Speculations on Hodge integrals related to moving elliptic targets

R. Pandharipande

May 2024

Based in part on discussions with

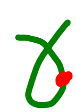
S. Canning
F. Greer
A. Iribar López
C. Lian
S. Molcho
D. Oprea
A. Pixton
H.-H. Tseng

My goal here is to present results and speculations concerning certain families of descendent Hodge integrals.

Let $\overline{M}_{1,1}$ be the moduli of stable pointed elliptic curves:

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ \overline{M}_{1,1} \arrow[u, bend right, "q"'] \end{tikzcd}$$

zero section

The fiber of π over [] is nodal.

We view π as a family of log targets.

Let

$$\begin{tikzcd}
 & \overline{\mathcal{M}}_{g,1}(\pi, d) \arrow[dl, "\varepsilon"'] \arrow[d, "\mu"] \arrow[r, "ev"] & \mathcal{E} \arrow[dl, "\pi"] \\
\overline{\mathcal{M}}_{g,1} & \overline{\mathcal{M}}_{1,1} &
\end{tikzcd}$$

be the Grothendieck π-relative space of stable maps to the fibers of π.

There is a virtual class for the families stable map space

$$\dim \left[\overline{\mathcal{M}}_{g,1}(\pi, d) \right]^{vir} = 2g-2+1+1 = 2g.$$

maps to an elliptic fiber

domain marking

$\dim \overline{\mathcal{M}}_{1,1}$

I consider here Hodge integrals of the following form:

$$\left\langle \tau_k(q)\, P_{2g-k-1}(\lambda) \right\rangle_{g,d}^{\pi}$$

$$\|$$

$$\int_{[\overline{M}_{g,1}(\pi,d)]^{vir}} \psi_1^k \cdot ev_1^*(q) \cdot P_{2g-k-1}(\lambda) .$$

$g \geq 1$ since $\overline{M}_{0,1}(\pi, d)$ is empty in genus 0.

$d \geq 0$

polynomial of degree $2g-k-1$ in $\lambda_1, \dots, \lambda_g$, $\lambda_i = c_i(\mathbb{E})$

Hodge bundle

• Constant maps $(d=0)$:

$$\left\langle \tau_k(q)\, P_{2g-k-1}(\lambda) \right\rangle^{\pi}_{g,0}$$

$$\parallel$$

$$\int_{\overline{M}_{g,1}} \psi_1^{k} \cdot P_{2g-k-1}(\lambda) \cdot \left(-\frac{1}{24} (-1)^{g-1} \lambda_{g-1} \right),$$

because we have

$$\overline{M}_{g,1}(\pi, 0) \supset ev_1^{-1}(q) \cong \overline{M}_{g,1} \times \overline{M}_{1,1}$$

with virtual class

tangent line on $\overline{M}_{1,1}$

$$\left[\overline{M}_{g,1} \times \overline{M}_{1,1} \right]^{vir} = \mathbb{E}^{\vee} \boxtimes \mathrm{Tan}_1$$

$$= (-1)^{g} \lambda_g - \psi_1 (-1)^{g-1} \lambda_{g-1}$$

- Case of the integrand

$$\tau_1(q) \cdot c(\mathbb{E}^\vee \otimes t_1)\, c(\mathbb{E}^\vee \otimes t_2) \,/\, t_1 t_2$$

t_1, t_2 variables

The integral is

$$\left\langle \tau_1(q)\, P_{2g-2}(\lambda) \right\rangle^{\pi}_{g,d}$$

where $P_{2g-2}(\lambda)$ is the degree $2g-2$ part,

$$P_{2g-2}(\lambda) = \left[\frac{c(\mathbb{E}^\vee \otimes t_1)\, c(\mathbb{E}^\vee \otimes t_2)}{t_1 t_2} \right]_{2g-2},$$

$$\left[\frac{c(\mathbb{E}^\vee \otimes t_1)\, c(\mathbb{E}^\vee \otimes t_2)}{t_1 t_2} \right]_{2g-2} ,$$

$$\parallel$$

$$\left[\begin{array}{l} \left((-1)^g \lambda_g + (-1)^{g-1} \lambda_{g-1} t_1 + (-1)^{g-2} \lambda_{g-2} t_1^2 \right) \cdot \\ \left((-1)^g \lambda_g + (-1)^{g-1} \lambda_{g-1} t_2 + (-1)^{g-2} \lambda_{g-2} t_2^2 \right) \end{array} \right]_{2g-2} \frac{1}{t_1 t_2}$$

$$\parallel$$

$$\frac{(t_1 + t_2)^2}{t_1 t_2} \lambda_{g-2} \lambda_g .$$

In the last equality, we have used Mumford's relation:

$$\lambda_{g-1}^2 = 2 \lambda_{g-2} \lambda_g .$$

Theorem (Iribar López - P - Tseng 2024):

$$\sum_{d\geq 0} Q^d \langle \tau_1(q)\, \lambda_{g-2}\lambda_g \rangle^{\pi}_{g,d} = \frac{1}{24}\, |B_{2g-2}| \cdot \binom{2g}{2}\, C_{2g}(Q)$$

Definitions:

$$C_{2g}(Q) = -\frac{B_{2g}}{2g \cdot 2g!} + \frac{2}{2g!} \sum_{n\geq 1} \sigma_{2g-1}(n)\, Q^n,$$

Sum of $2g-1^{\text{th}}$ powers of divisors of n

$$C_{2g}(Q) = -\frac{B_{2g}}{2g \cdot 2g!}\, E_{2g}(Q).$$

Eisenstein Series

The series sits at the center of various correspondences

NL cycle theory of A_g

Families Hodge integrals for π

$$\left\langle \tau_1(q)\, \lambda_{g-2}\lambda_g \right\rangle^{\pi}_{g,d}$$

Genus 1 GW invariants of $\mathrm{Hilb}(\mathbb{C}^2, d)$

See https://people.math.ethz.ch/~rahul/Leiden.pdf

Genus 1 GW invariants of $Hilb(\mathbb{C}^2, d)$:

Calculation of $\langle \tau_1(q)\, \lambda_{g-2}\lambda_g \rangle^{\pi}_{g,d}$ takes the form $(d \geq 1)$:

Conjecture (H.-H. Tseng - P 2023)

$$- \langle (2) \rangle_1^{Hilb(\mathbb{C}^2, d)} =$$

$$-\frac{1}{24}\,\frac{(t_1+t_2)^2}{t_1 t_2}\left(Tr_d + \sum_{k=2}^{d-1} \frac{\sigma(d-k)}{d-k}\, Tr_k \right).$$

In particular, the calculation of $\langle \tau_1(q)\, \lambda_{g-2}\lambda_g \rangle^{\pi}_{g,d}$ proves the above Conjecture.

Definitions:

Let $D = c_1(\mathcal{O}/_I) \in H^2(\mathrm{Hilb}(\mathbb{C}^2, k))$.

Let $M_{D,k}$ be the operator of quantum multiplication | $D = -(2)$

$$M_{D,k} = D \ast : H^*(\mathrm{Hilb}(\mathbb{C}^2,k)) \to H^*(\mathrm{Hilb}(\mathbb{C}^2,k)).$$

computed explicitly by Okounkov-P (2010)

Let $\mathrm{Tr}_k = \dfrac{1}{t_1+t_2} \,\mathrm{Trace}\left(M_{D,k}\right)$,

$$M_D = (t_1+t_2)\sum_r \left(\frac{r}{2}\,\frac{(-q)^r+1}{(-q)^r-1} - \frac{1}{2}\,\frac{(-q)+1}{(-q)-1}\right) \alpha_{-r}\alpha_r$$

+ off diagonal terms.

NL cycle theory of $\mathcal{A}_g$:

Calculation of $\langle \tau_1(q)\, \lambda_{g-2}\lambda_g \rangle^{\pi}_{g,d}$ takes the form ($d \geq 1$):

Theorem (Iribar López 2024)

$$\mathrm{Pr}_A([NL_{g,d}]) = \hat{c}_{g,d} \cdot \lambda_{g-1} \in R^*(\mathcal{A}_g),$$

$$\hat{c}_{g,d} = d^{2g-1} \prod_{\substack{p \mid d \\ p \text{ prime}}} (1 - p^{-2g+2}) \cdot \frac{g}{6|B_{2g}|}$$

At the moment, the causal connection of the above calculation to $\langle \tau_1(q)\, \lambda_{g-2}\lambda_g \rangle^{\pi}_{g,d}$ has a conjectural step.

Definitions :

Let $NL_{g,d} \to A_g$ be the moduli of pairs :

Condition : $E \cdot \Theta = d$

theta divisor of X

$$NL_{g,d} \ni [E \hookrightarrow X]$$

1 dim subgroup, an elliptic curve

PPAV of dimension g

Pr_A is the projection operator

Canning Oprea P Molcho

$$Pr_A : CH^*(A_g) \to R^*(A_g).$$

- Case of the integrand

$$\tau_k(q) \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g$$

is very similiar to the

$\tau_1(q)\ \lambda_{g-2}\lambda_g$ case.

NL cycle theory of A_g $\Rightarrow$

Families Hodge integrals for π $\Rightarrow$

$$\left\langle \tau_k(q) \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g \right\rangle_{g,d}^{\pi}$$

An explicit evaluation is obtained by another transformation:

uses K3 geometry

$$\int_{[\overline{M}_{g,1}(\pi, d)]^{vir}} \tau_k(q) \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g$$

$$\parallel$$

$$\frac{1}{24} \int_{[\overline{M}_{g,1}(E, d)]^{vir}} \tau_k(q) \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_{g-1}$$

Well studied theory

Okounkov-P, Pixton.

Generating series are quasimodular forms

fixed elliptic target,

$\mathrm{vdim} = 2g-2+1 = 2g-1.$

Noether-Lefschetz geometry for $\mathcal{A}_g$

Consider the fiber product:

$$\begin{tikzcd}
\mathrm{Tor}_1^{-1}(NL_{g,d}) \arrow[r] \arrow[d] & NL_d \arrow[d] \\
\mathcal{M}_{g,1}^{ct} \arrow[r, "\mathrm{Tor}_1"] & \mathcal{A}_g
\end{tikzcd}$$

$\mathrm{Tor}_1^{-1}(NL_{g,d})$ has a direct interpretation from the point of view of Gromov-Witten theory.

(A) stable maps

Let

$$\begin{tikzcd} & \mathcal{M}^{ct}_{g,1}(\pi, d) \arrow[dl, "\varepsilon"'] \arrow[d, "\mu"] \arrow[r, "ev"] & \mathcal{E} \arrow[dl, "\pi^{ct}"] \\ \mathcal{M}^{ct}_{g,1} & \mathcal{M}_{1,1} & \end{tikzcd}$$

$\mathcal{M}^{ct}_{1,1} = \mathcal{M}_{1,1}$

be the restriction of the family π to compact type curves.

compact type

$$[f : (C, p) \to (E, q)] \in \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)$$

there is a discrete invariant :

$$f^* : \mathrm{Jac}_0(E) \to \mathrm{Jac}_0(C),$$

$$\mathrm{Jdeg}\, f = d \,/\, |\ker f^*| .$$

$\mathrm{Jdeg}_f \in \{1, 2, \dots, d\}$ must divide d

and is a discrete invariant of f,

$$\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d) = \coprod_{\mathrm{Jdeg}\ \hat{d}} \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^{\hat{d}}.$$

maps with $\mathrm{Jdeg}_f = \hat{d}$

There is a canonical isomorphism

$$\mathrm{Tor}_1^{-1}(NL_{g,d}) \cong ev_1^{-1}(q)^d$$

$$\cap$$

$$\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^d.$$

Locus of map where the marking maps to q

(B) Virtual classes

- $\mathrm{Tor}_1^{-1}(NL_{g,d})$ carries a virtual fundamental class from the fiber diagram:

$$\left[\mathrm{Tor}_1^{-1}(NL_{g,d})\right]^{vir} = \mathrm{Tor}_1^{!}\,\left[NL_{g,d}\right]$$

- $ev_1^{-1}(q)^d \subset \mathcal{M}_{g,1}^{ct}(\pi^{ct}, d)^d$

Carries a GW virtual class

$$\left[ev_1^{-1}(q)^d\right]^{vir}$$

from the deformation theory of maps.

Theorem (Greer-Lian 2024)

Via the isomorphism

$$\mathrm{Tor}_1^{-1}(NL_{g,d}) \cong ev_1^{-1}(q)^d \subset M^{ct}_{g,1}(\pi^{ct}, d)^d,$$

We have $\left[\mathrm{Tor}_1^{-1}(NL_{g,d})\right]^{vir} = \left[ev_1^{-1}(q)^d\right]^{vir}$.

$$\int_{[\overline{M}_{g,1}(\pi,d)]^{vir}} \tau_k(q) \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g$$

$$\|$$

$$\int_{\overline{M}_{g,1}} \varepsilon_*\left[ev_1^{-1}(q)\right]^{vir} \cdot \psi_1^k \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g ,$$

Cotangent lines on $\overline{M}_{g,1}(\pi, d)$ and $\overline{M}_{g,1}$ agree since there are no maps of positive degree $\mathbb{P}^1 \to E$ or $\gamma_{\log}$.

$$\int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_*\left[ev_1^{-1}(q)\right]^{vir} \cdot \psi_1^k \cdot \widehat{P}_{g-k-1}(\lambda) \cdot \lambda_g ,$$

$$\|$$

Choice of closure over $\partial \mathcal{M}_{g,1}^{ct}$ does not affect the integral

$$\sum_{\hat{d} \mid d} \sigma\left(\frac{d}{\hat{d}}\right) \cdot \int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_* \left[\overline{ev^{-1}(q)^{\hat{d}} \cdot \psi_1^k}\right]^{vir} \cdot \widehat{P}_{g-k-1}(\lambda) \cdot \lambda_g$$

Count of $(\hat{E}, \hat{q}) \xrightarrow{\deg\left(\frac{d}{\hat{d}}\right)} (E, q)$,

vanishes on $\partial \mathcal{M}_{g,1}^{ct}$

$$\sigma(x) = \sum_{\ell \mid x} \ell$$

$$\int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_* \overline{\left[ev^{-1}(q)^d \cdot \psi_1^k \right]}^{vir} \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g$$

$$||$$

$$\int_{\overline{\mathcal{M}}_{g,1}} \overline{Tor_1^*\left([NL_{g,d}] \right)} \cdot \psi_1^k \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g$$

$$||$$

using projection operator and a homomorphism property

$$\int_{\overline{\mathcal{M}}_{g,1}} \hat{C}_{g,d} \cdot \lambda_{g-1} \cdot \psi_1^k \cdot \hat{P}_{g-k-1}(\lambda) \cdot \lambda_g$$

Conclusion: We have proven equalities

$$\int_{[\overline{\mathcal{M}}_{g,1}(\pi,d)]^{vir}} \tau_k(q)\cdot \hat{P}_{g-k-1}(\lambda)\cdot \lambda_g$$

$$\|$$

$$\sum_{\hat{d}\mid d} \sigma\left(\frac{d}{\hat{d}}\right)\cdot \hat{C}_{g,d}\cdot \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^k\, \hat{P}_{g-k-1}(\lambda)\cdot \lambda_g\lambda_{g-1}$$

$$\|$$

$$\frac{1}{24}\int_{[\overline{\mathcal{M}}_{g,1}(E,d)]^{vir}} \tau_k(q)\cdot \hat{P}_{g-k-1}(\lambda)\cdot \lambda_{g-1}\ .$$

- Speculations in case of the integrand

$T_k(q) \cdot P_{2g-k-1}(\lambda)$ λ_g now not necessarily present!

Consider the Noether-Lefshetz diagram for the perfect cone compactification

Studied by Iribar López

$$\begin{tikzcd}
\mathrm{Tor}_1^{-1}(\overline{NL}_{g,d}) \arrow[r] \arrow[d] & \overline{NL}_{g,d} \arrow[d] \\
\overline{\mathcal{M}}_{g,1} \arrow[r, "\overline{\mathrm{Tor}_1}"] & \overline{\mathcal{A}}_g
\end{tikzcd}$$

extended by Alexeev

Perfect cone

We can also consider other compactifications for which Tor_1 extends.

There is a commutative diagram for tautological projection operators:

see Canning-Molcho-Oprea, P

$$\begin{tikzcd} Pr_{\bar{A}} : CH^*(\bar{A}_g) \arrow[r] \arrow[d] & R^*(\bar{A}_g) \arrow[d] \\ Pr_{A} : CH^*(A_g) \arrow[r] & R^*(A_g). \end{tikzcd}$$

Since $R^{g-1}(\bar{A}_g) \cong R^{g-1}(A_g)$, the result of Iribar López yields:

$$Pr_{\bar{A}}([\overline{NL}_{g,d}]) = \hat{C}_{g,d} \cdot \lambda_{g-1} \in R^*(\bar{A}_g),$$

$$\hat{C}_{g,d} = d^{2g-1} \prod_{\substack{p \mid d \\ p \text{ prime}}} (1-p^{-2g+2}) \cdot \frac{g}{6|B_{2g}|}.$$

The basic question here is :

$$\int_{[\overline{\mathcal{M}}_{g,1}(\pi,d)]^{vir}} \tau_k(q) \cdot P_{2g-k-1}(\lambda)$$

$$\| \; ?$$

$$\sum_{\hat{d} \mid d} \sigma\left(\frac{d}{\hat{d}}\right) \cdot \hat{c}_{g,\hat{d}} \cdot \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^k \, P_{2g-k-1}(\lambda) \cdot \lambda_{g-1}$$

When $\lambda_g \mid P_{2g-k-1}(\lambda)$, we have seen that the equality holds by studying the geometry of $NL_{g,d}$.

For general $P_{2g-k-1}(\lambda)$, further geometric compatibility over the boundry $\bar{A}_g \setminus A_g$ is required.

log geometry, log stable maps

If the following 4 properties hold

- $\mathrm{Tor}_1^{-1}(\overline{NL}_{g,\hat{d}}) \to \overline{\mathcal{M}}_{1,1}$,
- $\phi_{\hat{d}} : \mathrm{Tor}_1^{-1}(\overline{NL}_{g,\hat{d}}) \times_{\overline{\mathcal{M}}_{1,1}} \mathrm{Hur}_1(d/\hat{d})$

$$\downarrow$$

$$ev_1^{-1}(q) \subset \overline{\mathcal{M}}_{g,1}(\pi, d),$$

where $\mathrm{Hur}_1(d/\hat{d})$ is the compact Hurwitz space of $\deg(d/\hat{d})$ covers

map to $\overline{\mathcal{M}}_{1,1}$ from domain

$$(\hat{E}, \hat{q}) \to (E, q).$$

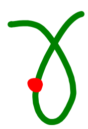
- $$\sum_{\hat{d} | d} \phi_{\hat{d}\,*} \left[\mathrm{Tor}_1^{-1}(\overline{NL}_{g,\hat{d}}) \right]^{vir} = \left[ev_1^{-1}(q) \right]^{vir},$$

- $$Pr_{\bar{A}}\left(\overline{\mathrm{Tor}}_{1*}[\bar{\mathcal{M}}_{g,1}] \cdot [\overline{NL}_{g,d}] \right) = Pr_{\bar{A}}\left(\overline{\mathrm{Tor}}_{1*}[\bar{\mathcal{M}}_{g,1}] \right) \cdot Pr_{\bar{A}}\left([\overline{NL}_{g,d}] \right),$$

then we obtain

$$\int_{[\bar{\mathcal{M}}_{g,1}(\pi,d)]^{vir}} \tau_k(q) \cdot P_{2g-k-1}(\lambda) = \sum_{\hat{d}|d} \sigma\left(\frac{d}{\hat{d}}\right) \cdot \hat{c}_{g,\hat{d}} \cdot \int_{\bar{\mathcal{M}}_{g,1}} \psi_1^k \, P_{2g-k-1}(\lambda) \cdot \lambda_{g-1}$$

for all polynomials $P_{2g-k-1}(\lambda)$.

These 4 properties hold when the geometry is restricted to $\mathcal{M}_{g,1}^{ct}$ and A_g, but there certainly could be corrections related to the Gromov-Witten theory of  .

Greer and Lian speculate:

It is possible the NL geometry is still very well behaved over the partial compactification of A_g corresponding to rank 1 degenerations.

Then, we would obtain

$$\int_{[\overline{\mathcal{M}}_{g,1}(\pi,d)]^{vir}} \tau_k(q)\cdot P_{2g-k-1}(\lambda) = \sum_{\hat{d}\,|\,d} \sigma\left(\frac{d}{\hat{d}}\right)\cdot \hat{C}_{g,\hat{d}}\cdot \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^k\, P_{2g-k-1}(\lambda)\cdot \lambda_{g-1}$$

for polynomials $P_{2g-k-1}(\lambda)$ which satisfy $\lambda_{g-1} \mid P_{2g-k-1}(\lambda)$.

The investigation of the geometric claims in rank 1 and the λ_{g-1} formula is an important direction.

A final remark:

for calculation of further GW series for $\mathrm{Hilb}(\mathbb{C}^2, d)$, essentially all quadratic polynomials

$$p(\lambda) = \lambda_i \lambda_j$$

are required.

Another requirement (to be taken up later) is the inclusion of more complicated descendent insertions.

The End

5 May 2024