Virasoro constraints for target curves

A. Okounkov and R. Pandharipande

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Abstract

We prove generalized Virasoro constraints for the relative Gromov-Witten theories of all nonsingular target curves. Descendents of the even cohomology classes are studied first by localization, degeneration, and completed cycle methods. Descendents of the odd cohomology are then controlled by monodromy and geometric vanishing relations. As an outcome of our results, the relative theories of target curves are completely and explicitly determined.

Contents

0 Introduction **2**
- 0.1 Overview . . . 2
- 0.2 The relative Gromov-Witten theory of curves . . . 3
- 0.3 Plan of the paper . . . 11
- 0.4 Acknowledgments . . . 12

1 Virasoro constraints for even classes **13**
- 1.1 Overview . . . 13
- 1.2 Virasoro reactions . . . 13
- 1.3 Degeneration . . . 16

2 The equivariant theory of the cap **21**
- 2.1 Overview . . . 21
- 2.2 Theories of the tube . . . 21
- 2.3 Localization for the cap . . . 22
- 2.4 Rubber calculus I . . . 25

2.5 Virasoro for the tube 27
2.6 Outlook 29

3 The operator formalism 30
3.1 Review of the infinite wedge space 30
3.2 Rubber calculus II 33
3.3 The equivariant relative cap revisited 35

4 Virasoro constraints for the cap 37
4.1 Plan of the proof 37
4.2 The nonequivariant limit 37
4.3 Analysis of the operator formula 47
4.4 Virasoro constraints 50

5 Odd classes 56
5.1 Overview 56
5.2 Elliptic invariants 56
5.3 Monodromy relations 58
5.4 Elliptic vanishing relations 60
5.5 Proof of Proposition 5.2 62

6 Virasoro constraints for the full theory 76
6.1 Overview 76
6.2 Proof of Theorems 2 and 3 76
6.3 Proof of Theorem 4 77

0 Introduction

0.1 Overview

We present here the last in a sequence three papers devoted to the Gromov-Witten theory of nonsingular target curves X. In the first paper [11], we considered the stationary sector of the theory formed by the descendents of the Poincaré dual of the point class. The stationary sector was identified in [11] with the Hurwitz theory of X with completed cycles insertions. In the second paper [12], we found an explicit operator formalism for the equivariant Gromov-Witten theory of $\mathbf{P}^1$ in terms of the infinite wedge representation.

As a consequence, we proved the equivariant theory is governed by a 2–Toda hierarchy.

We study here the conjectured Virasoro constraints for target curves X. The standard Virasoro constraints apply only to the absolute Gromov-Witten theory of X and only provide rules for removing the descendents of the identity class [1]. We strengthen the standard constraints in two directions:

(i) Virasoro constraints for the *relative* Gromov-Witten theory of target curves X are found,

(ii) Constraints providing rules for removing the descendents of the *odd* cohomology of X in the relative theory are found.

Our main result is a proof of the strengthened constraints (i)-(ii) for the relative Gromov-Witten theory of target curves. The Virasoro conjecture for curves is obtained as a special case of (i).

A complete description of the relative theory of curves is obtained from the strengthened constraints and the GW/Hurwitz correspondence of [11]. Our goal in the Introduction is to present our view of the relative theory of X.

0.2 The relative Gromov-Witten theory of curves

0.2.1

Let X be a nonsingular target curve of genus g. All curves in the paper are projective over $\mathbb{C}$. Let

$$\begin{gathered} 1 \\ \alpha_1, \ldots, \alpha_g, \ \beta_1, \ldots, \beta_g \\ \omega \end{gathered}$$

be a basis of $H^*(X, \mathbb{C})$ with the following properties:

(i) the class $1 \in H^0(X, \mathbb{C})$ is the identity,

(ii) the classes $\alpha_i \in H^{1,0}(X, \mathbb{C})$ and $\beta_j \in H^{0,1}(X, \mathbb{C})$ determine a symplectic basis of $H^1(X, \mathbb{C})$,

$$\int_X \alpha_i \cup \beta_j = \delta_{ij},$$

(iii) the class $\omega \in H^2(X, \mathbb{C})$ is the Poincaré dual of the point.

0.2.2

We will study the Gromov-Witten theory of X relative to a finite set of distinct points $q_1, \ldots, q_m \in X$. Let $\eta^1, \ldots, \eta^m$ be partitions of d. The moduli space

$$\overline{M}_{g,n}(X, \eta^1, \ldots, \eta^m)$$

parameterizes connected, genus g, n-pointed stable relative maps with monodromy η^i at q_i. Foundational developments of relative Gromov-Witten theory in symplectic and algebraic geometry can be found in [2, 6, 8, 9]. The absolute Gromov-Witten theory of X is recovered if $m = 0$.

The (nonequivariant) connected Gromov-Witten invariants of X relative to $q_1, \ldots, q_m$ are:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i), \eta^1, \ldots, \eta^m \right\rangle_{g,d}^{\circ X} = \int_{[\overline{M}_{g,n}(X,\eta^1,\ldots,\eta^m)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} \operatorname{ev}_i^*(\gamma_i), \qquad (0.1)$$

where $\tau_k(\gamma)$ denotes the kth descendent of the cohomology class $\gamma \in H^*(X, \mathbb{C})$. The order of the descendent insertions in (0.1) is important for the classes γ_i of odd degree.

The superscript $\circ$ in (0.1) denotes the connected theory. The corresponding disconnected theory will be denoted by the bracket $\langle\ \rangle^{\bullet}$. As we will also use the bare bracket $\langle\ \rangle$ for the disconnected theory, the superscript $\bullet$ will be used only for emphasis. We will be primarily interested in the disconnected theory.

If the target X is $\mathbf{P}^1$, we will omit the superscript $\mathbf{P}^1$. Most of the paper will be devoted to the study of the relative theories of $\mathbf{P}^1$ and the elliptic curve E.

The subscripted genus may be omitted in the notation (0.1) by the dimension constraint in the nonequivariant theory. If the set of relative points is nonempty, the subscripted degree may be also omitted.

0.2.3

We first review the formula for the descendents of ω obtained from the Gromov-Witten/Hurwitz correspondence and the theory of completed cycles [11].

Let d be a non-negative integer. Let λ be a partition of d,

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots .$$

Define the completed cycle $\mathbf{p}_l(\lambda)$ by the formula:

$$\mathbf{p}_l(\lambda) = \sum_{i=1}^{\infty} \left[(\lambda_i - i + \tfrac{1}{2})^l - (-i + \tfrac{1}{2})^l \right] + l! c_{l+1} \,, \tag{0.2}$$

for $l > 0$. The constants c_{l+1} are defined by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{z/2}{\sinh(z/2)}.$$

Notice that $\mathbf{p}_l(\lambda)$ is actually defined by a finite sum for each partition λ.

Let η be a partition of d. Let $C_\eta \subset S(d)$ be the associated conjugacy class of the symmetric group. Let $|C_\eta|$ denote the size of C_η. Define the function $\mathbf{f}_\eta(\lambda)$ by

$$\mathbf{f}_\eta(\lambda) = |C_\eta| \, \frac{\chi^\lambda_\eta}{\dim \lambda}, \tag{0.3}$$

where χ^λ_η is the character of any element of C_η in the representation of $S(d)$ corresponding to λ.

We have the following formula for the descendents of ω from the GW/H correspondence [11].

Theorem 1.

$$\left\langle \tau_{z_1}(\omega) \dots \tau_{z_l}(\omega), \eta^1, \dots, \eta^m \right\rangle^{\bullet X}_d = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g} \prod_{i=1}^{l} \frac{\mathbf{p}_{z_i+1}(\lambda)}{(z_i+1)!} \prod_{j=1}^{m} \mathbf{f}_{\eta^j}(\lambda) \,. \tag{0.4}$$

0.2.4

Next, we present our formula governing odd descendents in the presence of descendents of ω.

Let $[2k]$ denote the ordered set of integers $(1, \dots, 2k)$. Let $I(2k)$ denote the set of fixed point free involutions of $[2k]$. For each element $\sigma \in I(2k)$, let $o^\sigma_1, \dots, o^\sigma_k$ denote the orbits of σ. Each orbit o^σ_i is a two element ordered set (e_{i1}, e_{i2}). A canonical sign $\epsilon(\sigma)$ is associated to σ by the parity of the permutation

$$(e_{11}, e_{12}, e_{21}, e_{22}, \dots, e_{k1}, e_{k2})$$

in the symmetric group $S(2k)$.

The Gromov-Witten invariants (0.1) vanish by the dimension constraint if an odd number of odd descendent classes are inserted. Let

$$\gamma_1, \ldots, \gamma_{2k} \in H^1(X, \mathbb{C})$$

be an even number of odd classes.

Theorem 2.

$$\left\langle \prod_{i=1}^{2k} \tau_{y_i}(\gamma_i) \prod_{j=1}^{l} \tau_{z_j}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X}$$

$$= \sum_{\sigma \in I(2k)} \epsilon(\sigma) \prod_{i=1}^{k} \binom{y_{e_{i1}} + y_{e_{i2}}}{y_{e_{i1}}} \int_X \gamma_{e_{i1}} \cup \gamma_{e_{i2}} \quad \times$$

$$\left\langle \prod_{i=1}^{k} \tau_{y_{e_{i1}}+y_{e_{i2}}-1}(\omega) \prod_{j=1}^{l} \tau_{z_j}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X} .$$

We easily see the above formula is skew-symmetric in the insertions $\tau_{y_i}(\gamma_i)$. Theorem 2 will be proven in Section 6 of the paper.

0.2.5

Finally, we present our Virasoro constraints for the Gromov-Witten theory of a genus g curve X relative to $q_1, \ldots, q_m \in X$.

The Virasoro constraints are written here as explicit differential equations for the Gromov-Witten partition function. An useful interpretation of the constraints in terms of *Virasoro reactions* is given in Section 1.

Let X^* denote the punctured manifold,

$$X^* = X \setminus \{q_1, \ldots, q_m\},$$

with topological Euler characteristic $\chi(X^*) = 2 - 2g - m$.

We introduce four sets of variables corresponding to the descendents of the classes $1, \alpha_i, \beta_j$, and ω respectively:

$$\begin{gathered} t_0^0, t_1^0, t_2^0, \ldots, \\ s_0^i, s_1^i, s_2^i, \ldots, \;\; \bar{s}_0^j, \bar{s}_1^j, \bar{s}_2^j, \ldots, \\ t_0^1, t_1^1, t_2^1, \ldots . \end{gathered} \tag{0.5}$$

The even variables t^0_i, t^2_j commute, and the odd variables $s^i_k, \bar{s}^j_l$ supercommute. Let ξ denote the formal sum,

$$\xi = \sum_{k\geq 0} t^0_k \tau_k(1) + \sum_{i=1}^{g} \sum_{k\geq 0} \Big(s^i_k \tau_k(\alpha_i) + \bar{s}^i_k \tau_k(\beta_i) \Big) + \sum_{k\geq 0} t^1_k \tau_k(\omega).$$

The Gromov-Witten partition function $Z_d[\eta^1, \ldots, \eta^m]$ is the generating series of disconnected invariants with fixed relative conditions:

$$Z_d[\eta^1, \ldots, \eta^m] = \sum_{n\geq 0} \frac{1}{n!} \left\langle \xi^n, \eta^1, \ldots, \eta^m \right\rangle^{\bullet X}_d .$$

The bracket on the right is expanded multilinearly in the variables (0.5). The supercommutativity of the odd variables must not be forgotten in the expansion of $Z_d[\eta^1, \ldots, \eta^m]$.

We will consider differential operators D in the variables (0.5) acting on the series $Z_d[\eta^1, \ldots, \eta^m]$. The operators will contain only first order derivatives in the odd variables. The derivative

$$\frac{\partial}{\partial s^i_k} f$$

is defined for monomials f by supercommuting the variable s^i_k to the left and removing s^i_k. The same convention holds for $\bar{s}^j_l$.

We define the first two Virasoro operators for the relative theory of X by the following equations:

$$\begin{aligned} L_{-1} &= -\frac{\partial}{\partial t^0_0} \\ &\quad + \sum_{l\geq 0} \left(t^0_{l+1} \frac{\partial}{\partial t^0_l} + \sum_{i=1}^{g} \left(s^i_{l+1} \frac{\partial}{\partial s^i_l} + \bar{s}^i_{l+1} \frac{\partial}{\partial \bar{s}^i_l} \right) + t^1_{l+1} \frac{\partial}{\partial t^1_l} \right) \\ &\quad + t^0_0 t^1_0 + \sum_i s^i_0 \bar{s}^i_0, \end{aligned}$$

$$
\begin{aligned}
L_0 \quad = \quad & -\frac{\partial}{\partial t^0_1} \\
& -\chi(X^*)\frac{\partial}{\partial t^1_0} \\
& +\sum_{l\geq 0}\left(l t^0_l \frac{\partial}{\partial t^0_l} + \sum_{i=1}^{g}\left((l+1)s^i_l\frac{\partial}{\partial s^i_l} + l\bar{s}^i_l\frac{\partial}{\partial \bar{s}^i_l}\right) + (l+1)t^1_l\frac{\partial}{\partial t^1_l}\right) \\
& +\chi(X^*)\sum_{l\geq 0} t^0_{l+1}\frac{\partial}{\partial t^1_l} \\
& +\frac{\chi(X^*)}{2}t^0_0 t^0_0.
\end{aligned}
$$

The Virasoro operators L_{-1} and L_0 for the relative theory specialize to the corresponding Virasoro operators for the absolute theory if $m = 0$. The string, dilaton, and divisor equations for the relative theory imply the first two Virasoro constraints:

$$
\begin{aligned}
L_{-1}\ Z_d[\eta^1, \ldots, \eta^m] &= 0, \\
L_0\ Z_d[\eta^1, \ldots, \eta^m] &= 0.
\end{aligned}
$$

The derivation of the above constraints is identical to the corresponding derivation for the absolute theory.

In the definition of the remaining Virasoro operators, we will use the Pochhammer symbol,

$$
(a)_b = \frac{(a+b-1)!}{(a-1)!}.
$$

For $k > 0$, the operators L_k for the relative theory of X are defined by:

$$\begin{aligned} L_k &= -(k+1)!\ \frac{\partial}{\partial t^0_{k+1}} \\ &\quad -\chi(X^*)\ (k+1)!\ \sum_{r=1}^{k+1} \frac{1}{r}\ \frac{\partial}{\partial t^1_k} \\ &\quad +\sum_{l\geq 0} \left((l)_{k+1}\ t^0_l \frac{\partial}{\partial t^0_{k+l}} + (l+1)_{k+1}\ t^1_l \frac{\partial}{\partial t^l_{k+l}} \right) \\ &\quad +\sum_{l\geq 0} \sum_{i=1}^{g} \left((l+1)_{k+1}\ s^i_l \frac{\partial}{\partial s^i_{k+l}} + (l)_{k+1}\ \bar{s}^i_l \frac{\partial}{\partial \bar{s}^i_{k+l}} \right) \\ &\quad +\chi(X^*) \sum_{l\geq 0} (l)_{k+1}\ \sum_{r=l}^{k+l} \frac{1}{r}\ t^0_l \frac{\partial}{\partial t^1_{k+l-1}} \\ &\quad +\frac{\chi(X^*)}{2} \sum_{l\geq 0}^{k-2} (l+1)!(k-l-1)! \frac{\partial}{\partial t^1_l} \frac{\partial}{\partial t^1_{k-l-2}}. \end{aligned}$$

The operators L_k are easily seen to satisfy the Virasoro bracket,

$$[L_n, L_m] = (n-m)L_{n+m},$$

and thus determine a representation of the subalgebra of the Virasoro algebra spanned by holomorphic vector fields,

$$\mathcal{V} = \left\{ -z^{k+1} \frac{\partial}{\partial z} \right\}_{k\geq -1}.$$

A central result of the paper is a proof of the Virasoro constraints for the relative theory of X.

Theorem 3. *For all $k \geq -1$, $L_k Z_d[\eta^1, \ldots, \eta^m] = 0$.*

The proof is presented in two parts. The Virasoro constraints for the descendents of the even cohomology classes of X are proven first in Sections 1-4. The full constraints are established in Section 6. The Virasoro constraints for the absolute theory of X are obtained if $m = 0$.

Theorems 1 – 3 uniquely determine the relative Gromov-Witten theory of X. Every relative invariant of X can be efficiently calculated.

0.2.6

The Virasoro constraints for the relative Gromov-Witten theory of X represent a strengthening of the standard Virasoro constraints. The operator L_k provides a rule for the removal of the descendent $\tau_k(1)$ in the relative theory of X.

We define additional differential operators D^i_k and $\bar{D}^i_k$ for $k \geq -1$ by:

$$
\begin{aligned}
D^i_k = -(k+1)!\,\frac{\partial}{\partial \bar{s}^i_{k+1}} &+ \sum_{l=0}^{\infty} (l)_{k+1}\, t^0_l\, \frac{\partial}{\partial s_{k+l}} \\
&+ \sum_{l=0}^{\infty} (l+1)_{k+1}\, \bar{s}^i_l\, \frac{\partial}{\partial t^1_{k+l}}
\end{aligned}
$$

$$
\begin{aligned}
\bar{D}^i_k = -(k+1)!\,\frac{\partial}{\partial \bar{s}^i_{k+1}} &+ \sum_{l=0}^{\infty} (l)_{k+1}\, t^0_l\, \frac{\partial}{\partial \bar{s}^i_{k+l}} \\
&- \sum_{l=0}^{\infty} (l+1)_{k+1}\, s^i_l\, \frac{\partial}{\partial t^1_{k+l}}
\end{aligned}
$$

These operators annihilate the generating series $Z_d[\eta^1, \ldots, \eta^m]$ and provide rules for the removal of the descendents $\tau_k(\alpha_i)$ and $\tau_k(\beta_i)$.

Theorem 4. *For all $k \geq -1$,*

$$
\begin{aligned}
D^i_k\; Z_d[\eta^1, \ldots, \eta^m] &= 0, \\
\bar{D}^i_k\; Z_d[\eta^1, \ldots, \eta^m] &= 0.
\end{aligned}
$$

Theorem 4 is derived from Theorems 1 – 3 in Section 6 and represents our second strengthening of the standard Virasoro constraints.

0.2.7

The operators $L_k, D^i_k, \bar{D}^i_k$ satisfy the following commutation relations:

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m}, \\
[L_n, D^i_m] &= -(m+1)D^i_{n+m}, \\
[L_n, \bar{D}^i_m] &= (n-m)\bar{D}^i_{n+m}.
\end{aligned}
$$

The odd operators all anti-commute:

$$\{D^i_n, D^j_m\} = \{D^i_n, \bar{D}^j_m\} = \{\bar{D}^i_n, \bar{D}^j_m\} = 0.$$

Let the operators $\{L_k\}_{k\geq -1}$ be identified with the Lie algebra of holomorphic vector fields $\mathcal{V}$. Then, the operators $\{D^i_k\}_{k\geq -1}$ define a $\mathcal{V}$-module isomorphic to

$$\{-z^{k+1}\}_{k\geq -1}$$

with the action defined by differentiation, and the operators $\{\bar{D}^i_k\}_{k\geq -1}$ define a $\mathcal{V}$-module isomorphic to the adjoint representation.

0.3 Plan of the paper

The Virasoro constraints for the even theory are studied by degeneration in Section 1. The basic building blocks of the degeneration scheme are the cap, the tube, and the pair of pants. From the algebraic perspective, the building blocks may be viewed as $\mathbf{P}^1$ relative to 1, 2, and 3 points respectively. The main result of Section 1 is Proposition 1.5, the reduction of the even Virasoro conjecture to the case of the cap.

The theory of the cap is studied by localization in Section 2. The theory of the tube arises in the vertex integrals of the localization formula for the cap. A formula for the theory of the cap in terms of Hodge integrals and tube integrals is obtained in Proposition 2.1.

The relationship between the theories of the cap and tube plays an essential role in our proof of the even Virasoro constraints. The relative Gromov-Witten theories of the cap and tube are filtered by the number of insertions of $\tau_k(1)$ termed the depth. The results of Section 1 determine the depth r tube theory in terms of the depth r cap theory. The results of Section 2 determine the depth r cap theory in terms of the depth $r-1$ tube theory. The opposite determination thus specify the full theories of the cap and the tube in terms of their stationary sectors.

In Section 3, the theory of the cap is expressed in terms of vacuum expectations of operators in the infinite wedge representation $\Lambda^{\frac{\infty}{2}}V$. The Virasoro constraints for the cap are derived in Section 4 using the operator formalism.

The full relative theory of target curves including descendents of the odd classes is studied in Section 5. Proposition 5.1 reduces the study of the full theory of arbitrary curves to the study of an elliptic curve relative to 1 point. Several techniques including monodromy invariance and geometric vanishing relations are used to determine the full elliptic theory in terms of the even theory. The proofs of Theorems 2 – 4 are completed in Section 6.

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1 Virasoro constraints for even classes

1.1 Overview

In Sections 1 – 4, we will consider the relative Gromov-Witten theory of X with only the descendents $\tau_k(\gamma)$ of the even cohomology classes

$$\gamma \in H^{2\bullet}(X, \mathbb{C}).$$

The odd theory will be studied in Sections 5 and 6.

The Virasoro constraints for the relative Gromov-Witten theory of X are easily seen to respect the even classes. Let ξ^* denote the even sum,

$$\xi^* = \sum_{k\geq 0} t_k^0 \tau_k(1) + \sum_{k\geq 0} t_k^1 \tau_k(\omega),$$

and let $Z_d^*[\eta^1, \ldots, \eta^m]$ be the generating series of even relative invariants:

$$Z_d^*[\eta^1, \ldots, \eta^m] = \sum_{n\geq 0} \frac{1}{n!} \left\langle (\xi^*)^n, \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X}.$$

Let L_k^* denote the restricted Virasoro operator,

$$L_k^* = L_k|_{\{s_p^i, \bar{s}_q^i = 0\}}.$$

The full Virasoro constraints *imply* the even Virasoro constraints:

$$L_k^* \; Z_d^*[\eta^1, \ldots, \eta^m] = 0, \tag{1.1}$$

for all $k \geq -1$. Our goal in Sections 1 – 4 is to prove the even Virasoro constraints for all relative target curves X.

1.2 Virasoro reactions

1.2.1

The Virasoro constraints provide rules for removing $\tau_k(1)$ insertions in the relative Gromov-Witten theory of target curves X. We will describe the Virasoro rule for the removal of $\tau_k(1)$ from

$$\left\langle \tau_k(1) \prod_i \tau_{l_i}(\gamma_i) \right\rangle^X \tag{1.2}$$

as a *reaction*.

The descendent $\tau_k(1)$ is viewed as unstable and subject to decay. The descendent $\tau_k(1)$ decays via five types of reactions with the other insertions of (1.2). These reactions are detailed in the table below. The columns show the number, the type, and the actual formulas for the reactions.

(i)	$1+1 \to 1$	$\tau_k(1)\,\tau_l(1) \to \quad \binom{k+l-1}{k}\,\tau_{k+l-1}(1)$
(ii)	$1+1 \to \omega$	$\tau_k(1)\,\tau_l(1) \to \quad \chi(X^*)\binom{k+l-1}{k}\left(\sum_{j=l}^{k+l-1}\frac{1}{j}\right)\tau_{k+l-2}(\omega)$
(iii)	$1+\omega \to \omega$	$\tau_k(1)\,\tau_l(\omega) \to \quad \binom{k+l}{k}\,\tau_{k+l-1}(\omega)$
(iv)	$1 \to \omega$	$\tau_k(1) \to \quad -\chi(X^*)\left(\sum_{j=1}^{k}\frac{1}{j}\right)\tau_{k-1}(\omega)$
(v)	$1 \to \omega+\omega$	$\tau_k(1) \to \quad \frac{\chi(X^*)}{2k}\sum_{i=0}^{k-3}\binom{k-1}{i+1}^{-1}\tau_i(\omega)\,\tau_{k-i-3}(\omega)$

Recall X^* is the manifold obtained by removing the relative points of X, and $\chi(X^*)$ is the topological Euler characteristic. The reactions (ii), (iv), (v), whose intensity involves a factor of $\chi(X^*)$, are *extensive* reactions. The remaining reactions (i) and (iii) are *intensive* reactions.

The Virasoro rule for the removal of $\tau_{k+1}(1)$ from (1.2) for $k \geq 1$ is to sum over all the invariants arising as outputs of the five decay reactions. For example,

$$\langle \tau_2(1)\tau_3(\omega)\rangle^X = \binom{5}{2}\langle \tau_4(\omega)\rangle^X - \frac{3}{2}\,\chi(X^*)\,\langle \tau_1(\omega)\tau_3(\omega)\rangle^X ,$$

by rules (iii) and (iv).

The five reactions cover all the terms of the even Virasoro operators L_k^* for $k \geq 1$. The Virasoro reactions are therefore *equivalent* to the even Virasoro constraints (1.1) for $k \geq 1$. We leave the elementary verification to the reader. We will prove the Virasoro rules are valid for the even relative Gromov-Witten theory of X.

For $\tau_0(1)$ and $\tau_1(1)$ the Virasoro rules can be supplemented to incorporated the constant terms. However, since the Virasoro constraints L_{-1}^* and L_0^* are proven, we will not investigate them further.

1.2.2

The stationary theory of X relative to $q_1, \ldots, q_m$ is determined by Theorem 1 via the GW/H correspondence. The Virasoro constraints uniquely determine an even theory from the stationary sector.

Proposition 1.1. *There exists a unique solution to the even Virasoro constraints,*

$$L_k^* \bar{Z}_d^*[\eta^1, \ldots, \eta^m] = 0, \quad \forall k \geq -1,$$

which extends the stationary Gromov-Witten theory of X.

Proof. The coefficients of $\bar{Z}_d^*[\eta^1, \ldots, \eta^m]$ determine a new bracket

$$\left\langle \prod_i \tau_{k_i}(\gamma_i) \right\rangle^{-},$$

for even classes γ_i. The solution is said to *extend* the stationary theory of X if the bracket $\langle , \rangle^-$ agrees with the relative Gromov-Witten bracket $\langle , \rangle$ in case all insertions are descendents of ω.

The uniqueness of the solution $\bar{Z}_d^*[\eta^1, \ldots, \eta^m]$ is clear from the Virasoro reactions. After repeated applications, the reactions remove all the descendents of the identity class from $\langle , \rangle^-$ and leave only the stationary descendents.

To prove existence, we must prove the Virasoro rules are compatible. Given a bracket

$$\left\langle \prod_i \tau_{k_i}(\gamma_i) \right\rangle^{-},$$

we must prove the reduction of the bracket to the stationary theory is independent of the *order* of application of the Virasoro rules.

The compatibility is easily obtained by induction on the number of descendents of the identity in the bracket $\langle , \rangle^-$ and the commutation relation

$$[L_n^*, L_m^*] = (n-m) L_{n+m}^*$$

of the Virasoro operators. □

1.3 Degeneration

1.3.1

Let X be a target curve with relative points $q_1, \ldots, q_m$. We will consider nodal degenerations of X of two types:

(i) X degenerates to $X' \cup X''$ intersecting in a node q_*. The relative points are distributed in the degeneration to $q'_1, \ldots, q'_{m'}$ on X' and $q''_1, \ldots, q''_{m''}$ on X''.

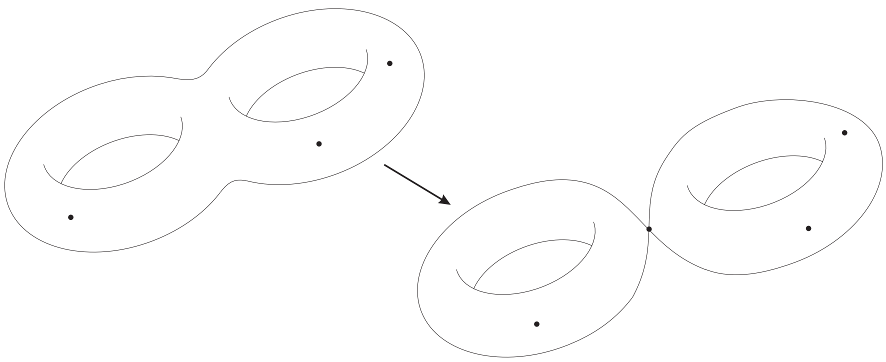

Figure 1: Nodal degeneration of type (i)

(ii) X degenerates to an irreducible curve X' of geometric genus

$$g(X') = g(X) - 1$$

with node q_*.

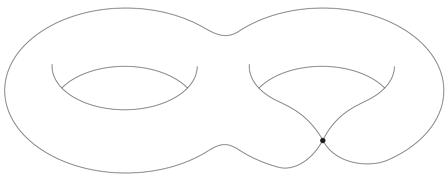

Figure 2: Nodal degeneration of type (ii)

The degeneration formula expresses the relative Gromov-Witten invariants of X in terms of the relative theory of the degenerations. Consider the relative invariant

$$\left\langle \prod_{i=1}^{k} \tau_{y_i}(1) \prod_{j=1}^{l} \tau_{z_j}(\omega), \eta^1, \ldots, \eta^m \right\rangle^X . \tag{1.3}$$

Let $T \subset \{1, \ldots, l\}$ be a subset. The type (i) degeneration formula for the invariant (1.3) is:

$$\sum_{S \subset \{1,\ldots,k\}} \sum_{|\mu|=d} \left\langle \prod_{i \in S} \tau_{k_i}(1) \prod_{j \in T} \tau_{l_j}(\omega), \eta^1, \ldots, \eta^{m'}, \mu \right\rangle^{X'} \mathfrak{z}(\mu) \quad \times$$
$$\left\langle \prod_{i \notin S} \tau_{k_i}(1) \prod_{j \notin T} \tau_{l_j}(\omega), \eta^1, \ldots, \eta^{m''}, \mu \right\rangle^{X''} . \tag{1.4}$$

The type (ii) degeneration formula for the invariant (1.3) is:

$$\sum_{|\mu|=d} \mathfrak{z}(\mu) \left\langle \prod_{i} \tau_{k_i}(1) \prod_{j} \tau_{l_j}(\omega), \eta^1, \ldots, \eta^n, \mu, \mu \right\rangle^{X'} . \tag{1.5}$$

Here, the automorphism factor $\mathfrak{z}(\mu)$ is defined by:

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)} \mu_i \,, \tag{1.6}$$

where $\mathrm{Aut}(\mu)$ is the symmetry group permuting equal parts of μ.

Proofs of the degeneration formulas (1.4) and (1.5) can be found in [6, 8, 9].

Proposition 1.2. *If the relative theories of the degenerations of either type satisfy the even Virasoro constraints, then the original relative theory of X satisfies the even Virasoro constraints.*

Proof. The Proposition is obtained by an elementary verification of the compatibility of the Virasoro reactions with the degeneration formulas. □

1.3.2

The cap $\mathbf{C}$ is the target determined by $\mathbf{P}^1$ relative to $\infty \in \mathbf{P}^1$. We will see the relative theory of the cap governs the even relative theories all of target curves.

Motivated by the operator formalism of the infinite wedge representation, we will denote the relative invariants of $\mathbf{C}$ by:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \middle| \eta \right\rangle .$$

The bracket $\langle\ \rangle^{\mathbf{C}}$ may be used to emphasize the target.

1.3.3

Let X be a target curve with relative points $q_1, \ldots, q_m$. Consider the type (i) degeneration of X to $X \cup \mathbf{C}$, where $\mathbf{C}$ is the cap and all the relative points

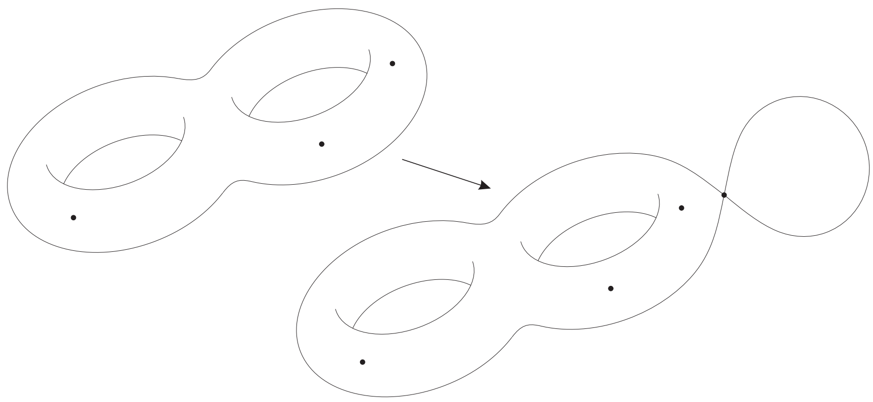

Figure 3: Degeneration used in the proof of Proposition 1.3

$q_1, \ldots, q_m$ remain on X, see Fig. 3. We will use the degeneration formulas and the GW/H correspondence to prove the following result.

Proposition 1.3. *The even theory of X relative to $q_1, \ldots, q_m$ and the theory of the cap uniquely determine the even theory of X relative to $q_1, \ldots, q_m, q_*$.*

Proof. Let $p(d)$ be the number of partitions of size d. Consider the $\infty \times p(d)$ matrix M_d, indexed by monomials L in the descendents of ω and partitions μ of d, with coefficient

$$\langle L \mid \mu \rangle_d^{\mathbf{C}}$$

in position (L, μ). Since the completed cycles,

$$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots,$$

generate the algebra of shifted symmetric functions, the matrix M_d has full rank equal to $p(d)$ by the GW/H correspondence, see [11].

We prove the Proposition by induction on the number of descendents of the identity in the bracket $\langle\ \rangle^X$ relative to $m+1$ points. The base case and the induction step are proven simultaneously.

Consider the invariants of X relative to $m+1$ points,

$$\left\langle \prod_i \tau_{k_i}(1) \prod_j \tau_{l_j}(\omega), \eta^1, \ldots, \eta^m, \mu \right\rangle_d^X, \tag{1.7}$$

defined by fixing the descendent insertions and varying μ among all partitions of d.

We will determine the invariants (1.7) from the invariants of X relative to m points,

$$\left\langle L \ \prod_i \tau_{k_i}(1) \prod_j \tau_{l_j}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^X, \tag{1.8}$$

defined by all monomials L in the descendents of ω.

We apply the type (i) formula for the degeneration of Fig. 3 to the invariants (1.8). All the descendents of ω remain on X in the degeneration except for those in L which distribute to $\mathbf{C}$. By induction, we need only analyze the terms of the degeneration formula in which the descendents of the identity remain on X. Then, since M_d has full rank, the invariants (1.7) are determined by the invariants (1.8). $\square$

The *depth* r theory of X relative to $q_1, \ldots, q_m$ consists of the even invariants with at most r descendents of the identity class. In particular, the depth 0 theory coincides with the stationary theory of X. The proof of Proposition 1.3 yields a refined result.

Corollary 1.4. *The depth r theory of X relative to m points and the depth r theory of the cap uniquely determine the depth r theory of X relative to $m+1$ points.*

1.3.4

Proposition 1.3 implies the theory of the cap determines the even relative theories of all curves.

Proposition 1.5. *The Virasoro constraints for the cap imply the even Virasoro constraints for the even relative theories of all curves.*

Proof. Assume the relative Gromov-Witten theory of the cap satisfies the Virasoro constraints. We will prove by induction on m that the Gromov-Witten theory of $\mathbf{P}^1$ relative to m points also satisfies the Virasoro constraints. The Proposition is then a consequence of Proposition 1.2 since every curve X admits a sequence of type (ii) degeneration to $\mathbf{P}^1$. The base of the induction is satisfied since $\mathbf{P}^1$ relative to 1 point is the cap.

Let $m > 1$. The Virasoro constraints define a unique extension of the stationary Gromov-Witten theory of $\mathbf{P}^1$ relative to m points. By Proposition 1.2 and the induction hypothesis, the extension defined by the Virasoro constraints is compatible with the degeneration formulas.

By Proposition 1.3, the Gromov-Witten theory of $\mathbf{P}^1$ relative to m points is uniquely determined by the Gromov-Witten theory of $\mathbf{P}^1$ relative to $m-1$ points by considering the type (i) degeneration of $\mathbf{P}^1$ to $\mathbf{P}^1 \cup \mathbf{C}$. Therefore, the extension defined by the Virasoro constraints coincides with the Gromov-Witten theory of $\mathbf{P}^1$ relative to m points. $\square$

The proof of Proposition 1.5 and Corollary 1.4 together yield a refined result.

Corollary 1.6. *The Virasoro constraints for the depth r theory of the cap implies the even Virasoro constraints for the depth r relative theories of all target curves.*

2 The equivariant theory of the cap

2.1 Overview

The theory of the cap governs the even relative theory of all target curves. We will now study the equivariant relative Gromov-Witten theory of the cap by localization. The results will be cast in the operator formalism of $\Lambda^{\frac{\infty}{2}}V$ in Section 3.

2.2 Theories of the tube

The tube $\mathbf{T}$ is the target determined by $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$. The automorphism group of the tube fixing the relative points is $\mathbb{C}^*$. There are two relative theories of the tube:

(i) The *parameterized* theory concerns integration over the moduli space $\overline{M}_{g,n}(\mathbf{P}^1, \mu, \nu)$.

(ii) The *unparameterized* theory concerns integration over the moduli space $\overline{M}^{\sim}_{g,n}(\mathbf{P}^1, \mu, \nu)$ obtained by identifying maps which differ by an automorphism of the target.

The standard relative Gromov-Witten theory of the tube is the parameterized theory .

The unparameterized theory is special to the geometry of the tube. Let

$$U_{g,n}(\mathbf{P}^1, \mu, \nu) \subset \overline{M}_{g,n}(\mathbf{P}^1, \mu, \nu)$$

be the locus of maps with finite $\mathbb{C}^*$-stabilizers for the induced action on the moduli of maps. The moduli space $\overline{M}^{\sim}_{g,n}(\mathbf{P}^1, \mu, \nu)$ is defined by:

$$\overline{M}^{\sim}_{g,n}(\mathbf{P}^1, \mu, \nu) = U_{g,n}(\mathbf{P}^1, \mu, \nu) \ / \ \mathbb{C}^*. \tag{2.1}$$

Since the tube dilates with the $\mathbb{C}^*$-action, we view $\overline{M}^{\sim}_{g,n}(\mathbf{P}^1, \mu, \nu)$ as a moduli space of maps to *rubber*.

The unparameterized theory of the tube arises naturally in the vertex integrals of the localization formula for the cap.

2.3 Localization for the cap

2.3.1

Let $V = \mathbb{C} \oplus \mathbb{C}$. Let the algebraic torus $\mathbb{C}^*$ act on V with weights $(0,1)$:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2)\,.$$

Let $\mathbf{P}^1$ denote the projectivization $\mathbf{P}(V)$. There is a canonically induced $\mathbb{C}^*$-action on $\mathbf{P}^1$ with fixed points $0, \infty \in \mathbf{P}^1$.

The $\mathbb{C}^*$-equivariant cohomology ring of a point is $\mathbb{C}[t]$ where t is the first Chern class of the standard representation. The $\mathbb{C}^*$-equivariant cohomology ring $H^*_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{C})$ is canonically a $\mathbb{C}[t]$-module. The (localized) equivariant cohomology of $\mathbf{P}^1$ is spanned by the classes of the fixed points,

$$[0], [\infty] \in H^*_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{C}).$$

2.3.2

We will study here the equivariant relative Gromov-Witten theory of the cap. Let $\infty \in \mathbf{P}^1$ be the relative point of the cap. The $\mathbb{C}^*$-action lifts to the moduli space $\overline{M}^\bullet_{g,n}(\mathbf{P}^1, \nu)$. As the virtual structure is canonically equivariant, we may define equivariant relative invariants by equivariant integration:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(0) \prod_{j=1}^m \tau_{l_j}(\infty) \middle| \nu \right\rangle_g = \int_{[\overline{M}^\bullet_{g,n+m}(\mathbf{P}^1,\nu)]^{vir}} \prod_{i=1}^n \psi_i^{k_i} \mathrm{ev}^*([0]) \prod_{j=1}^m \psi_j^{l_j} \mathrm{ev}^*([\infty]).$$

The equivariant relative invariants take values in $\mathbb{C}[t]$.

The virtual class is defined by an (equivariant) perfect obstruction theory [5, 9]. The virtual localization formula of [4] is applied to the case of relative maps in [5]. A presentation of the localization formula for relative maps to $\mathbf{P}^1$ can also be found in [3].

2.3.3

The virtual localization formula for the cap will be used to study the equivariant relative invariants. The formula is best expressed in terms of the generating series of equivariant relative invariants,

$$\mathsf{G}(z_1, \dots, z_n, w_1, \dots, w_m|\nu) = \sum_g \sum_{k_i} \sum_{l_j} \prod_{i=1}^{n} z_i^{k_i+1} \prod_{j=1}^{m} w_j^{l_j+1} \left\langle \prod \tau_{k_i}(0) \prod \tau_{l_j}(\infty) \Big| \nu \right\rangle_g . \quad (2.2)$$

As disconnected invariants are considered, the genus g may be negative in the outer sum. The genus variable u of [12] is omitted here without any loss of information (by dimension considerations).

The nonequivariant specialization of G will be denoted by G':

$$\mathsf{G}'(x_1, \dots, x_n, y_1, \dots, y_m|\,\nu) = \sum_g \sum_{k_i \geq 0} \sum_{l_j \geq 0} \prod_{i=1}^{n} x_i^{k_i+1} \prod_{j=1}^{m} y_j^{l_j+1} \left\langle \prod \tau_{k_i}(\omega) \prod \tau_{l_j}(1) \Big| \nu \right\rangle_g . \quad (2.3)$$

In the nonequivariant case, the genus g is determined from the insertions and relative conditions by the dimension constraint.

The function G' is obtained from the function G by using the relations

$$\omega = [0],$$
$$1 = \frac{[0] - [\infty]}{t}, \quad (2.4)$$

and then letting $t \to 0$. The procedure is very much like differentiating G with respect to t.

The depth r, nonequivariant, relative theory of the cap is determined by the set of functions,

$$\{\mathsf{G}(z_1, \dots, z_n, w_1, \dots, w_m|\nu)\}_{n<\infty, m\leq r},$$

by nonequivariant specialization.

2.3.4

To achieve uniformity in the localization formulas and the operator formalism, we will include unstable contributions in the generating functions such as (2.2). We follow the conventions of [12] concerning the unstable contributions. The true Gromov-Witten invariants are obtained from the coefficients of terms of positive degree in all the variables.

2.3.5

The localization formula expresses G in terms of Hodge integrals over 0 and rubber integrals over ∞. We first introduce the generating functions associated to these vertex integrals.

Let $\mathsf{H}(z_1, \ldots, z_n, u)$ denote the disconnected n-point function of λ-linear Hodge integrals defined in [12]. We follow here the Hodge integral conventions of [12] governing the unstable cases. The function H is identified as a vacuum expectation in $\Lambda^{\frac{\infty}{2}} V$ in [12] and is fully determined.

The rubber integrals which arise over ∞ are of the following form:

$$\left\langle \mu, k \left| \prod_{i=1}^{n} \tau_{k_i} \right| \nu \right\rangle_g^{\sim} = \int_{[\overline{M}^{\bullet\sim}_{g,n}(\mathbf{P}^1,\mu,\nu)]^{vir}} \psi^k \prod_{i=1}^{n} \psi_i^{k_i},$$

where ψ is the cotangent line to the target at the relative point. The superscript $\sim$ indicates the rubber target. Let

$$\mathsf{G}^{\sim}(\mu, s \mid w_1, \ldots, w_m \mid \nu) = \sum_g \sum_{k \geq 0} \sum_{l_j \geq 0} s^{k-m+1} \prod_{j=1}^{m} w_j^{l_j+1} \left\langle \mu, k \left| \prod \tau_{l_j} \right| \nu \right\rangle_g^{\sim}. \quad (2.5)$$

A direct application of the virtual localization formula for the cap yields the following result.

Proposition 2.1.

$$\mathsf{G}(z_1, \ldots, z_n, w_1, \ldots, w_m|\nu) = \sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)} \frac{(\frac{1}{t})^{\ell(\mu)}}{t^{d+n}} \prod \frac{\mu_i^{\mu_i}}{\mu_i!} \mathsf{H}(\mu, tz, \tfrac{1}{t})\ \mathfrak{z}(\mu)\ \mathsf{G}^{\sim}(\mu, -\tfrac{1}{t} \mid w_1, \ldots, w_m \mid \nu).$$

Here, $\mathfrak{z}(\eta)$ is the automorphism factor defined in (1.6).

The series $\mathsf{G}(z_1, \ldots, z_n, w_1, \ldots, w_m|\nu)$ will be expressed in terms of the operator formalism of the infinite wedge representation in Section 3.

2.4 Rubber calculus I

2.4.1

The rubber integrals,

$$\left\langle \mu, k \left| \prod_{i=1}^{n} \tau_{k_i} \right| \nu \right\rangle_g^{\sim} = \int_{[\overline{M}^{\bullet \sim}_{g,n}(\mathbf{P}^1,\mu,\nu)]^{vir}} \psi^k \prod_{i=1}^{n} \psi_i^{k_i}, \tag{2.6}$$

arise in the localization formula for the cap. The relative Gromov-Witten invariants of the tube are:

$$\left\langle \mu \left| \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right| \nu \right\rangle_g = \int_{[\overline{M}^{\bullet}_{g,n}(\mathbf{P}^1,\mu,\nu)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i), \tag{2.7}$$

where $\gamma_i \in H^*(\mathbf{P}^1, \mathbb{C})$. It will be important for us to express the rubber integrals (2.6) in terms of the tube theory (2.7) *without* cotangent line classes at the relative points.

If the marking set is nonempty, the rubber integral (2.6) is immediately expressed as a tube integral with cotangent line classes at the relative points.

Lemma 2.2. *For $n > 0$,*

$$\left\langle \mu, k \left| \prod_{i=1}^{n} \tau_{k_i} \right| \nu \right\rangle^{\sim} = \left\langle \mu, k \left| \tau_{k_1}(\omega) \prod_{i=2}^{n} \tau_{k_i}(1) \right| \nu \right\rangle. \tag{2.8}$$

Proof. The conceptually simplest proof of the Lemma is by a direct identification of moduli spaces,

$$\overline{M}^{\bullet \sim}_{g,n}(\mathbf{P}^1, \mu, \nu) \stackrel{\sim}{=} \mathrm{ev}_1^{-1}(x) \subset \overline{M}^{\bullet}_{g,n}(\mathbf{P}^1, \mu, \nu),$$

by the quotient description (2.1). Here, $x \in \mathbf{P}^1$ is any point not equal to 0 or ∞. The Lemma then follows after an identification of obstruction theories and virtual classes.

Alternatively, a formal derivation of the Lemma from a localization calculation of the tube integral can be found. The details are left to the reader. □

Let q denote the relative point on the target lying over the fixed point $0 \in \mathbf{P}^1$. The cotangent line ψ at the relative point q is easily analyzed on the moduli space $\overline{M}^{\bullet}_{g,n}(\mathbf{P}^1, \mu, \nu)$. Since q lies over 0, we obtain

$$\psi = \psi_0 + \Delta, \tag{2.9}$$

where ψ_0 is the trivial cotangent line at $0 \in \mathbf{P}^1$ and Δ is the divisor corresponding to proper degenerations in the Artin stack of degeneration of the relative space $0 \in \mathbf{P}^1$.

Lemma 2.2, equation (2.9), and the splitting formula for the relative theory together yield the following recursive relation in case $n > 0$ and $k > 0$:

$$\left\langle \mu, k \left| \prod_{i=1}^{n} \tau_{k_i} \right| \nu \right\rangle^{\sim} = \sum_{S \subset \{2,\ldots,n\}} \sum_{\eta} \left\langle \mu, k-1 \left| \prod_{i \notin S} \tau_{k_i} \right| \eta \right\rangle^{\sim} \mathfrak{z}(\eta) \left\langle \eta \left| \tau_{k_1}(\omega) \prod_{i \in S} \tau_{k_i}(1) \right| \nu \right\rangle , \quad (2.10)$$

where the summation is over all subsets S and all intermediate partitions η of the same size as μ and ν.

After repeated application, the recursive relation (2.10) expresses the original rubber integrals (2.6) in terms of two types of integrals: rubber integrals *without* markings and tube integrals without cotangent line classes at the relative points.

The following Lemma completes the reduction of the rubber integrals (2.6) to the tube theory (2.7).

Lemma 2.3. *For $n = 0$,*

$$\langle \mu, k \,||\, \nu \rangle^{\sim} = \frac{1}{(k+1)!} \left\langle \mu \left| \tau_1(\omega)^{k+1} \right| \nu \right\rangle . \quad (2.11)$$

Proof. The dilaton equation for the relative theory yields:

$$\begin{aligned} \langle \mu, k \,|\tau_1|\, \nu \rangle^{\sim}_g &= (2g-2+\ell(\mu)+\ell(\nu)) \, \langle \mu, k \,||\, \nu \rangle^{\sim}_g \\ &= (k+1) \, \langle \mu, k \,||\, \nu \rangle^{\sim}_g \end{aligned}$$

The second equality is obtained by matching k with the dimension of the moduli space $\overline{M}^{\bullet\sim}_{g,0}(\mu,\nu)$. If $k = 0$, the Lemma is proven by the dilaton equation and Lemma 2.2.

We proceed by induction. For $k > 0$, the induction step is proven by applying the recursion (2.10) to the left side of the dilaton equation and using the degeneration formula. $\square$

The reduction of the rubber integrals to the tube theory will be expressed in terms of the operator formalism of the infinite wedge representation in Section 3. The following convention will simplify our formulas:

$$\langle \mu, -1 \,||\, \nu \rangle^{\sim} = \langle \mu \,||\, \nu \rangle = \frac{\delta_{\mu,\nu}}{\mathfrak{z}(\mu)} \,. \tag{2.12}$$

For example, the recursion (2.10) is valid for $k = 0$ with the above convention.

2.4.2

The outcome of the rubber calculus is a determination of the depth n rubber integral (2.6) in terms of the depth $n-1$ theory of the tube.

Proposition 2.4. *The depth r theory of the tube determines the depth $r+1$ theory of the cap.*

Proof. By Proposition 2.1, the equivariant theory of the cap is determined by the localization formula. After nonequivariant specialization, the depth $r+1$ theory of the cap is determined by depth $r+1$ rubber integrals. By the rubber calculus, the latter integrals are determined by the depth r theory of the tube. □

2.5 Virasoro for the tube

The Virasoro constraints for the tube take a very simple form. Since

$$\chi(\mathbf{T}^*) = 0,$$

there are no extensive Virasoro reactions.

Proposition 2.5. *The Virasoro constraints for the theory of the tube are equivalent to the following relation:*

$$\left\langle \mu \left| \tau_l(\omega) \prod \tau_{k_i}(1) \right| \nu \right\rangle = \binom{\sum k_i + l}{k_1, \dots, k_n, l} \left\langle \mu \left| \tau_{l + \sum (k_i - 1)}(\omega) \right| \nu \right\rangle \,. \tag{2.13}$$

Proof. Equality (2.13) follows from the Virasoro constraints by induction using the following elementary identity for multinomial coefficients:

$$\binom{\sum k_i + l}{k_1, \dots, k_n, l} = \binom{k_1 + l}{k_1} \binom{\sum k_i + l - 1}{k_2, \dots, k_n, k_1 + l - 1} +$$
$$\sum_{j=2}^{n} \binom{k_1 + k_j - 1}{k_1} \binom{\sum k_i + l - 1}{k_2, \dots, k_j + k_1 - 1, \dots, k_n, l} .$$

Indeed, relation (2.13) is equivalent to validity of the Virasoro constraints for relative invariants of the tube with exactly 1 stationary descendent.

Relation (2.13), together with the degeneration formula, uniquely determines the entire relative theory,

$$\left\langle \mu \left| \prod_i \tau_{k_i}(\omega) \prod_j \tau_{l_j}(1) \right| \nu \right\rangle , \tag{2.14}$$

in terms of the stationary theory:

(i) If there are no stationary descendents, then integral (2.14) vanishes. Proofs of the vanishing are easily obtained by degeneration arguments or the localization formula for the tube.

(ii) If there is at least one stationary descendent, we can degenerate the tube $\mathbf{T}$ into a chain of tubes, each containing exactly one stationary descendent.

Since the Virasoro constraints are compatible with degeneration, relation (2.13) implies the validity of the Virasoro constraints for the entire theory of the tube. □

We will only require the nonequivariant generating series for the tube:

$$\mathsf{G}'(\mu \,|x_1, \dots, x_n, y_1, \dots, y_m|\, \nu) =$$
$$\sum_g \sum_{k_i \geq 0} \sum_{l_j \geq 0} \prod_{i=1}^{n} x_i^{k_i+1} \prod_{j=1}^{m} y_j^{l_j+1} \left\langle \mu \left| \prod \tau_{k_i}(\omega) \prod \tau_{l_j}(1) \right| \nu \right\rangle_g .$$

Relation (2.13) is equivalent to the following equation:

$$\begin{aligned}\mathsf{G}'(\mu|x, y_1, \ldots, y_m|\nu) = \\ x y_1 \cdots y_m \left(x + \sum y_j\right)^{m-1} \mathsf{G}'\left(\mu \,\middle|\, x + \sum y_j \,\middle|\, \nu\right) . \end{aligned} \quad (2.15)$$

The tree function

$$\begin{aligned}\mathsf{T}(x_1, \ldots, x_k) &= x_1 \ldots x_k \left(\sum x_i\right)^{k-2} \\ &= \sum_T \prod x_i^{\mathrm{val}_T(i)} \end{aligned} \quad (2.16)$$

will occur often. The sum in (2.16) is over all trees T with vertex set $\{1, \ldots, k\}$. The valence of the vertex i of T is $\mathrm{val}_T(i)$. The factorization of the tree sum is a classical result due to Cayley.

2.6 Outlook

By Proposition 1.3, the depth r theory of the cap determines the depth r theory of the tube. By Proposition 2.4, the depth r theory of the tube determines the depth $r + 1$ theory of the cap. Together, the results uniquely determine the entire relative theories of both the cap and the tube from their stationary theories. We prove the Virasoro constraints for **C** and **T** by showing the Virasoro operators are compatible with the opposite determinations.

The proof of the Virasoro constraints for the full relative theory of target curves requires several additional techniques for handling the odd cohomology.

3 The operator formalism

3.1 Review of the infinite wedge space

3.1.1

We present here a brief review of the infinite wedge representation $\Lambda^{\frac{\infty}{2}}V$. An expository treatment can be found in [7]. Several operators on $\Lambda^{\frac{\infty}{2}}V$ will be required for the study of the cap. We refer the reader to [11, 12] for the geometric motivations behind the definitions.

3.1.2

Let V be a linear space with basis $\{\underline{k}\}$ indexed by the half-integers:

$$V = \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}\,\underline{k}.$$

For each subset $S = \{s_1 > s_2 > s_3 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$ satisfying:

(i) $S_+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right)$ is finite,

(ii) $S_- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \setminus S$ is finite,

we denote by v_S the following infinite wedge product:

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots\,. \tag{3.1}$$

By definition,

$$\Lambda^{\frac{\infty}{2}}V = \bigoplus \mathbb{C}\, v_S$$

is the linear space with basis $\{v_S\}$. Let $(\,\cdot\,,\,\cdot\,)$ be the inner product on $\Lambda^{\frac{\infty}{2}}V$ for which $\{v_S\}$ is an orthonormal basis.

3.1.3

The fermionic operator ψ_k on $\Lambda^{\frac{\infty}{2}}V$ is defined by wedge product with the vector $\underline{k}$,

$$\psi_k \cdot v = \underline{k} \wedge v\,.$$

The operator ψ_k^* is defined as the adjoint of ψ_k with respect to the inner product $(\,\cdot\,,\,\cdot\,)$.

These operators satisfy the canonical anti-commutation relations:

$$\psi_i \psi_j^* + \psi_i^* \psi_j = \delta_{ij} \,, \tag{3.2}$$

$$\psi_i \psi_j + \psi_j \psi_i = \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0. \tag{3.3}$$

The *normally ordered* products are defined by:

$$:\psi_i \, \psi_j^* := \begin{cases} \psi_i \, \psi_j^* \,, & j > 0 \,, \\ -\psi_j^* \, \psi_i \,, & j < 0 \,. \end{cases} \tag{3.4}$$

3.1.4

Let E_{ij}, for $i, j \in \mathbb{Z} + \frac{1}{2}$, be the standard basis of matrix units of $\mathfrak{gl}(\infty)$. The assignment

$$E_{ij} \mapsto :\psi_i \, \psi_j^*: \quad ,$$

defines a projective representation of the Lie algebra $\mathfrak{gl}(\infty) = \mathfrak{gl}(V)$ on $\Lambda^{\frac{\infty}{2}} V$.

The *charge* operator C corresponding to the identity matrix of $\mathfrak{gl}(\infty)$,

$$C = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{kk},$$

acts on the basis v_S by:

$$C \, v_S = (|S_+| - |S_-|) v_S \,.$$

The kernel of C, the zero charge subspace, is spanned by the vectors

$$v_\lambda = \underline{\lambda_1 - \tfrac{1}{2}} \wedge \underline{\lambda_2 - \tfrac{3}{2}} \wedge \underline{\lambda_3 - \tfrac{5}{2}} \wedge \dots$$

indexed by all partitions λ. We will denote the kernel by $\Lambda_0^{\frac{\infty}{2}} V$.

The eigenvalues on $\Lambda_0^{\frac{\infty}{2}} V$ of the *energy* operator,

$$H = \sum_{k \in \mathbb{Z} + \frac{1}{2}} k \, E_{kk},$$

are easily identified:

$$H \, v_\lambda = |\lambda| \, v_\lambda \,.$$

The vacuum vector

$$v_\emptyset = \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \dots$$

is the unique vector with the minimal (zero) eigenvalue of H.

The *vacuum expectation* $\langle A \rangle$ of an operator A on $\Lambda^{\frac{\infty}{2}}V$ is defined by the inner product:

$$\langle A \rangle = (Av_{\emptyset}, v_{\emptyset}).$$

3.1.5

For $r \in \mathbb{Z}$, we define

$$\mathcal{E}_r(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})} E_{k-r,k} + \frac{\delta_{r,0}}{\varsigma(z)}, \tag{3.5}$$

where the function $\varsigma(z)$ is defined by

$$\varsigma(z) = e^{z/2} - e^{-z/2}. \tag{3.6}$$

Definition (3.5) and an elementary calculation yield

$$\mathcal{E}_r(z)^* = \mathcal{E}_{-r}(z),$$

where the adjoint is taken with respect to the standard inner product on $\Lambda^{\frac{\infty}{2}}V$.

Define the operators $\mathcal{P}_k$ for $k > 0$ by:

$$\frac{\mathcal{P}_k}{k!} = [z^k]\, \mathcal{E}_0(z), \tag{3.7}$$

where $[z^k]$ stands for the coefficient of z^k. The operator,

$$\mathcal{F}_2 = \frac{\mathcal{P}_2}{2!} = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{k^2}{2} E_{k,k},$$

will play a special role.

3.1.6

The operators $\mathcal{E}$ satisfy the following fundamental commutation relation:

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma\left(\det\left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right]\right) \mathcal{E}_{a+b}(z+w). \tag{3.8}$$

Equation (3.8) automatically incorporates the central extension of the $\mathfrak{gl}(\infty)$-action, which appears as the constant term in $\mathcal{E}_0$ when $r = -s$.

3.1.7

The operators $\mathcal{E}$ specialize to the standard bosonic operators on $\Lambda^{\frac{\infty}{2}}V$:

$$\alpha_k = \mathcal{E}_k(0)\,, \quad k \neq 0\,.$$

The commutation relation (3.8) specializes to the following equation

$$[\alpha_k, \mathcal{E}_l(z)] = \varsigma(kz)\, \mathcal{E}_{k+l}(z)\,. \tag{3.9}$$

When $k + l = 0$, equation (3.9) has the following constant term:

$$\frac{\varsigma(kz)}{\varsigma(z)} = \frac{e^{kz/2} - e^{-kz/2}}{e^{z/2} - e^{-z/2}}\,.$$

Letting $z \to 0$, we recover the standard relation:

$$[\alpha_k, \alpha_l] = k\, \delta_{k+l}\,.$$

3.2 Rubber calculus II

3.2.1

We will find an operator formula for rubber integrals using the calculus of Section 2.4 and Proposition 2.5. In particular, the operator formula here *will depend upon the Virasoro constraints for the tube.*

3.2.2

For a partition ν, let the vector $|\nu\rangle \in \Lambda^{\frac{\infty}{2}}V$ be defined by

$$|\nu\rangle = \frac{1}{\mathfrak{z}(\nu)} \prod \alpha_{-\nu_i}\, v_\emptyset\,.$$

The operator

$$\sum_{|\eta|=d} \Big|\eta\Big\rangle\, \mathfrak{z}(\eta)\, \Big\langle\eta\Big|$$

is the orthogonal projection onto the subspace of the Fock space $\Lambda^{\frac{\infty}{2}}V_0$ corresponding to partitions of size d.

3.2.3

The 1-point stationary generating series of the tube has been calculated in [11] by the GW/H correspondence:

$$\mathsf{G}'\left(\mu\,|x|\,\nu\right) = \left\langle \mu\,|\mathcal{E}_0(x)|\,\nu\right\rangle \,.$$

Introduce the following operator:

$$\mathcal{E}_0(x_1,\dots,x_k) = \mathsf{T}(x_1,\dots,x_k)\,\mathcal{E}_0(x_1+\dots+x_k)\,,$$

where T is the tree function. Then, equation (2.15) yields:

$$\mathsf{G}'(\mu|x,y_1,\dots,y_m|\nu) = \left\langle \mu\,|\mathcal{E}_0(x,y_1,\dots,y_m)|\,\nu\right\rangle \,. \tag{3.10}$$

3.2.4

Similarly, by Lemma 2.3,

$$\sum_{k\geq -1} s^{k+1}\,\left\langle \mu,k\,||\,\nu\right\rangle^{\sim} = \sum_{k\geq -1} \frac{s^{k+1}}{(k+1)!}\,\left\langle \mu\,\middle|\mathcal{F}_2^{k+1}\middle|\,\nu\right\rangle = \left\langle \mu\,\middle|e^{s\mathcal{F}_2}\middle|\,\nu\right\rangle \,. \tag{3.11}$$

Here, we follow convention (2.12).

3.2.5

Let $\mathsf{E}(x_1,\dots,x_m,s)$ denote the following operator:

$$\mathsf{E}(x_1,\dots,x_m,s) = \sum_{\pi} s^{m-\ell(\pi)} \prod_{k=1}^{\ell(\pi)} \mathcal{E}_0(x_{\pi_k})\,, \tag{3.12}$$

where the summation is over all partitions

$$\pi = \pi_1 \sqcup \dots \sqcup \pi_\ell$$

of the set $\{1,\dots,m\}$ into nonempty disjoint subsets. Here, x_{α_k} denotes the variables x_i with indices in the subset π_k. For example

$$\begin{aligned}\mathsf{E}(x_1,x_2,x_3,s) = {} & \mathcal{E}_0(x_1)\,\mathcal{E}_0(x_2)\,\mathcal{E}_0(x_3) + s\,\mathcal{E}_0(x_1,x_2)\,\mathcal{E}_0(x_3)\\ & s\,\mathcal{E}_0(x_1,x_3)\,\mathcal{E}_0(x_2) + s\,\mathcal{E}_0(x_2,x_3)\,\mathcal{E}_0(x_1) + s^2\,\mathcal{E}_0(x_1,x_2,x_3)\,.\end{aligned}$$

Since the operators $\mathcal{E}_0$ commute, the ordering of the blocks of π is not important.

The recursive relation (2.10) together with formulas (3.10) and (3.11) directly yields an operator formula for the generating series $\mathsf{G}^\sim$ of rubber integrals.

Proposition 3.1.

$$\mathsf{G}^\sim(\mu, s|x_1, \dots, x_m|\nu) = \left\langle \mu \left| e^{s\mathcal{P}_2} \mathsf{E}(x_1, \dots, x_m, \tfrac{1}{s}) \right| \nu \right\rangle .$$

The validity of Proposition 3.1 depends upon the Virasoro constraints for the depth $m-1$ theory of the tube.

3.3 The equivariant relative cap revisited

Recall the basic operator $\mathsf{A}(z)$ defined in [12]:

$$\mathsf{A}(z) = \mathcal{S}(z)^{tz} \left(\sum_{k \in \mathbb{Z}} \frac{\varsigma(z)^k}{(tz+1)_k} \, \mathcal{E}_k(z) \right) , \tag{3.13}$$

where

$$\varsigma(z) = e^{z/2} - e^{-z/2} , \quad \mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{\sinh z/2}{z/2}$$

and

$$(a+1)_k = \frac{(a+k)!}{a!} = \begin{cases} (a+1)(a+2)\cdots(a+k) , & k \geq 0 , \\ (a(a-1)\cdots(a+k+1))^{-1} , & k \leq 0 . \end{cases}$$

Here, the genus variable u of [12] is set to 1.

Proposition 2.1 together with the operator formulas for H of Section 5 of [12] and the operator formula for $\mathsf{G}^\sim$ yields:

$$\mathsf{G}(z_1, \dots, z_n, w_1, \dots, w_m|\nu) = \\ \sum_\eta \left\langle \prod \mathsf{A}(z_i) \, e^{\alpha_1} \, e^{\frac{1}{t}\mathcal{F}_2} \middle| \eta \right\rangle \mathfrak{z}(\eta) \left\langle \eta \middle| e^{-\frac{1}{t}\mathcal{F}_2} \, \mathsf{E}(w_1, \dots, w_m, -t) \middle| \nu \right\rangle , \tag{3.14}$$

where the summation is over all partitions η such that $|\eta| = |\nu|$.

The orthogonal projection,

$$\sum_{|\eta|=d} \left| \eta \right\rangle \mathfrak{z}(\eta) \left\langle \eta \right|$$

commutes with $\mathcal{F}_2$ and E and fixes the vector $|\nu\rangle$. After commuting the projection to the far right, we obtain the following the operator formula for the equivariant relative theory of the cap.

Proposition 3.2.

$$\mathsf{G}(z_1,\dots,z_n,w_1,\dots,w_m|\nu) = \left\langle \prod \mathsf{A}(z_i)\, e^{\alpha_1}\, \mathsf{E}(w_1,\dots,w_m,-t) \right| \left. \nu \right\rangle . \quad (3.15)$$

The validity of Proposition 3.2 depends upon the Virasoro constraints for the depth $m-1$ theory of the tube.

4 Virasoro constraints for the cap

4.1 Plan of the proof

We prove the Virasoro constraints for the cap by induction on depth. The base of the induction is trivial as the Virasoro constraints are empty for the depth 0 theory of the cap. Assume the Virasoro constraints hold for the depth r theory of the cap. By Proposition 1.5, the Virasoro constraints hold for the depth r theory of the tube. Then, Proposition 3.2 provides operator formulas for the equivariant series,

$$\mathsf{G}(z_1, \ldots, z_n, w_1, \ldots, w_m | \nu), \tag{4.1}$$

for $m \leq r+1$. Via the nonequivariant limit, the series (4.1) determine the depth $r+1$ nonequivariant theory of the cap. To complete the induction step, we must prove the nonequivariant limits of the operator formulas for (4.1) satisfy the Virasoro rules.

4.2 The nonequivariant limit

4.2.1

Our goal now is to study the nonequivariant limit of the operator formula (3.15) for

$$\mathsf{G}(z_1, \ldots, z_n, w_1, \ldots, w_m | \nu).$$

We will obtain an operator formula for the nonequivariant limit involving the first two terms, $\mathsf{A}^0(z)$ and $\mathsf{A}^1(z)$, in the expansion of the operator $\mathsf{A}(z)$,

$$\mathsf{A}(z) = \sum_{k \geq 0} t^k \, \mathsf{A}^k(z),$$

in powers of the parameter t. The operator

$$\mathsf{A}^0(z) = \mathsf{A}(z)\Big|_{t=0}$$

is related to the operator $\mathcal{E}_0(z)$ via

$$e^{-\alpha_1} \, \mathsf{A}^0(z) \, e^{\alpha_1} = \mathcal{E}_0(z) \,, \tag{4.2}$$

see [12]. The above relation will play an important role in our analysis of formula (3.15).

4.2.2

We will often encounter the operator $\mathsf{A}^\vee$ constructed from the operators A^0 and A^1:

$$\mathsf{A}^\vee(z_1,\dots,z_n) = \mathsf{T}(z_1,\dots,z_n) \Bigg[z_n \ln\left(1+\frac{z_1+\dots+z_{n-1}}{z_n}\right) \mathsf{A}^0(z_1+\dots+z_n) + \frac{z_n}{z_1+\dots+z_n}\,\mathsf{A}^1(z_1+\dots+z_n)\Bigg]. \quad (4.3)$$

The expression in square brackets is the coefficient of t in the expansion of

$$\left(\frac{z_1+\dots+z_n}{z_n}\right)^{tz_n} \mathsf{A}(tz_n, z_1+\dots+z_n)$$

in powers of t, where the 2-parameter operator $\mathsf{A}(a,b)$ is defined by

$$\mathsf{A}(a,b) = \mathcal{S}(b)^a \left(\sum_{k\in\mathbb{Z}} \frac{\varsigma(b)^k}{(a+1)_k}\,\mathcal{E}_k(b)\right). \quad (4.4)$$

4.2.3

An operator formula for the nonequivariant limit of $\mathsf{G}(z,w)$ is obtained from Proposition 3.2 and the relation,

$$1 = \frac{[0]-[\infty]}{t},$$

in the localized equivariant cohomology of $\mathbf{P}^1$:

$$\mathsf{G}'(z_1,\dots,z_n,w_1,\dots,w_m|\nu) = [t^m]\left\langle \prod \mathsf{A}(z_i)\,\mathsf{M}(w_1,\dots,w_m)\middle|\,\nu\right\rangle, \quad (4.5)$$

where $[t^m]$ denotes the coefficient of t^m. The operator $\mathsf{M}(w)$ is defined by:

$$\mathsf{M}(w_1,\dots,w_m) = \sum_{S\subset\{1,\dots,m\}} (-1)^{|S|} \prod_{i\notin S} \mathsf{A}(w_i)\, e^{\alpha_1}\,\mathsf{E}(w_S,-t). \quad (4.6)$$

where the summation is over all subsets S of the index set $\{1, \ldots, m\}$. Here, w_S denotes the variables w_j such that $j \in S$.

Inside the brackets, the order of the operators in the product

$$\prod_{i \notin S} \mathsf{A}(w_i)$$

does not matter by the symmetry of the equivariant series G. However, the operators themselves do *not* commute. We order the operators in the above product by their indices i.

4.2.4

We now express the leading t coefficient of the operator M in terms of the operators $\mathsf{A}^\vee$ introduced in Section 4.2.2.

Proposition 4.1. *We have*

(i) for $k < m$,

$$[t^k]\, \mathsf{M}(w_1, \ldots, w_m) = 0\,, \tag{4.7}$$

(ii) for $k = m$,

$$[t^m]\, \mathsf{M}(w_1, \ldots, w_m) = \sum_\pi \left(\overrightarrow{\prod_{p=1,\ldots,\ell(\pi)}} \mathsf{A}^\vee(w_{\pi_p}) \right) e^{\alpha_1}\,. \tag{4.8}$$

The summation in (4.8) is over all partitions π of the ordered set $\{1, \ldots, m\}$. The argument w_{π_p} denotes the set of variables in the induced order with indices in the part π_p of π. The operators $\mathsf{A}^\vee$ are ordered left-to-right by increasing maximal elements of π_p. Since the operators $\mathsf{A}^\vee$ do not commute, their ordering is important.

4.2.5

The operator $\mathsf{A}(z)$ is obtained from $\mathsf{A}(a, b)$ by the evaluation

$$\mathsf{A}(z) = \mathsf{A}(tz, z)\,.$$

Therefore,

$$\mathsf{A}^0(z) = \mathsf{A}(0, z)\,.$$

The following result will play an important role in the proof of Proposition 4.1.

Lemma 4.2. *We have*

$$\left[\mathsf{A}^0(z), \mathsf{A}(a,b)\right] = \varsigma(za)\,\left(1+\frac{z}{b}\right)^a \mathsf{A}(a, z+b)\,. \tag{4.9}$$

Proof. The basic commutation relation for the operators $\mathcal{E}$ is:

$$[\mathcal{E}_k(z), \mathcal{E}_l(b)] = \varsigma(kb - lz)\,\mathcal{E}_{k+l}(z+b)\,. \tag{4.10}$$

Therefore,

$$\left[\mathsf{A}^0(z), \mathsf{A}(a,b)\right] = \mathcal{S}(b)^a \sum_{k\geq 0,\, l\in\mathbb{Z}} \frac{\varsigma(z)^k\,\varsigma(b)^l\,\varsigma(kb-lz)}{k!\,(a+1)_l}\,\mathcal{E}_{k+l}(z+b)\,.$$

The coefficient of $\mathcal{E}_m(z+b)$ in the above series equals

$$\mathcal{S}(b)^a\,\frac{\varsigma(b)^m}{(a+1)_m}\sum_{k\geq 0}\binom{m+a}{k}\varsigma(kb+kz-mz)\,\left(\frac{\varsigma(z)}{\varsigma(b)}\right)^k\,. \tag{4.11}$$

Since

$$\varsigma(kb+kz-mz)\,\left(\frac{\varsigma(z)}{\varsigma(b)}\right)^k = e^{-mz/2}\,\left(\frac{e^z-1}{1-e^{-b}}\right)^k - e^{mz/2}\,\left(\frac{1-e^{-z}}{e^b-1}\right)^k,$$

the binomial series in (4.11) sums to

$$e^{-mz/2}\,\left(\frac{e^z-e^{-b}}{1-e^{-b}}\right)^{m+a} - e^{mz/2}\,\left(\frac{e^b-e^{-z}}{e^b-1}\right)^{m+a} = \varsigma(za)\,\frac{\varsigma(b+z)^{m+a}}{\varsigma(b)^{m+a}}\,.$$

Thus, the coefficient (4.11) of $\mathcal{E}_m(z+b)$ in the commutator equals

$$\frac{\varsigma(za)}{b^a}\,\frac{\varsigma(z+b)^{m+a}}{(a+1)_m}\,,$$

as was to be shown. ☐

Corollary 4.3. *We have*

$$\left[\mathsf{A}^0(z_1), \left[\mathsf{A}^0(z_2), \left[\dots, \left[\mathsf{A}^0(z_n), \mathsf{A}(a,b)\right]\dots\right]\right]\right] = \prod \varsigma(z_i a)\,\left(1+\frac{\sum z_i}{b}\right)^a \mathsf{A}\left(a, \sum z_i + b\right)\,. \tag{4.12}$$

4.2.6

The strategy of our proof of Proposition 4.1 will be presented here. The proof is given in Sections 4.2.7 – 4.2.9.

We must extract coefficients of t from the operator M defined by formula (4.6). Consider the summand of the formula indexed by S:

$$(-1)^{|S|} \prod_{i \notin S} \mathsf{A}(w_i)\, e^{\alpha_1}\, \mathsf{E}(w_S, -t). \tag{4.13}$$

Let $\mathsf{A}(w_i)$ be the leftmost operator in (4.13) where

$$i = \min\{j, j \in \complement S\},$$

and $\complement S$ denotes the complement of S. We start by selecting a coefficient of t from $\mathsf{A}(w_i)$. There are two possibilities. Either we select the minimal power t^0, corresponding to the operator $\mathsf{A}^0(w_i)$, or we select *all* higher powers of t:

(i) If we select the minimal power t^0 in $\mathsf{A}(w_i)$, we commute the resulting operator $\mathsf{A}^0(w_i)$ to the right of the operator e^{α_1}. The operator $\mathsf{A}^0(w_i)$ commutes through the operators $\mathsf{A}(w_j)$ via the commutation relation

$$\left[\mathsf{A}^0(w_i), \mathsf{A}(w_j)\right] = \varsigma(tw_iw_j) \left(1 + \frac{w_i}{w_j}\right)^{tw_j} \mathsf{A}(tw_j, w_i + w_j), \tag{4.14}$$

obtained from Lemma 4.2.

The prefactors in equation (4.14) play an important role. In particular, since

$$\varsigma(tw_iw_j) = tw_iw_j + o(t^2),$$

the minimal positive power of t that can be extracted from the operator (4.14) is t^1, which yields the operator $w_iw_j\, \mathsf{A}^0(w_i + w_j)$.

When the commutator (4.14) appears, we say the ith operator interacts with the jth operator. The final commutation of $\mathsf{A}(w_i)$ with e^{α_1} is determined by relation (4.2). When the relation (4.2) is used, we say the ith operator interacts with ∞.

(ii) If the higher powers of t in $\mathsf{A}(w_i)$ are chosen at the start, then we do nothing further with $\mathsf{A}(w_i)$.

Once step (i) or (ii) has been completed, we proceed to treat the leftmost operators of the remaining terms, repeating the above cycle using step (i) for the minimal power of t and step (ii) for the higher powers. The cycle is repeated until all the operators A of all the terms are treated. The outcome is a transformation of formula (4.6). Proposition 4.1 is proven by finding a large cancellation in the transformed formula.

4.2.7

The following diagrammatic technique is useful for understanding the terms arising in the transformation of formula (4.13) discussed in Section 4.2.6. The terms of the transformed formula will be indexed by graphs $\mathcal{I}$ of the kind shown in Fig. 4 which we call *interaction diagrams*. An interaction

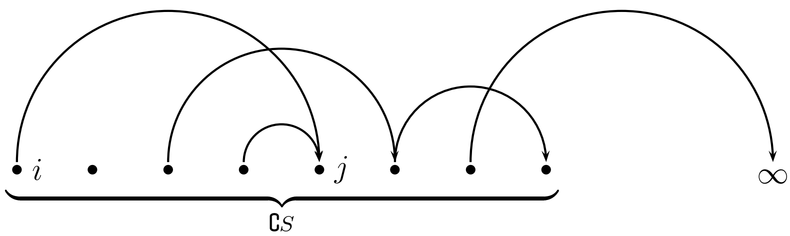

Figure 4: An interaction diagram

diagram has vertices indexed by the set $\complement S$ plus an additional vertex marked by the symbol ∞. If the ith operator interacts with the jth operator, we draw an arrow from i to j. The notation $i \to j$ will also be used. If the ith operator interacts with ∞, we draw an arrow from i to ∞. A vertex with no outgoing arrows in an interaction diagram marks an occurrence of the step (ii) of Section 4.2.6.

Abstractly, interaction diagrams $\mathcal{I}$ can be defined as trees with vertex set $\complement S \cup \{\infty\}$ such that for every $i \in \complement S$ there is at most one edge (i, j) with $j > i$. An interaction diagram $\mathcal{I}$ gives the set $\complement S$ a natural partial order $\trianglelefteq$ in which by definition $i \trianglelefteq j$ if i can be connected to j by arrows of $\mathcal{I}$. Note that this order is compatible with the usual ordering of natural numbers.

4.2.8

For any $k \in \complement S$, let $\mathcal{I}(k)$ be the maximal connected subdiagram of $\mathcal{I}$ with maximal vertex k. Clearly,

$$\text{vertices}(\mathcal{I}(k)) = \{i \,|\, i \trianglelefteq k\}\,. \tag{4.15}$$

The operators $\mathsf{A}(w_i)$, $i \trianglelefteq k$, are precisely the operators that influence the operator $\mathsf{A}(w_k)$ in the transformation of the term (4.13).

Concretely, the operator $\mathsf{A}(w_k)$ is transformed by the interactions specified by $\mathcal{I}(k)$ to the operator:

$$\mathbf{w}(\mathcal{I}(k)) \left(1 + \frac{\sum_{i \triangleleft k} w_i}{w_k}\right)^{tw_k} \mathsf{A}\left(tw_k, \sum_{i \trianglelefteq k} w_i\right)\,, \tag{4.16}$$

where $\mathbf{w}(\mathcal{I}(k))$ is the following product over all arrows $i \to j$ in $\mathcal{I}(k)$

$$\mathbf{w}(\mathcal{I}(k)) = \prod_{i \to j} \varsigma\left(t\left(\sum_{i' \trianglelefteq i} w_{i'}\right) w_j\right)\,.$$

In (4.16), only the factor $\mathbf{w}(\mathcal{I}(k))$ depends on the actual interaction diagram $\mathcal{I}(k)$ and not only on the vertex set (4.15). Therefore, it is natural to sum the weights $\mathbf{w}(\mathcal{I}(k))$ over all interaction diagram with the same vertex set K, where

$$K = \{k_1 < k_2 < \cdots < k_r = k\}\,,$$

is an arbitrary subset of $\complement S$ with maximal element k.

Lemma 4.4. *We have,*

$$\sum_{\text{vertices}(\mathcal{I}(k))=K} \mathbf{w}(\mathcal{I}(k)) = t^{r-1}\,\mathsf{T}(w_{k_1}, \dots, w_{k_r}) + o(t^r)\,, \tag{4.17}$$

where T *is the tree function defined by* (2.16).

Proof. We present a combinatorial proof of the following equivalent identity:

$$\sum_{\text{vertices}(\mathcal{I}(k))=K} \prod_{i \to j} \left(\sum_{i' \trianglelefteq i} w_{i'}\right) w_j = \mathsf{T}(w_{k_1}, \dots, w_{k_r})\,. \tag{4.18}$$

We will construct a map σ from the set of trees T on K to the set of interaction diagrams $\mathcal{I}(k)$ with the following property:

$$\prod_{i \to j} \left(\sum_{i' \trianglelefteq i} w_{i'} \right) w_j = \sum_{T \in \sigma^{-1}(\mathcal{I}(k))} \prod w_i^{\mathrm{val}_T(i)} . \tag{4.19}$$

Identity (4.18) is then deduced from the definition of T.

Let T be an arbitrary tree on the vertex set K with edge set E. What needs to be constructed is the edge set of the interaction diagram $\sigma(T)$. Denote this edge set by $\sigma(E)$. The construction of $\sigma(E)$ will be inductive.

Let $(i, j) \in E$ be an edge of T and assume that all edges $(a, b) \in E$ with $b < j$ have been already considered. Then we add to $\sigma(E)$ the edge $i' \to j$, where i' is the maximal vertex connected to i by edges already in $\sigma(E)$. In other words, the new edge is obtained from the edge (i, j) by moving the origin as far as we can along the edges already in $\sigma(E)$. This is illustrated in Fig. 5.

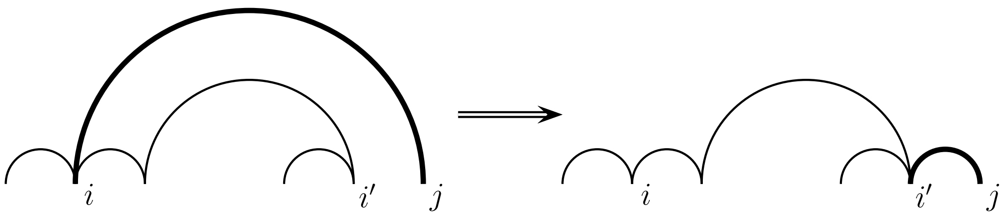

Figure 5: From trees to interaction diagrams

The (multivalued) inverse of the map σ is constructed similarly. The edge set E of a tree $T \in \sigma^{-1}(\mathcal{I}(k))$ is constructed inductively. Suppose all edges with endpoints less than $j \in K$ have already been constructed. Let $i' \to j$ be an arrow in $\mathcal{I}(k)$ with endpoint j. Then, we add to E an arbitrary edge (i, j) where the vertex i is connected to i' by the edges already in E. The last condition is equivalent to $i' \trianglelefteq i$.

Property (4.19) follows immediately from the construction of the map σ. $\square$

4.2.9

The diagrammatic technique for analyzing the term (4.13) can be formalized as follows

$$\mathsf{M}(w_1,\dots,w_m) = \sum_{S\subset\{1,\dots,m\}} \sum_{\mathfrak{I}} \operatorname{Contr}(S,\mathfrak{I})\,, \tag{4.20}$$

where $\operatorname{Contr}(S,\mathfrak{I})$ denotes the contribution of the interaction diagram $\mathfrak{I}$ to the transformation of term indexed by S in (4.6).

Proposition 4.1 is proven by finding a large cancellation in the expansion (4.20). We will call an interaction diagram $\mathfrak{I}$ *finite* if no interaction with infinity occur. The terms in (4.20) with $S=\emptyset$ and finite $\mathfrak{I}$ will be called the *principal terms*. Proposition 4.1 follows from Lemmas 4.5 and 4.6 below.

Lemma 4.5. *The minimal t power in the sum of the principal terms is:*

$$t^m \sum_{\pi} \left(\overrightarrow{\prod_{k=1,\dots,\ell(\pi)}} \mathsf{A}^\vee(w_{\pi_k}) \right) e^{\alpha_1}$$

where the sum is over all partitions π of $\{1,\dots,m\}$.

Proof. Let $\mathfrak{I}$ be a finite interaction diagram and let k a vertex without outgoing edges (that is, a maximal element of the associated partial ordering $\trianglelefteq$). Let

$$K = \{k_1 < k_2 < \cdots < k_r = k\}\,, \tag{4.21}$$

be the vertices of $\mathfrak{I}(k)$. Since $k<\infty$, the procedure of Section 4.2.6 requires a selection of a nonminimal power of t from the operator (4.16). Using Lemma 4.4 we can sum the first nonminimal powers of t over all diagrams $\mathfrak{I}(k)$ with the vertex set K. This yields

$$t^r\, \mathsf{A}^\vee(w_{k_1},\dots,w_{k_r})\,, \tag{4.22}$$

completing the proof of the Lemma. □

Lemma 4.6. *The sum of the non-principal terms is 0.*

Proof. We explain the cancellation of non-principal terms corresponding to an interaction diagrams $\mathfrak{I}$ with exactly one infinite interaction,

$$k \to \infty. \tag{4.23}$$

The derivation of the general case of the cancellation is identical.

As before, let (4.21) denote the vertex set of $\mathcal{I}(k)$. In contrast to the case considered in Lemma 4.5, we now want to extract the *minimal* nonzero power of t from the operator (4.16). Summed over all $\mathcal{I}(k)$ with vertex set K, this gives

$$t^{r-1}\,\mathsf{T}(w_{k_1},\dots,w_{k_r})\,\mathsf{A}^0(w_{k_1}+\cdots+w_{k_r})\,. \tag{4.24}$$

After commuting past e^{α_1}, the operator (4.24) is transformed to

$$t^{r-1}\,\mathsf{T}(w_{k_1},\dots,w_{k_r})\,\mathcal{E}(w_{k_1}+\cdots+w_{k_r})\,. \tag{4.25}$$

This contribution of the set K is canceled by terms in the operator E occurring in the definition of M. Consider the summand of formula (4.6) indexed by

$$S' = S \cup K.$$

Consider the associated summand of $\mathsf{E}(w_S,-t)$ indexed by

$$\pi' = \pi \sqcup K.$$

It produces the identical result, the only difference being that the prefactor of the K contribution,

$$t^{|K|-1}\,(-1)^{|S|},$$

is replaced by the factor

$$(-t)^{|K|-1}\,(-1)^{|S'|}\,,$$

resulting in a cancellation. □

The following result is an immediate consequence of equation (4.5) and Proposition 4.1. Let

$$\mathsf{M}'(w_1,\dots,w_n) = \sum_{\pi}\left(\overrightarrow{\prod_{k=1,\dots,\ell(\pi)}}\mathsf{A}^\vee(w_{\pi_k})\right)e^{\alpha_1}\,. \tag{4.26}$$

Corollary 4.7.

$$\mathsf{G}'(z_1,\dots,z_n,w_1,\dots,w_m|\nu) = \left\langle \prod \mathsf{A}^0(z_i)\,\mathsf{M}'(w_1,\dots,w_m)\right|\nu\Big\rangle\,. \tag{4.27}$$

4.3 Analysis of the operator formula

4.3.1

We study here the operator formula (4.27) for G'. The operator M' is constructed from $\mathsf{A}^\vee$-operators which in turn are defined in terms of A^0-operators and A^1-operators. Individual relative invariants of the cap are obtained by extracting specific powers of the parameters z_i and w_j. Therefore, the relative invariants involve the operators A^0_i and A^1_i defined by

$$\mathsf{A}^k(z) = \sum_{i \in \mathbb{Z}} \mathsf{A}^k_i \, z^{i+1} \,, \quad k = 0, 1 \,. \tag{4.28}$$

We will prove a rule for the removal of the A^1-operators from formula (4.27). The result expresses the relative invariants of the cap purely in terms of A^0-operators. By the GW/H correspondence of [11], the vacuum expectations of A^0-operators coincide with the stationary invariants of the cap. We will see in Section 4.4 that the rule for removal of the A^1-operators is equivalent to Virasoro constraints for the cap.

4.3.2

Our strategy for removing the A^1-operators from formula (4.27) is the following. We commute all the A^1-operators, starting from the leftmost operator, to the left until they reach the vacuum, that is, until they reach the "$\langle$" symbol. Along the way, we will encounter commutators of A^0-operators with A^1-operators. As we shall see in Lemma 4.8, the commutators produce new A^0-operators. The commutators are related to the Virasoro reactions of type (iii).

Once a A^1-operator has reached the vacuum $v_\emptyset$, the A^1-operator is exchanged for a sum of A^0-operators. Since the vectors

$$\left(\prod \mathsf{A}^0_{m_i} \right)^* v_\emptyset \,, \quad m_1 \geq m_2 \geq \cdots \geq 0 \,, \tag{4.29}$$

span a basis of $\Lambda^{\frac{\infty}{2}} V_0$, we can express the vector $(\mathsf{A}^1_k)^* \, v_\emptyset$ as a linear combination of the vectors (4.29). The linear combination, described in Proposition 4.11, is related to the Virasoro reactions of type (iv) and (v) of Section 1.2.1.

4.3.3

We first study the commutation relation between the operators $\mathsf{A}^k(z)$, or, equivalently, their coefficients (4.28).

The commutation of the operators $\mathsf{A}^0(z)$ for different values of z can be seen, for example, from equation (4.2). Hence,

$$\big[\mathsf{A}^0_k, \mathsf{A}^0_l\big] = 0\,.$$

The commutation relation between $\mathsf{A}^0(z)$ and $\mathsf{A}^1(w)$ can be obtained by extracting the t^1 term from equation (4.14).

Lemma 4.8. *We have*

$$\big[\mathsf{A}^0(z), \mathsf{A}^1(w)\big] = zw\,\mathsf{A}^0(z+w)\,, \tag{4.30}$$

or, equivalently,

$$\big[\mathsf{A}^0_k, \mathsf{A}^1_l\big] = \binom{k+l}{k}\mathsf{A}^0_{k+l-1}\,, \quad k \geq 0\,. \tag{4.31}$$

While k is nonnegative in equation (4.31), $k+l-1$ may be negative. From the definitions,

$$\mathsf{A}^0_k = \begin{cases} 1\,, & k=-2\,,\\ 0\,, & k=-1,-3,-4,\dots\,, \end{cases}$$

see [11], [12] for a discussion. Since

$$\binom{-1}{k} = (-1)^k\,, \tag{4.32}$$

we find

$$\big[\mathsf{A}^0_k, \mathsf{A}^1_{-k-1}\big] = (-1)^k\,, \quad k \geq 0\,.$$

In equation (4.32) and below, we view

$$\binom{l}{k} = \frac{l(l-1)\dots(l-k+1)}{k!}$$

as a polynomial of degree k in l which can be evaluated at any l and differentiated with respect to l.

For the proof of Lemma 4.10, we will need the commutation relation between $\mathsf{A}^0(z)$ and $\mathsf{A}^2(w)$ obtained by extracting the t^2 term from equation (4.14).

Lemma 4.9. *We have*

$$\big[\mathsf{A}^0(z), \mathsf{A}^2(w)\big] = \frac{zw}{1+\frac{z}{w}}\, \mathsf{A}^1(z+w) + zw^2 \,\ln\left(1+\frac{z}{w}\right)\, \mathsf{A}^0(z+w)\,, \qquad (4.33)$$

or, equivalently,

$$\big[\mathsf{A}^0_k, \mathsf{A}^2_l\big] = \binom{k+l-1}{k}\, \mathsf{A}^1_{k+l-1} + \left[\frac{\partial}{\partial l}\binom{k+l-1}{k}\right]\, \mathsf{A}^0_{k+l-2}\,,\ \ k \geq 0. \quad (4.34)$$

Equation (4.33) is obtained from equation (4.34) by rewriting the coefficient of the second term,

$$\begin{aligned}
[z^{k+1}\, w^{l+1}]\, zw^2 \,\ln\left(1+\frac{z}{w}\right)\, (z+w)^{k+l-1} &= [z^k]\, \ln(1+z)\,(1+z)^{k+l-1} \\
&= \frac{\partial}{\partial l}\,[z^k]\,(1+z)^{k+l-1} \\
&= \frac{\partial}{\partial l}\binom{k+l-1}{k}.
\end{aligned}$$

Finally, we determine the commutation relations between A^1_k and A^1_l. The commutator of $\mathsf{A}(z)$ with $\mathsf{A}(w)$, computed in [12], is linear in t. Hence,

$$[t^2]\ [\mathsf{A}(z), \mathsf{A}(w)] = 0\,.$$

The following result is an immediate consequence.

Lemma 4.10. *We have*

$$\big[\mathsf{A}^1_k, \mathsf{A}^1_l\big] = \big[\mathsf{A}^0_l, \mathsf{A}^2_k\big] - \big[\mathsf{A}^0_k, \mathsf{A}^2_l\big]\ . \qquad (4.35)$$

In particular, for $k, l \geq 0$, we find

$$\left[\frac{\mathsf{A}^1_k}{k!}, \frac{\mathsf{A}^1_l}{l!}\right] = (k-l)\,\frac{\mathsf{A}^1_{k+l-1}}{(k+l-1)!} + \big(\dots\big)\,\mathsf{A}^0_{k+l-2}\,.$$

The first term on the right is the commutation relation of the Virasoro subalgebra $\mathcal{V}$ of the holomorphic vector fields on the line (with shifted index).

4.3.4

When an A^1-operator reaches the vacuum "$\langle$", we can exchange the A^1-operator for A^0-operators by the following result.

Proposition 4.11. *We have*

$$\left\langle \mathsf{A}^1_k \ldots \right\rangle = -\left(\sum_{j=1}^{k} \frac{1}{j}\right) \left\langle \mathsf{A}^0_{k-1} \ldots \right\rangle + \frac{1}{2} \sum_{i=0}^{k-3} \frac{(i+1)!\,(k-i-2)!}{k!} \left\langle \mathsf{A}^0_i\, \mathsf{A}^0_{k-i-3} \ldots \right\rangle \,, \quad (4.36)$$

where the dots stand for arbitrary A^0 *and* A^1*-operators.*

Proof. For $k \leq 0$, the vanishing of both sides of equation (4.36) is obvious. For $k = 1, 2, 3$, we can check equation (4.36) by an explicit calculation.

Assume equation (4.36) holds for $k \geq 3$. Then, using the commutation relation

$$\left[\mathsf{A}^1_2, \mathsf{A}^1_k\right] = \frac{(k+1)(2-k)}{2}\, \mathsf{A}^1_{k+1} + \left[(k+1) \sum_{j=2}^{k+1} \frac{1}{j} - k - \frac{1}{2}\right] \mathsf{A}^0_k \,,$$

we verify equation (4.36) holds for $k+1$. □

Lemma 4.8 and Proposition 4.11 together provide a rule for the systematic removal of A^1-operators from formula (4.27) for G'.

4.4 Virasoro constraints

4.4.1

To complete the proof of the Virasoro constraints for the cap, we must prove the formula,

$$\mathsf{G}'(z_1, \ldots, z_n, w_1, \ldots, w_m | \nu) = \left\langle \prod \mathsf{A}^0(z_i)\, \mathsf{M}'(w_1, \ldots, w_m) \middle|\, \nu \right\rangle \,, \quad (4.37)$$

implies the Virasoro constraints for the depth m theory. The Virasoro constraints provide rules for expressing the depth m theory in terms of the stationary theory. We will prove our rules for the removal of the A^1-operators from formula (4.37) yields precisely the *same* reduction of the depth m theory to the stationary theory. Therefore, the unique depth m theory determined by the Virasoro constraints from the stationary theory equals the depth m theory determined by formula (4.37).

4.4.2

Consider the reduction of the following relative Gromov-Witten invariant,

$$\langle \tau_{l_1}(\omega)\, \tau_{l_2}(\omega) \ldots \tau_{k_1}(1)\, \tau_{k_2}(1)\, \tau_{k_3}(1) \ldots |\nu\, \rangle \ , \tag{4.38}$$

via the Virasoro constraints. We apply the Virasoro reactions for the removal of the insertions $\tau_k(1)$ successively from left to right starting with $\tau_{k_1}(1)$. Of course, the ordering in the bracket (4.38) is immaterial. However, since we will interpret the Virasoro reactions in the noncommutative operator formalism of $\Lambda^{\frac{\infty}{2}} V$, the ordering will play a crucial role.

When an insertion $\tau_{k_i}(1)$ decays by interaction with another insertion $\tau_r(\gamma)$, the result is placed in the location in the bracket determined by $\tau_r(\gamma)$. When an insertion $\tau_{k_i}(1)$ decays alone, via Virasoro reactions of type (iv) or (v), the result replaces $\tau_{k_i}(1)$ in the bracket.

Virasoro reactions of type (iii) in the reduction of the relative invariant (4.38) can be separated into two subtypes. A type (iii.a) reaction takes place when an insertion $\tau_{k_i}(1)$ interacts with a insertion $\tau_r(\omega)$ occurring further *right* in the bracket. A type (iii.b) reaction takes place when an insertion $\tau_{k_i}(1)$ interacts with a insertion $\tau_r(\omega)$ occurring further *left* in the bracket.

For our analysis, we separate the Virasoro reactions in the reduction of the relative invariant (4.38) into two classes:

(I) Virasoro reactions of type (i), (ii), and (iii.a),

(II) Virasoro reactions of type (iii.b), (iv), and (v).

We will first analyze the class I interactions. The class II interactions will be considered afterwards. We will see the total result of all class I interactions is incorporated in formula (4.27). The class II interactions corresponds to the rules for removing the A^1-operators derived in Section 4.3.

4.4.3

We study here the total effect of all class I reactions on the bracket (4.38).

As a first step, we analyze the effects of type (i) reactions. Consider the following insertions,

$$\langle \dots \tau_{r_1}(1) \dots \tau_{r_2}(1) \dots \tau_{r_s}(1) \dots |\nu\rangle \,, \tag{4.39}$$

in the bracket (4.38). The position determined by the last insertion $\tau_{r_s}(1)$ will be called the *last* position. A sequence of type (i) reactions which cluster the insertions (4.39) yield a result proportional to the bracket

$$\Big\langle \dots \tau_{\sum r_i - s+1}(1) \dots |\nu \Big\rangle \,, \tag{4.40}$$

with the new insertion placed in the last position.

Lemma 4.12. *After summation over all possible sequences of type (i) reactions which cluster the insertions,*

$$\langle \dots \tau_{r_1}(1) \dots \tau_{r_2}(1) \dots \tau_{r_s}(1) \dots |\nu\rangle \to \binom{\sum r_i - 1}{r_1, \dots, r_{s-1}, r_s - 1} \Big\langle \dots \tau_{\sum r_i - s+1}(1) \dots |\nu \Big\rangle \,. \tag{4.41}$$

Proof. The Lemma is easily obtained from the elementary identity,

$$\binom{\sum r_i - 1}{r_1, \dots, r_{s-1}, r_s - 1} = \binom{\sum r_i - 2}{r_2, \dots, r_{s-1}, r_1 + r_s - 2} \binom{r_1 + r_s - 1}{r_1} + \sum_{j=2}^{s-1} \binom{\sum r_i - 2}{r_2, \dots, r_1 + r_j - 1, r_{s-1}, r_s - 1} \binom{r_1 + r_j - 1}{r_1} \,,$$

by induction. □

For fixed values $r_1, \dots, r_{s-1}$, the coefficient in formula (4.41) and the coefficients of the individual type (i) reactions are polynomials in r_s. Hence, we may differentiate the coefficient of formula (4.41) with respect to the variable r_s,

$$\frac{\partial}{\partial r_s} \binom{\sum r_i - 1}{r_1, \dots, r_{s-1}, r_s - 1}. \tag{4.42}$$

The differentiation (4.42) provides a summation of over all Virasoro reactions involving the insertions (4.40) of the following form: a sequence of type (i) interactions, followed by a single type (ii) interaction *involving the last position*, followed by a sequence of type (iii.a) interactions.

The coefficient in formula (4.41) represents a sum over all type (i) interactions. The differentiation (4.42) can be analyzed by considering the individual reactions. From each sequence of type (i) reactions involving the insertions (4.40), the differentiation singles out a particular one *involving the last position* and transforms the interaction coefficient by:

$$\binom{a+b-1}{a} \mapsto \frac{\partial}{\partial b}\binom{a+b-1}{a} = \binom{a+b-1}{a}\left[\frac{1}{b}+\cdots+\frac{1}{a+b-1}\right].$$

where a and b stand for disjoint sums of the numbers r_i with r_s occurring in b.

Lemma 4.12 and our interpretation of the differentiation (4.42) yield a proof of the following result.

Proposition 4.13. *After summation over all sequences of class I interactions which cluster the insertions*

$$\left\langle \dots \tau_{r_1}(1) \dots \tau_{r_2}(1) \dots \tau_{r_s}(1) \dots \right| \nu \rangle ,$$

we obtain

$$\binom{\sum r_i - 1}{r_1, \dots, r_{s-1}, r_s - 1} \left\langle \dots \tau_{\sum r_i - s + 1}(1) \dots \right| \nu \rangle + \frac{\partial}{\partial r_s} \binom{\sum r_i - 1}{r_1, \dots, r_{s-1}, r_s - 1} \left\langle \dots \tau_{\sum r_i - s}(\omega) \dots \right| \nu \rangle . \quad (4.43)$$

4.4.4

Proposition 4.13 has a nice interpretation in terms of generating functions. First, we have

$$\sum_{r_1,\dots,r_s \geq 0} \binom{\sum r_i - 1}{r_1, \dots, r_{s-1}, r_s - 1} \tau_{\sum r_i - s + 1}(1) \prod w_i^{r_i+1} = \mathsf{T}(w_1, \dots, w_s) \frac{w_s}{w_1 + \cdots + w_s} \sum_{k \geq 0} \tau_k(1) \left(\sum w_i\right)^{k+1} ,$$

which precisely matches the form of the A^1-term in $\mathsf{A}^\vee$. Second, we have

$$\frac{\partial}{\partial r_s}\binom{\sum r_i - 1}{r_1, \ldots, r_{s-1}, r_s - 1}$$
$$= \frac{\partial}{\partial r_s}\left[w_1^{r_1} w_2^{r_2} \ldots w_{s-1}^{r_{s-1}}\right]\left(\sum_{i=1}^{s-1} w_i + 1\right)^{\sum_1^s r_i - 1}$$
$$= \left[w_1^{r_1} w_2^{r_2} \ldots w_{s-1}^{r_{s-1}}\right] \ln\left(1 + \sum_{i=1}^{s-1} w_i\right)\left(\sum_{i=1}^{s-1} w_i + 1\right)^{\sum_1^s r_i - 1}$$
$$= \left[w_1^{r_1} w_2^{r_2} \ldots w_s^{r_s - 1}\right] \ln\left(1 + \frac{\sum_1^{s-1} w_i}{w_s}\right)\left(\sum_{i=1}^{s} w_i\right)^{\sum_1^s r_i - 1}.$$

Therefore,

$$\sum_{r_1,\ldots,r_s \geq 0} \frac{\partial}{\partial r_s}\binom{\sum r_i - 1}{r_1, \ldots, r_{s-1}, r_s - 1} \tau_{\sum r_i - s}(\omega) \prod w_i^{r_i+1} =$$
$$\mathsf{T}(w_1, \ldots, w_s)\, w_s \ln\left(1 + \frac{\sum_1^{s-1} w_i}{w_s}\right) \sum_{k \geq 0} \tau_k(\omega) \left(\sum w_i\right)^{k+1},$$

which precisely matches the form of the A^0 term of $\mathsf{A}^\vee$.

4.4.5

Consider the generating function of depth m relative invariants of the cap,

$$\mathsf{G}'(z_1, \ldots, z_n, w_1, \ldots, w_m|\nu) =$$
$$\sum_{k_i \geq 0} \sum_{l_j \geq 0} \prod_{i=1}^{n} z_i^{k_i+1} \prod_{j=1}^{m} w_j^{l_j+1} \left\langle \prod \tau_{k_i}(\omega) \prod \tau_{l_j}(1) \middle| \nu \right\rangle.$$

We reduce the relative invariants on the right via all Virasoro reactions of class I. By the results of Sections 4.4.3 and 4.4.4, the outcome exactly equals the operator formula,

$$\left\langle \prod \mathsf{A}^0(z_i)\, \mathsf{M}'(w_1, \ldots, w_m) \middle| \nu \right\rangle, \tag{4.44}$$

after the following substitutions,

$$\tau_k(1) \to \mathsf{A}^1_k\,, \quad \tau_k(\omega) \to \mathsf{A}^0_k\,.$$

To complete the Virasoro reduction of the depth m theory to the stationary theory, we must apply Virasoro reactions (iii.b), (iv), and (v). By Lemma 4.8 and Proposition 4.11, these Virasoro reactions exactly correspond to the rules for the removal of the A^1-operators from formula (4.44).

We have proven the reduction of the depth m theory of the cap to the stationary theory by the Virasoro constraints *equals* the reduction of formula (4.44) by our operator methods. Hence, the Virasoro constraints for the cap are proven. □

By Proposition 1.5, the proof of the even Virasoro constraints for the relative theories of all target curves is complete. The treatment of the odd classes will be presented in Sections 5 and 6.

5 Odd classes

5.1 Overview

The even relative Gromov-Witten theory of target curves X is completely determined by the GW/H correspondence and the even Virasoro constraints.

The full relative Gromov-Witten theory of target curves includes the descendents of both even and odd cohomology classes. We will prove the full relative theory of target curves is uniquely determined from the even theory by the following four properties:

(i) Algebraicity of the virtual class,

(ii) Degeneration formulas for the relative theory in the presence of odd cohomology,

(iii) Monodromy invariance of the relative theory,

(iv) Elliptic vanishing relations.

The Virasoro constraints for the full theory are proven by establishing their compatibility with the above properties (i)-(iv).

5.2 Elliptic invariants

Let E be an elliptic target with a relative point e. Let

$$\alpha \in H^{1,0}(E, \mathbb{C}),$$

$$\beta \in H^{0,1}(E, \mathbb{C}),$$

span a symplectic basis with

$$\int_E \alpha \cup \beta = 1.$$

Consider the set of relative elliptic invariants with odd insertions and *without* descendents of a point:

$$\left\langle \prod_{h \in H} \tau_{o_h}(1) \prod_{i \in I} \tau_{n_i}(\alpha) \prod_{j \in J} \tau_{m_j}(\beta) \,\middle|\, \eta \right\rangle^E . \tag{5.1}$$

The above invariant is defined by integration against

$$[\overline{M}_{g,n}(E,\eta)]^{vir}.$$

Since, by property (i), the virtual class is algebraic, the invariant (5.1) vanishes if $|I| \neq |J|$. The balance $|I| = |J|$ is the *only* consequence of the algebraicity which will be used. Since the bracket is skew-symmetric in the odd insertions, we require I and J to be ordered sets to fix the sign.

Proposition 5.1. *The full relative theory of target curves is uniquely determined from the even theory by the elliptic invariants* (5.1) *and the degeneration property (ii).*

Proof. Consider a relative Gromov-Witten invariant on a target curve X of genus g,

$$\left\langle \prod \tau_{r_i}(\gamma_i) \,|\, \eta^1, \ldots, \eta^m \right\rangle^X . \tag{5.2}$$

If X is rational, the theory is even.

If $g > 0$, we may degenerate X to a rational curve with g elliptic tails, see Fig. 6. We may specialize the relative points of X and the descendents

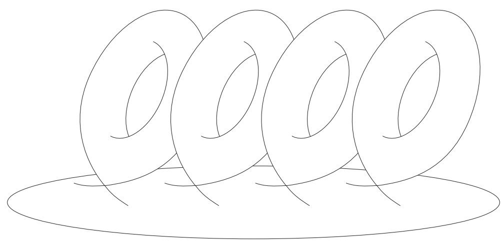

Figure 6: A rational curve with 4 elliptic tails

$\tau_r(\omega)$ to the rational component.

The degeneration does not alter $H^1(X,\mathbb{C})$. The odd cohomology of X can be written in terms of a symplectic basis which is the union of the bases of the odd cohomologies of the elliptic components. The descendents of the odd basis elements then specialize to the corresponding elliptic factors.

By the degeneration formula, the original invariant (5.2) is expressed in terms of the relative invariants of the degenerate components after all possible distributions of the descendents $\tau_r(1)$. Hence, the invariant (5.2) is expressed purely in terms of relative invariants of the rational component and invariants of type (5.1) of the elliptic components. $\square$

After introducing the monodromy and elliptic vanishing relations in Sections 5.3 and 5.4, we will prove the following uniqueness result in Section 5.5.

Proposition 5.2. *The elliptic invariants* (5.1) *are uniquely determined from the even theory by properties (i-iv).*

Together, Propositions 5.1 and 5.2 show the full relative theory of target curves is uniquely determined from the even theory by properties (i-iv).

5.3 Monodromy relations

5.3.1

We will now find relations for the absolute Gromov-Witten theory of an elliptic target E obtained from the monodromy invariance property (iii).

Using the moduli of elliptic curves, we can find a monodromy transformation on $H^1(E,\mathbb{C})$ satisfying:

$$\alpha \mapsto \alpha, \quad \beta \mapsto \alpha + \beta. \tag{5.3}$$

In fact, the monodromy group is $SL_2(\mathbb{Z})$, but we will only require the above transformation.

Let Ψ denote the set $\{\psi^0, \psi^1, \psi^2, \ldots\}$. Let I and J be disjoint ordered index sets such that $|I| > 0$ and $|I| = |J|$. Let

$$\mathbf{n} : I \to \Psi, \quad i \mapsto \psi^{n_i},$$

$$\mathbf{m} : J \to \Psi, \quad j \mapsto \psi^{m_j},$$

be descendent assignments.

Let $\delta \subset I$ be a subset. Let $S(\delta)$ denote the set of subsets of $I \cup J$ of cardinality $|I|$ containing δ. For $D \in S(\delta)$, Let

$$\tau_{\mathbf{n},\mathbf{m}}(D) = \prod_{i\in I} \tau_{n_i}(\gamma_i^D) \prod_{j\in J} \tau_{m_j}(\gamma_j^D).$$

Here, for $\xi \in I \cup J$,

$$\gamma_\xi^D = \alpha \text{ or } \beta$$

if $\xi \in D$ or $\xi \notin D$ respectively.

The monodromy invariant monomial insertion,

$$N = \prod_{h\in H} \tau_{o_h}(1) \prod_{h'\in H'} \tau_{o'_{h'}}(\omega),$$

will be an idle prefactor in the relations below.

Proposition 5.3. *For every proper subset $\delta \subset I$, the descendent relation,*

$$\sum_{D\in S(\delta)} \langle N\ \tau_{\mathbf{n},\mathbf{m}}(D)\rangle_d^E = 0,$$

holds for the Gromov-Witten theory of E.

Proof. The proof is a straightforward application of the monodromy transformation (5.3). Certainly, the invariant

$$\left\langle N \prod_{i\in I} \tau_{n_i}(\gamma_i^\delta) \prod_{j\in J} \tau_{m_j}(\beta) \right\rangle_d^E \tag{5.4}$$

vanishes due to the imbalance of the odd insertions (since δ is a proper subset of I). The Proposition is obtained by simply applying transformation (5.3) to the vanishing invariant (5.4). □

Let $R_d(N, \mathbf{n}, \mathbf{m}, \delta)$ denote the monodromy relation of Proposition 5.3:

$$\sum_{D\in S(\delta)} \langle N\ \tau_{\mathbf{n},\mathbf{m}}(D)\rangle_d^E = 0. \tag{5.5}$$

5.3.2

We will require a formal generalization of the monodromy relation obtained in Section 5.3.1.

Let $\Psi_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space with basis given by the set Ψ. Let functions $\mathbf{n}, \mathbf{m}$ take values in $\Psi_{\mathbb{Q}}$.

$$\mathbf{n} : I \to \Psi_{\mathbb{Q}}, \quad \mathbf{m} : J \to \Psi_{\mathbb{Q}}.$$

Instead of simply assigning to each marking in $i \in I$ a descendent ψ^{n_i}, the function $\mathbf{n}$ assigns to i a finite linear combination,

$$i \mapsto c_0^i \psi^0 + c_1^i \psi^1 + c_2^i \psi^2 + \dots .$$

Similarly, the function $\mathbf{m}$ assigns to j a finite linear combination,

$$j \mapsto c_0^j \psi^0 + c_1^j \psi^1 + c_2^j \psi^2 + \dots .$$

A richer monodromy relation $R_d(N, \mathbf{n}, \mathbf{m}, \delta)$ is defined by setting

$$\tau_{\mathbf{n},\mathbf{m}}(D) = \prod_{i \in I} \Big(\sum_{q \geq 0} c_q^i \tau_q(\gamma_i^D) \Big) \prod_{j \in J} \Big(\sum_{q \geq 0} c_q^j \tau_q(\gamma_j^D) \Big)$$

on the right side of equation (5.5). The richer relation $R_d(N, \mathbf{n}, \mathbf{m}, \delta)$ is proven by expanding and using Proposition 5.3.

5.4 Elliptic vanishing relations

We present here geometric vanishing relations which constrain the absolute Gromov-Witten theory of E.

Let K be an ordered index set satisfying $|K| > 0$. Let P be a set partition of K with parts of size at least 2. Let $P_1, \dots, P_\ell$ be the parts of P.

Let $\overline{M}_{g,S}(E, d)$ be a moduli space of stable maps with possibly disconnected domains for which the marking set S contains K. Let

$$\phi_i : \overline{M}_{g,S}(E, d) \to E^{|P_i|}$$

be the product evaluation map determined by the ordered part P_i. Let

$$\mathbf{l} : K \to \Psi, \quad k \mapsto \psi^{l_k},$$

be a descendent assignment.

The *small diagonal* of the r-fold product E^r is defined by:

$$\{(x, \dots, x) \mid x \in E\} \subset E^r.$$

Let $\triangle_r \in H^*(E^r, \mathbb{C})$ denote the Poincaré dual of the small diagonal.

The monomial insertion of descendents of the identity,

$$M = \prod_{h \in H} \tau_{o_h}(1),$$

will be an idle prefactor in the elliptic vanishing relations below. The descendents $\tau_k(\omega)$ do *not* appear in M.

Proposition 5.4. *The elliptic vanishing relation $V_d(M, P, \mathbf{l})$ holds:*

$$\int_{[\overline{M}_{g,H+K}(E,d)]^{vir}} \prod_{h\in H} \psi_h^{o_h} \prod_{k\in K} \psi_k^{l_k} \prod_{i=1}^{\ell} \phi_i^*(\triangle_{|P_i|}) \quad = 0.$$

While the Proposition is true for all g, the vanishing is trivial unless g is determined from the rest of the data by the dimension constraint.

Proof. The integral is proven to vanish in two steps. First, the virtual fundamental class is analyzed. Second, the integrand is analyzed. A similar elliptic vanishing is proven by the same method in [13].

The moduli space of maps $\overline{M}_{g,H+K}(E,d)$ is equipped with an algebraic translation action of the elliptic curve E. There exists an algebraic quotient of this free action:

$$\overline{M}_{g,H+K}(E,d)/E = \mathrm{ev}_\xi^{-1}(0) \subset \overline{M}_{g,H+K}(E,d),$$

where ξ is any marking and $0 \in E$ is the neutral element. In fact $\overline{M}_{g,H+K}(E,d)$ is E-equivariantly isomorphic to a product of E by the quotient.

The virtual fundamental class of $\overline{M}_{g,H+K}(E,d)$ is pulled-back from the quotient $\overline{M}_{g,H+K}(E,d)/E$. The pull-back property is obtained easily from the construction of the virtual fundamental class. Since no $\tau_k(\omega)$ insertions are allowed, the integrand is also pulled-back from the quotient space. Hence, we may use the push-pull formula to conclude the integral vanishes. □

The elliptic vanishing relations can be expressed in terms of the absolute Gromov-Witten theory of E via the Künneth decompositions of the classes $\triangle_r$. The Künneth decompositions of $\triangle_2$ and $\triangle_3$ are

$$\triangle_2 = 1 \otimes \omega + \omega \otimes 1 - \alpha \otimes \beta + \beta \otimes \alpha.$$

$$\begin{aligned} \triangle_3 \;\; = \;\; & 1 \otimes \omega \otimes \omega + \omega \otimes 1 \otimes \omega + \omega \otimes \omega \otimes 1 \\ & -\alpha \otimes \beta \otimes \omega + \beta \otimes \alpha \otimes \omega \\ & -\alpha \otimes \omega \otimes \beta + \beta \otimes \omega \otimes \alpha \\ & -\omega \otimes \alpha \otimes \beta + \omega \otimes \beta \otimes \alpha. \end{aligned}$$

We will separate the Künneth components of $\triangle_r$ into two groups,

$$\triangle_r = \triangle_r^{even} + \triangle_r^{odd}.$$

The summand $\triangle_r^{even}$ consists of r terms in which the identity class and $r-1$ copies of ω are tensored in all distinct orders,

$$\triangle_r^{even} = 1 \otimes \omega^{r-1} + \ldots + \omega^{r-1} \otimes 1.$$

The summand $\triangle_r^{odd}$ consists of $2\binom{r}{2}$ terms. For each pair of indices $i < j$ two terms occur:

- $-\alpha$ in the ith factor, β in the jth factor and $r-2$ copies of ω in all the other tensor factors,
- β in the ith factor, α in the jth factor and $r-2$ copies of ω in all the other tensor factors.

We will be primarily interested in the summand $\triangle_r^{odd}$.

The simplest example occurs when $|K| = 2$ and P has one part. After expanding $V_d(M, P, \mathbf{l})$ using the Künneth decomposition of $\triangle_2$, we find

$$\begin{aligned} & \langle M\ \tau_{l_1}(1)\tau_{l_2}(\omega)\rangle_d^E + \langle M\ \tau_{l_1}(\omega)\tau_{l_2}(1)\rangle_d^E \\ - & \langle M\ \tau_{l_1}(\alpha)\tau_{l_2}(\beta)\rangle_d^E + \langle M\ \tau_{l_1}(\beta)\tau_{l_2}(\alpha)\rangle_d^E = 0. \end{aligned}$$

Descendents of the odd cohomology of E appear via the summand $\triangle_2^{odd}$.

The function $\mathbf{l}$ may take more general values for the elliptic vanishing relation:

$$\mathbf{l} : K \to \Psi_{\mathbb{Q}}, \quad k \mapsto c_0^k\psi^0 + c_1^k\psi^1 + c_2^k\psi^2 + \ldots.$$

The relation $V_d(M, P, \mathbf{l})$ is well-defined and true in the richer context.

5.5 Proof of Proposition 5.2

5.5.1

We must determine the relative elliptic invariants,

$$\left\langle \prod_{h\in H} \tau_{o_h}(1) \prod_{i\in I} \tau_{n_i}(\alpha) \prod_{j\in J} \tau_{m_j}(\beta) \mid \eta \right\rangle_d^E, \tag{5.6}$$

from the even theory by properties (i-iv) of Section 5.1.

By property (i), the invariants vanish unless $|I| = |J|$. We will proceed by induction on $|I|$.

If $|I| = 0$, then the invariant is even. We will start with a proof of Proposition 5.2 in case $|I| = 1$. The method for $|I| = 1$ will be generalized in Section 5.5.3 to establish the induction step.

Lemma 5.5. *The elliptic invariants* (5.6) *where* $|I| = 1$ *are uniquely determined from the even theory by properties (i)-(iv).*

For the proof of Lemma 5.5, we will require an auxiliary result derived from the GW/H correspondence.

Let $\mathcal{P}(d)$ be the set of partitions of d. Let $\mathbb{Q}^{\mathcal{P}(d)}$ denote the linear space of functions from $\mathcal{P}(d)$ to $\mathbb{Q}$. Let

$$\tilde{\tau}(\omega) = \sum_{q=0}^{\infty} c_q \tau_q(\omega)$$

be a *finite* linear combination of descendent of ω. For $v \geq 0$, define a function on $\mathcal{P}(d)$ by:

$$\gamma_v : \mathcal{P}(d) \to \mathbb{Q}, \quad \eta \mapsto \langle \tilde{\tau}(\omega)^v \mid \eta \rangle_d^{\mathbf{P}^1}.$$

The above bracket is defined by a multilinear expansion of the insertion $\tilde{\tau}(\omega)^v$.

Lemma 5.6. *For* $d \geq 0$*, there exists a linear combination* $\tilde{\tau}(\omega)$ *for which the set of functions,*

$$\{\gamma_0, \gamma_1, \gamma_2, \dots\},$$

spans $\mathbb{Q}^{\mathcal{P}(d)}$*.*

Proof. The formula,

$$\gamma_v(\eta) = \sum_{|\lambda|=d} \left(\frac{\dim\lambda}{d!} \right)^2 \left(\sum_{q=0}^{\infty} c_q \frac{\mathbf{p}_{q+1}(\lambda)}{(q+1)!} \right)^v \mathbf{f}_\eta(\lambda),$$

is a direct consequence of the GW/H correspondence [11].

Since $\mathbf{f}_\eta(\lambda)$ is proportional to the character of the conjugacy class C_η in the representation λ of the symmetric group S_d,

$$\mathbf{f}_\eta(\lambda) = |C_\eta| \frac{\chi_\eta^\lambda}{\dim\lambda},$$

the functions,

$$\eta \mapsto \mathbf{f}_\eta(\lambda),$$

span $\mathbb{Q}^{\mathcal{P}(d)}$ as λ varies.

To prove the Lemma, we require a $\tilde{\tau}(\omega)$ for which the functions,

$$\lambda \mapsto \left(\sum_{q=0}^{\infty} c_q \frac{\mathbf{p}_{q+1}(\lambda)}{(q+1)!} \right)^v,$$

span $\mathbb{Q}^{\mathcal{P}(d)}$ as v varies. By the Vandermonde determinant, we need only find a $\tilde{\tau}(\omega)$ for which the values

$$\sum_{q=0}^{\infty} c_q \frac{\mathbf{p}_{q+1}(\lambda)}{(q+1)!}$$

are distinct as λ varies in $\mathcal{P}(d)$.

Since λ is a partition of d, we may write

$$\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0.$$

We may recover λ from the set:

$$\lambda_1 - 1 + \frac{1}{2}, \lambda_2 - 2 + \frac{1}{2}, \ldots, \lambda_d - d + \frac{1}{2}. \tag{5.7}$$

On $\mathcal{P}(d)$, the function $\mathbf{p}_1$ is easily evaluated to yield a nonzero constant. By definition, the functions $\mathbf{p}_{q+1}(\lambda)$ are (up to nonzero constants) the $q+1$-power sums of the elements (5.7). Since the functions $\mathbf{p}_1, \mathbf{p}_2, \ldots$ include all the power sums, their values separate elements of $\mathcal{P}(d)$. Since $\mathcal{P}(d)$ is a finite set, we may find a finite linear combination of elements $\mathbf{p}_1, \mathbf{p}_2, \ldots$ which separate the set. $\square$

We now prove Lemma 5.5. We will start by proving the invariants

$$\langle \tau_n(\alpha)\tau_m(\beta) \mid \eta \rangle_d^E \tag{5.8}$$

are determined from the even theory by properties (i)-(iv).

Let $d \geq 0$. Let $\tilde{\psi} = \sum_{q \geq 0} c_q \psi^q$, where

$$\tilde{\tau}(\omega) = \sum_{q \geq 0} c_q \tau_q(\omega)$$

satisfies the conditions of Lemma 5.6 for d.

Let $v \geq 0$. Let K_v be an ordered index set with $v+2$ elements. Let P be the set partition of K_v with one part. Let the descendent assignment $\mathbf{l}$ on K_v take the value $\tilde{\psi}$ for all elements of K_v.

Consider the elliptic vanishing relation $V_d(1, P, \mathbf{l})$. The terms of $V_d(1, P, \mathbf{l})$ which contain odd classes from the Künneth decomposition of $\triangle_{v+2}$ are easily seen to equal:

$$\begin{aligned} - \quad & \binom{v+2}{2} \langle \tilde{\tau}(\omega)^v \ \tilde{\tau}(\alpha)\tilde{\tau}(\beta)\rangle_d^E \\ + \quad & \binom{v+2}{2} \langle \tilde{\tau}(\omega)^v \ \tilde{\tau}(\beta)\tilde{\tau}(\alpha)\rangle_d^E . \end{aligned}$$

After an application of the monodromy relation $R_d(\tilde{\tau}(\omega)^v, \{\tilde{\psi}\}, \{\tilde{\psi}\}, \emptyset)$, we may rewrite the odd terms as:

$$-2 \quad \binom{v+2}{2} \langle \tilde{\tau}(\omega)^v \ \tilde{\tau}(\alpha)\tilde{\tau}(\beta)\rangle_d^E .$$

As the remainder of the relation $V_d(1, P, \mathbf{l})$ consists of terms with only even descendent insertions, we may conclude the invariants

$$\langle \tilde{\tau}(\omega)^v \ \tilde{\tau}(\alpha)\tilde{\tau}(\beta)\rangle_d^E \tag{5.9}$$

are determined for all v.

We now study the invariants (5.9) via the degeneration formula:

$$\langle \tilde{\tau}(\omega)^v \ \tilde{\tau}(\alpha)\tilde{\tau}(\beta)\rangle_d^E = \sum_{|\eta|=d} \langle \tilde{\tau}(\alpha)\tilde{\tau}(\beta) \mid \eta\rangle_d^E \ \mathfrak{z}(\eta) \ \langle \eta \mid \tilde{\tau}(\omega)^v\rangle_d^{\mathbf{P}^1} , \tag{5.10}$$

where $\mathfrak{z}(\eta) = |\mathrm{Aut}(\eta)| \prod_i \eta_i$. Here, E degenerates to a nodal target

$$E \cup \mathbf{P}^1.$$

The v markings corresponding to the insertions $\tilde{\tau}(\omega)$ specialize to the component $\mathbf{P}^1$ in the degeneration.

We have seen the left side of (5.10) is determined for all v from the even theory by conditions (i)-(iv). The invariants

$$\langle \tilde{\tau}(\alpha)\tilde{\tau}(\beta) \mid \eta\rangle_d^E \tag{5.11}$$

are then uniquely determined by Lemma 5.6.

Let L be an arbitrary monomial in the descendents of ω,

$$L = \prod_{h' \in H'} \tau_{o'_{h'}}(\omega).$$

By the degeneration formula, the invariants

$$\langle L \ \tilde{\tau}(\alpha)\tilde{\tau}(\beta)\rangle^E_d \tag{5.12}$$

are determined by the invariants (5.11) and the even relative theory of $\mathbf{P}^1$.

Let K_v and P be as before. Let $\mathbf{l}_f$ take the value ψ^n on the first element of K_v and the value $\tilde{\psi}$ on the following elements. Consider the elliptic vanishing relation $V_d(1, P, \mathbf{l}_f)$. The terms of $V_d(1, P, \mathbf{l}_f)$ which contain odd classes from the Künneth decomposition are:

$$\begin{aligned}
- & \binom{v+1}{1} \langle \tilde{\tau}(\omega)^v \ \tau_n(\alpha)\tilde{\tau}(\beta)\rangle^E_d \\
+ & \binom{v+1}{1} \langle \tilde{\tau}(\omega)^v \ \tau_n(\beta)\tilde{\tau}(\alpha)\rangle^E_d \\
- & \binom{v+1}{2} \left\langle \tilde{\tau}(\omega)^{v-1}\tau_n(\omega) \ \tilde{\tau}(\alpha)\tilde{\tau}(\beta)\right\rangle^E_d \\
+ & \binom{v+1}{2} \left\langle \tilde{\tau}(\omega)^{v-1}\tau_n(\omega) \ \tilde{\tau}(\beta)\tilde{\tau}(\alpha)\right\rangle^E_d .
\end{aligned}$$

By the determination (5.12), only the first two terms need be analyzed. After application of the monodromy relation

$$R_d(\tilde{\tau}(\omega)^v, \{ \ \psi^n\}, \{\tilde{\psi}\}, \{1\}),$$

the first two odd terms equal:

$$-2 \ \binom{v+1}{1} \langle \tilde{\tau}(\omega)^v \ \tau_n(\alpha)\tilde{\tau}(\beta)\rangle^E_d .$$

As the remainder of the relation $V_d(1, P, \mathbf{l}_f)$ consists of even terms, we may conclude the invariants

$$\langle \tilde{\tau}(\omega)^v \ \tau_n(\alpha)\tilde{\tau}(\beta)\rangle^E_d$$

are determined for all v.

Now, by degeneration and Lemma 5.6, as before, we find the invariants,

$$\langle \tau_n(\alpha)\tilde{\tau}(\beta) \mid \eta\rangle_d^E\,, \quad \langle L\ \tau_n(\alpha)\tilde{\tau}(\beta)\rangle_d^E\,,$$

are determined.

Similarly, by studying the elliptic vanishing relation $V_d(1, P, \mathbf{l}_l)$ for the function $\mathbf{l}_l$ which takes the value ψ^m on the last element of K_v and $\tilde{\psi}$ on the preceding elements, we find the invariants,

$$\langle \tilde{\tau}(\alpha)\tau_m(\beta) \mid \eta\rangle_d^E\,, \quad \langle L\ \tilde{\tau}(\alpha)\tau_m(\beta)\rangle_d^E\,,$$

are determined.

Finally, we study the elliptic vanishing relations $V_d(1, P, \mathbf{l}_{fl})$ where the function $\mathbf{l}_{fl}$ takes the value $\psi^n, \psi^m, \tilde{\psi}$ on the first, last, and remaining elements of K_v respectively. We then find the invariants,

$$\langle \tau_n(\alpha)\tau_m(\beta)\ |\eta\rangle_d^E\,,$$

are determined.

To conclude the proof of Proposition 5.5, we must show the invariants

$$\langle \eta \mid M\ \tau_n(\alpha)\tau_m(\beta)\rangle_d^E$$

are determined for every monomial $M = \prod_{h\in H} \tau_{o_h}(1)$.

We proceed by induction on the degree of M. The degree 0 case has already been established. If $\deg(M) > 0$, we observe that M is a spectator in both the monodromy and elliptic vanishing relations. Hence, we may repeat the above argument based upon the elliptic vanishing relations

$$V_d(M, P, \mathbf{l}),\ V_d(M, P, \mathbf{l}_f),\ V_d(M, P, \mathbf{l}_l),\ V_d(M, P, \mathbf{l}_{fl}),$$

where the definitions of the functions $\mathbf{l}$, $\mathbf{l}_f$, $\mathbf{l}_l$, and $\mathbf{l}_{fl}$ on K_v are unchanged.

The only difference occurs in the degeneration formulas. Here, we must sum over all possible distributions of M over the degenerate components. However, if any factors of M are distributed to the component $\mathbf{P}^1$, the resulting $\tau_k(1)$ monomial on the component E will have strictly lower degree. Consequently, the terms in which factors of M are distributed to $\mathbf{P}^1$ are inductively determined. Hence, we need only consider terms in the degeneration formula for which the entire monomial M remains on the component E. Then, the induction step proceeds exactly as the degree 0 case. □

5.5.2

We will derive consequences of the monodromy and elliptic vanishing relations needed for the $|I|$ induction in the proof of Proposition 5.2.

Let I, J be disjoint ordered index sets satisfying $|I| = |J|$. Let $\mathbf{n}, \mathbf{m}$ be functions,

$$\mathbf{n} : I \to \Psi, \quad \mathbf{m} : J \to \Psi.$$

Let $K = I \cup J$. We order K by placing I before J. Let

$$\mathbf{l} : K \to \Psi$$

be determined by $\mathbf{n}, \mathbf{m}$.

We will consider two types of relations. Let $\sigma : I \to J$ be a bijection. Let P_σ be the set partition of K into doublets given by $\{i, \sigma(i)\}$. First, as σ varies, we find $|I|!$ relations,

$$V_d(M, P_\sigma, \mathbf{l}), \tag{5.13}$$

where $M = \prod_{h \in H} \tau_{o_h}(1)$ is a fixed monomial. Second, we have all the monodromy relations,

$$R_d(M, \mathbf{n}, \mathbf{m}, \delta), \tag{5.14}$$

for proper subsets $\delta \subset I$.

Lemma 5.7. *The relations* (5.13) *and* (5.14) *determine the invariant,*

$$\left\langle M \prod_{i \in I} \tau_{n_i}(\alpha) \prod_{j \in J} \tau_{m_j}(\beta) \right\rangle_d^E,$$

in terms of degree d invariants of E with strictly fewer odd insertions.

Proof. Let $\delta \subset I$ be a subset, and let $S(\delta)$ be the set of subsets $I \cup J$ of cardinality $|I|$ containing δ. For $D \in S(\delta)$, let

$$\tau_{\mathbf{n},\mathbf{m}}(D) = \prod_{i \in I} \tau_{n_i}(\gamma_i^D) \prod_{j \in J} \tau_{m_j}(\gamma_j^D),$$

following the notation of Section 5.3. Let $S^*(\delta) \subset S(\delta)$ denote the set of subsets D satisfying $D \cap I = \delta$.

Since we are only interested in invariants with $|I|+|J|$ odd insertions, we need only analyze the odd splittings of the $|I|$ distinct Künneth decompositions in the relation $V_d(M, P_\sigma, \mathbf{l})$. Since I and J are ordered, the function σ is canonically an element of the symmetric group and therefore has a sign. We easily compute the sum of the terms of

$$\sum_{\sigma}(-1)^{\binom{|I|}{2}}\text{sign}(\sigma)\ V_d(M, P_\sigma, \mathbf{l}) \tag{5.15}$$

with $|I|+|J|$ odd parts equals:

$$\sum_{\delta\subset I}\ \sum_{D\in S^*(\delta)} (-1)^{|I|-|\delta|}|\delta|!(|I|-|\delta|)!\ \left\langle M\ \prod_{i\in I}\tau_{n_i}(\gamma_i^D)\prod_{j\in J}\tau_{m_j}(\gamma_j^D)\right\rangle_d^E. \tag{5.16}$$

The invariant $\left\langle M\ \prod_{i\in I}\tau_{n_i}(\alpha)\prod_{j\in J}\tau_{m_j}(\beta)\right\rangle_d^E$ occurs in the $\delta = I$ summand of (5.16) with coefficient $|I|!$ as the invariant appears exactly once in each summand of (5.15). Let V denote the sum (5.16).

For every $\ell < |I|$, let $R(\ell)$ denote the monodromy relation sum,

$$\sum_{|\delta|=\ell} R_d(M, \mathbf{n}, \mathbf{m}, \delta).$$

We may expand $R(\ell)$ as:

$$\sum_{|\delta|\geq\ell}\ \sum_{D\in S^*(\delta)} \binom{|\delta|}{\ell}\ \left\langle M\ \prod_{i\in I}\tau_{n_i}(\gamma_i^D)\prod_{j\in J}\tau_{m_j}(\gamma_j^D)\right\rangle_d^E\ =\ 0. \tag{5.17}$$

Using the relations $R(0), \ldots, R(|I|-1)$, we can uniquely eliminate all terms of V except for the $\delta = I$ term,

$$\left\langle M\ \prod_{i\in I}\tau_{n_i}(\alpha)\prod_{j\in J}\tau_{m_j}(\beta)\right\rangle_d^E, \tag{5.18}$$

which we hope to determine. If the coefficient of the term (5.18) is not 0 after elimination, then the Lemma is proven.

We abstract the linear algebra arising in the above elimination. Let $\mathbb{Q}^{|I|+1}$ be a vector space with basis $e_0, e_1, \ldots, e_{|I|}$. Here, e_k corresponds to the sum,

$$\sum_{|\delta|=k}\ \sum_{D\in S^*(\delta)}\ \left\langle M\ \prod_{i\in I}\tau_{n_i}(\gamma_i^D)\prod_{j\in J}\tau_{m_j}(\gamma_j^D)\right\rangle_d^E.$$

Then, V is the vector,

$$V = \sum_{k=0}^{|I|} (-1)^{|I|-k} k!(|I|-k)!\ e_k.$$

For $0 \le \ell \le |I|$, let

$$R(\ell) = \sum_{k \ge \ell} \binom{k}{\ell}\ e_k.$$

For $\ell < |I|$, the vector $R(\ell)$ is the corresponding the monodromy relation.

The vectors $R(0), \ldots, R(|I|)$ span a basis of $\mathbb{Q}^{|I|+1}$. Hence,

$$V = \sum_{\ell=0}^{|I|} c_\ell R(\ell),$$

for unique coefficients c_ℓ. The coefficient of $e_{|I|}$ obtained after the canonical elimination of V by the vectors $R(0), \ldots, R(|I|-1)$ is simply $c_{|I|}$.

The column vectors $R(\ell)$ determine an $|I| \times |I|$ lower unitriangular matrix R with coefficients

$$R_{ab} = \binom{a}{b}.$$

It is well known that R^{-1} has coefficients

$$(R^{-1})_{ab} = (-1)^{a+b} \binom{a}{b}.$$

Since the column vector $(c_0, \ldots, c_{|I|})$ is obtained by the product of R^{-1} with the column vector V, we find

$$c_{|I|} = \sum_{k=0}^{|I|} (-1)^{|I|+k} \binom{|I|}{k} (-1)^{|I|-k} k!(|I|-k)! = (|I|+1)! \ne 0\,.$$

The proof of the Lemma is complete. □

Lemma 5.7 is valid in case the function $\mathbf{n}$ and $\mathbf{m}$ take more general values,

$$\mathbf{n} : I \to \Psi_{\mathbb{Q}}, \quad \mathbf{m} : J \to \Psi_{\mathbb{Q}},$$

since the monodromy and elliptic vanishing relations remain valid.

5.5.3

Consider the relative elliptic invariant

$$\left\langle \prod_{h\in H} \tau_{o_h}(1) \prod_{i\in I} \tau_{n_i}(\alpha) \prod_{j\in J} \tau_{m_j}(\beta) \mid \eta \right\rangle_d^E . \tag{5.19}$$

Assume such invariants with strictly fewer odd insertions are determined from the even theory by properties (i)-(iv). We now complete the proof of Proposition 5.2 by establishing the induction step. We will follow the proof of Lemma 5.5 using a variation of Lemma 5.7 for the monodromy and elliptic vanishing relations.

We will start by assuming the monomial,

$$M = \prod_{h\in H} \tau_{o,h}(1),$$

is degree 0 and proceed by induction on the degree of M.

Let $v \geq 0$. Let W be an ordered set disjoint from I and J satisfying $|W| = v$. Let K_v be defined by

$$K_v = I \cup W \cup J,$$

with the given order. Let $1 \in I$ denote the first element. For each bijection

$$\sigma : I \to J,$$

let P_σ be the set partition given by the part $\{1\} \cup W \cup \{\sigma(1)\}$ of order $v+2$ together with the doublets $\{i, \sigma(i)\}$ for $i \neq 1$. Let the function $\mathbf{l}$ take the value $\tilde{\psi}$ on all elements of K_v.

Let V be the sum of the terms of

$$\sum_{\sigma} \binom{v+2}{2}^{-1} (-1)^{\binom{|I|}{2}} \mathrm{sign}(\sigma)\ V_d(M, P_\sigma, \mathbf{l}) \tag{5.20}$$

with $|I| + |J|$ odd parts. The inverse binomial prefactor accounts for the multiplicity of choice in the Künneth decomposition absent for the doublets considered in Lemma 5.7. We find, V equals

$$\sum_{\delta\subset I} \sum_{D\in S^*(\delta)} (-1)^{|I|-|\delta|} |\delta|!(|I|-|\delta|)! \left\langle M\tilde{\tau}(\omega)^v \prod_{i\in I} \tilde{\tau}(\gamma_i^D) \prod_{j\in J} \tilde{\tau}(\gamma_j^D) \right\rangle_d^E , \tag{5.21}$$

following the notation of the proof of Lemma 5.7.

Next, the monodromy relations $R(0), \ldots, R(|I|-1)$ are considered with the induced descendent assignments $\mathbf{n}, \mathbf{m}$ and prefactor $M\tilde{\tau}(\omega)^v$. Elimination proves all the invariants

$$\left\langle M\tilde{\tau}(\omega)^v \prod_{i\in I} \tilde{\tau}(\alpha) \prod_{j\in J} \tilde{\tau}(\beta) \right\rangle_d^E,$$

are inductively determined. The elimination analysis exactly follows the proof of Lemma 5.7.

Degeneration, together with Lemma 5.6 and induction on the degree of M, then shows all the invariants

$$\left\langle M \prod_{i\in I} \tilde{\tau}(\alpha) \prod_{j\in J} \tilde{\tau}(\beta) \mid\eta \right\rangle_d^E,$$

$$\left\langle ML \prod_{i\in I} \tilde{\tau}(\alpha) \prod_{j\in J} \tilde{\tau}(\beta) \right\rangle_d^E, \tag{5.22}$$

are inductively determined. Here, $L = \prod_{h'\in H'} \tau_{o'_{h'}}(\omega)$ is an arbitrary monomial.

We will now repeat the analysis for several different assignment functions. Let $\mathbf{l}_{f[r]l[s]}$ take the values

$$\mathbf{l}_{f[r]l[s]}(\xi) = n_\xi$$

for the first r elements of I, and the values

$$\mathbf{l}_{f[r]l[s]}(\xi) = m_\xi$$

for the first s elements of J, and the value $\tilde{\psi}$ for the remaining elements of K_v. We have already considered the function $\mathbf{l}_{f[0]l[0]}$.

We first repeat the analysis for the assignment function $\mathbf{l}_{f[1]l[0]}$. Let V be the sum of the terms of

$$\sum_\sigma \binom{v+1}{1}^{-1} (-1)^{\binom{|I|}{2}} \mathrm{sign}(\sigma)\ V_d(M, P_\sigma, \mathbf{l}_{f[1]l[0]})$$

with $|I| + |J|$ odd parts *modulo the invariants* (5.22).

Consider the Künneth decomposition associated to the first part of P_σ. The terms with an odd class distributed to 1 contribute to V (and are normalized by the prefactor $\binom{v+1}{1}^{-1}$ since they occur with multiplicity). If the odd parts are distributed away from 1, then the resulting terms are of the form (5.22).

We may then eliminate V using the relations $R(0), \ldots, R(|I|-1)$ with the induced descendent assignments and prefactor $M\tilde{\tau}(\omega)^v$. Because of the normalization and the removal of the invariants (5.22), the elimination analysis exactly follows the proof of Lemma 5.7.

By degeneration, Lemma 5.6, and induction on the degree of M, we conclude the invariants

$$\left\langle M \; \tau_{n_1}(\alpha) \prod_{1\neq i\in I} \tilde{\tau}(\alpha) \prod_{j\in J} \tilde{\tau}(\beta) \mid \eta \right\rangle_d^E,$$

$$\left\langle ML \; \tau_{n_1}(\alpha) \prod_{1\neq i\in I} \tilde{\tau}(\alpha) \prod_{j\in J} \tilde{\tau}(\beta) \right\rangle_d^E, \tag{5.23}$$

are inductively determined for any n_1.

Next, we repeat the analysis for the assignment function $\mathbf{l}_{f[0]l[1]}$. Let V be the sum of the terms of

$$\sum_\sigma C_\sigma^{-1} (-1)^{\binom{|I|}{2}} \mathrm{sign}(\sigma) \; V_d(M, P_\sigma, \mathbf{l}_{f[0]l[1]})$$

with $|I|+|J|$ odd parts *modulo the invariants* (5.22).

Here, C_σ equals $\binom{v+1}{1}$ or $\binom{v+2}{2}$ if $\sigma(1)$ is the first element of J or not. The coefficients C_σ are used to correct for multiplicities.

Consider the Künneth decomposition associated to the first part of P_σ. If $\sigma(1)$ is the first element of J, then the terms with an odd class distributed to $\sigma(1)$ contribute to V (and are normalized by the prefactor $\binom{v+1}{1}^{-1}$). If the odd parts are distributed away from $\sigma(1)$, then the resulting terms are of the form (5.22). If $\sigma(1)$ is not the first element of J, then all the Künneth distributions contribute to V (and are normalized by the prefactor $\binom{v+2}{2}^{-1}$).

We may then eliminate V using the relations $R(0), \ldots, R(|I|-1)$ with the induced descendent assignments and prefactor $M\tilde{\tau}(\omega)^v$. Because of the normalization and the removal of the invariants (5.22), the elimination analysis exactly follows the proof of Lemma 5.7.

By degeneration, Lemma 5.6, and induction on M, we conclude the invariants

$$\left\langle M \prod_{i \in I} \tilde{\tau}(\alpha)\ \tau_{m_1}(\beta) \prod_{1 \neq j \in J} \tilde{\tau}(\beta) \mid \eta \right\rangle_d^E ,$$

$$\left\langle ML \prod_{1 \neq i \in I} \tilde{\tau}(\alpha)\ \tau_{m_1}(\beta) \prod_{1 \neq \in J} \tilde{\tau}(\beta) \right\rangle_d^E , \tag{5.24}$$

are inductively determined for any m_1.

We now analyze the assignment $\mathbf{l}_{f[r]l[s]}$ where $r + s > 1$. The *special* elements are the first r elements of I and the first s elements of J. Let V be the sum of the terms of

$$\sum_{\sigma} C_{\sigma}^{-1} (-1)^{\binom{|I|}{2}} \mathrm{sign}(\sigma)\ V_d(M, P_{\sigma}, \mathbf{l}_{f[r]l[s]}) \tag{5.25}$$

with $|I|+|J|$ odd parts modulo the invariants determined by the analysis for the assignments $\mathbf{l}_{f[r']l[s']}$ for $r' + s' < r + s$.

In the definition of V, the summands are normalized with prefactors C_{σ}^{-1} depending on the assignment function and $\sigma(1)$. The first part

$$\{1\} \cup W \cup \{\sigma(1)\}$$

of P_{σ} contains either 2,1, or 0 special elements:

- if P_{σ} contains 2 special elements, then $C_{\sigma} = 1$,
- if P_{σ} contains 1 special element, then $C_{\sigma} = \binom{v+1}{1}$,
- if P_{σ} contains 0 special elements, then $C_{\sigma} = \binom{v+2}{2}$.

If P_{σ} has special elements and the distribution of odd classes in the Künneth decomposition corresponding to P_{σ} in a term of (5.25) misses at least 1 special element, then the term has *fewer* than $r + s$ special elements with odd classes. Such terms are inductively determined by the analysis for the assignments $\mathbf{l}_{f[r']l[s']}$ for $r' + s' < r + s$.

We may then eliminate V using the relations $R(0), \ldots, R(|I| - 1)$ with the induced descendent assignments and prefactor $M\tilde{\tau}(\omega)^v$. Because of the normalization and the removal of the invariants with fewer special elements, the elimination analysis exactly follows the proof of Lemma 5.7.

Using degeneration, Lemma 5.6, and induction on M, the outcome for $\mathbf{l}_{f[r]l[s]}$ is a determination of all invariants

$$\left\langle M \prod_{i\leq r} \tau_{n_i}(\alpha) \prod_{r<i\in I} \tilde{\tau}(\alpha) \prod_{j\leq s} \tau_{m_j}(\beta) \prod_{s<j\in J} \tilde{\tau}(\beta)\ |\eta \right\rangle_d^E,$$

$$\left\langle ML \prod_{i\leq r} \tau_{n_i}(\alpha) \prod_{r<i\in I} \tilde{\tau}(\alpha) \prod_{j\leq s} \tau_{m_j}(\beta) \prod_{s<j\in J} \tilde{\tau}(\beta) \right\rangle_d^E.$$

By induction on $r+s$, we find the invariant (5.19) is determined from the even theory by properties (i)-(iv).

The induction on $|I|$ is therefore established and the proof of Proposition 5.2 is complete. □

6 Virasoro constraints for the full theory

6.1 Overview

We complete the proofs of the main results of the paper. Theorem 1 was proven in [11]. Theorems 2 and 3 are proven first. Theorem 4 is then derived as a corollary.

6.2 Proof of Theorems 2 and 3

We may define an *alternate* relative theory of target curves by the following construction. The alternate stationary sector is defined by the GW/H correspondence. The descendents of the odd classes are added to the alternate theory by the formula of Theorem 2. The Virasoro constraints of Theorem 3 then define a unique extension of the alternate theory including the descendents of the identity. The proof of the existence and uniqueness of the Virasoro solution here exactly follows the proof Proposition 1.1, the corresponding even result. The alternate theory of target curves is well-defined.

To prove Theorems 2 and 3, we must show the alternate relative theory coincides with the relative Gromov-Witten theory. Certainly, the two theories have equal stationary sectors by Theorem 1. In fact, the two theories have equal even sectors since we have proven the even relative Gromov-Witten satisfies the even Virasoro constraints.

We now establish properties (i)-(iv), studied in Section 5, hold for the alternate relative theory of target curve.

(i) Algebraicity of the virtual class.

The balance of descendents of type $(1, 0)$ and $(0, 1)$ is the only consequence of algebraicity used in Section 5. For odd classes in the presence of descendents of ω, the alternate theory satisfies the balance property by the formula of Theorem 2. Since the Virasoro constraints respect the balance, the entire alternate theory satisfies the balance property.

(ii) Degeneration.

The GW/H correspondence is compatible with degeneration. The formula of Theorem 2 for the addition of the odd classes is formally compatible with degeneration. The Virasoro constraints are also formally compatible with degeneration. Hence, the alternate theory satisfies the degeneration formula.

(iii) Monodromy invariance.

The stationary sector of the alternate theory is certainly monodromy invariant. Since monodromy invariance preserves the intersection form, the formula of Theorem 2 for the addition of the odd classes is monodromy invariant. The monodromy invariance of the Virasoro solution is not immediate since a polarization of $H^*(X, \mathbb{C})$ is required for the definition of the Virasoro operators. However, an elementary argument by expansion in terms of the stationary theory shows the elliptic monodromy relation,

$$R_d(N, \mathbf{n}, \mathbf{m}, \delta),$$

formally holds for the alternate theory. Only these monodromy relations were used in Section 5.

(iv) Elliptic vanishing relations.

An elementary argument by expansion in terms of the stationary theory shows the elliptic vanishing relation,

$$V_d(M, P, \mathbf{l}),$$

formally holds for the alternate theory.

In Section 5, we proved the relative Gromov-Witten theory of target curves is uniquely determined from the even theory by properties (i)-(iv). Therefore, since the alternate theory coincides with the relative Gromov-Witten theory on the even sector and satisfies properties (i)-(iv), the alternate theory equals the relative Gromov-Witten theory. □

6.3 Proof of Theorem 4

Consider first the relative Gromov-Witten theory of target curves *without* descendents of the identity. An explicit expansion shows the differential equations

$$\begin{aligned} D^i_k \; Z_d[\eta^1, \ldots, \eta^m] &= 0, \\ \bar{D}^i_k \; Z_d[\eta^1, \ldots, \eta^m] &= 0, \end{aligned}$$

when restricted to the zero locus of the ideal

$$I = (t^0_0, t^0_1, t^0_2, \ldots),$$

are equivalent to the formula of Theorem 2.

The differential equations for D^i_k and $\bar{D}^i_k$ are proven to hold on the zero locus of I^r by induction on r using the Virasoro constraints,

$$L_n \; Z_d[\eta^1, \ldots, \eta^m] = 0,$$

and the commutation relations,

$$[L_n, D^i_k] = -(k+1) D^i_{n+k},$$
$$[L_n, \bar{D}^i_k] = (n-k) \bar{D}^i_{n+k}.$$

The Theorem is then deduced from completeness with respect to the ideal I. $\square$

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Department of Mathematics
Princeton University
Princeton, NJ 08544
okounkov@math.princeton.edu

Department of Mathematics
Princeton University
Princeton, NJ 08544
rahulp@math.princeton.edu