFlat^{gp} \simeq \mathbb{Z} \times FFlat_\infty$ $\mathbb{A}^1$-equiv. on affines.
(gp wrt ⊔)

Thm C The forgetful map induces

5 $FFlat^{gp} \to Vect^{gp}$ $\mathbb{A}^1$-equiv. (on affines) of presheaves of $\mathcal{E}_\infty$-rings.

Rmk: uses Thm B. Enough: $FFlat_\infty \underset{\mathbb{A}^1}{\overset{\sim}{\to}} Vect_\infty$ colim $Hilb_d(\mathbb{A}^n)$

Cor 1 ∎ $K \underset{\mathbb{A}^1}{\simeq} FFlat^{gp} \underset{\mathbb{A}^1}{\simeq} \mathbb{Z} \times FFlat_\infty \underset{\mathbb{A}^1}{\simeq} \mathbb{Z} \times Hilb_\infty(\mathbb{A}^\infty)$,

and the canonical map $MGL \to KGL$ induces

5 $$\Omega^\infty_{\mathbb{P}^1} MGL \to \Omega^\infty_{\mathbb{P}^1} KGL = K$$

up to + [BEHKSY] $\to$

$$\begin{array}{ccc} \Omega^\infty_{\mathbb{P}^1} MGL & \to & \Omega^\infty_{\mathbb{P}^1} KGL \\ \wr\downarrow & & \wr \\ \mathbb{Z}\times \underbrace{\mathrm{Hilb}^{lci}_\infty(\mathbb{A}^\infty)} & \to & \mathbb{Z}\times \mathrm{Hilb}_\infty(\mathbb{A}^\infty) \end{array}$$

its motivic homotopy type (motive) would be important to understand! That was our motivation

Cor 2. K-theory is the universal motivic invariant

5 among those who have pushforwards wrt finite flat maps of schemes (FFlat obviously does).

Analogously, alg. cobordism is universal wrt finite flat lci transfers.

Natural question: what about hermitian K-thy?

S = field of char $\neq 2$.

$$\begin{array}{ccc} GW & \simeq & (\mathrm{Vect}^{Bil})^{gp} \\ \downarrow & & \downarrow \\ K & \simeq & \mathrm{Vect}^{gp} \end{array}$$

Vect^{Bil}: = v.b. with non-deg. symm. bil. forms

5 In order to obtain hermitian analogues of Corollaries 1 and 2 we prove

Thm D. $(\mathrm{FGor}^{or})^{gp} \xrightarrow[\mathbb{A}^1]{\sim} (\mathrm{Vect}^{Bil})^{gp}$

where FGor^{or} is the stack of (finite flat) oriented Gorenstein algebras (their families).

This means the dualizing module $\omega_{A/k} = \mathrm{Hom}_k(A, k)$ of a finite flat k-algebra A is an invertible A-module (Gorenstein), and it's given a trivialization (orientation).

So a trivialization is a k-linear $\varphi: A \to k$,

5 s.t. $B_\varphi(x,y) = \varphi(x \cdot y)$ is a non-deg. symm. bil. form,

that's why there's a forgetful map $FGor^{or} \to Vect^{bil}$, analogous to $FFlat \to Vect$.

Ex: $k[x]/x^2$, $\varphi(ax+b) = a$, $B_\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$