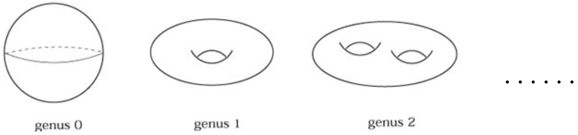

Cycles on the moduli space of curves

Rahul Pandharipande

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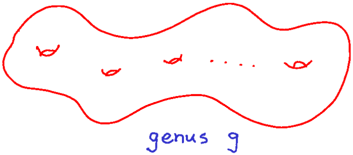
July 2015

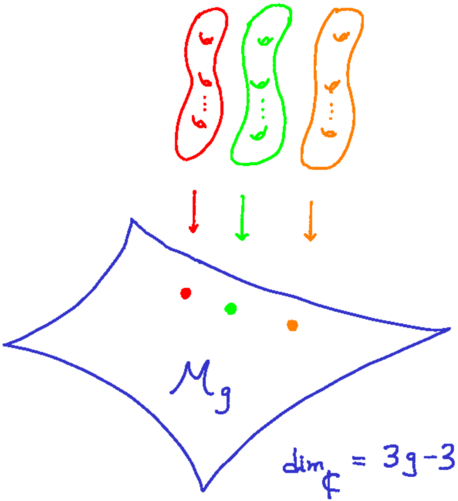
§I. **Nonsingular curves**

Let C be a complete, nonsingular, irreducible curve of genus $g \geq 2$:



The curve C has a complex structure which we can vary (while keeping the topology fixed).

Riemann studied the moduli space $\mathcal{M}_g$ of all genus g curves:



Riemann knew $\mathcal{M}_g$ was (essentially) a complex manifold of dimension 3g-3.

Theorie der *Abel*'schen Functionen.

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Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable pointed curves:

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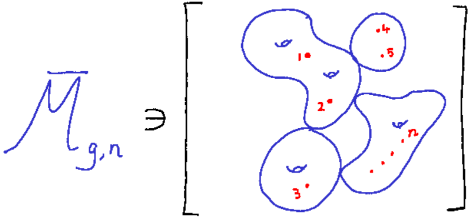
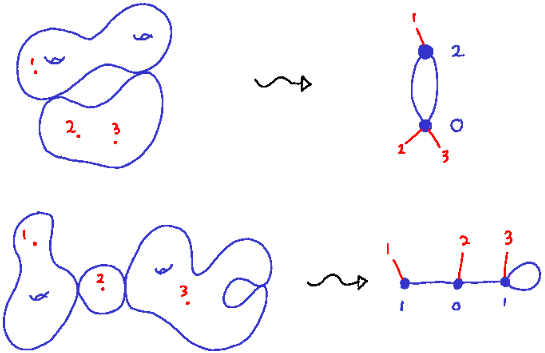


The boundary strata of the moduli $\overline{\mathcal{M}}_{g,n}$ of *fixed topological type* correspond to *stable graphs*.



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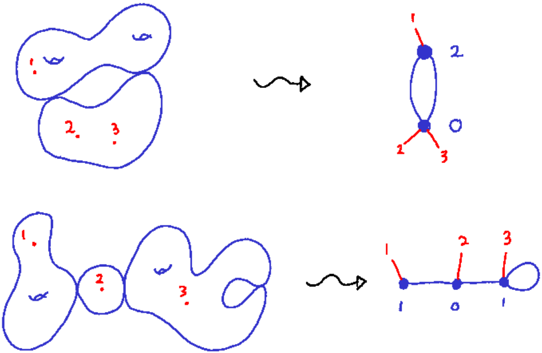

For a graph Γ, let $[\Gamma] \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the class of the closure of the stratum (with a multplicity related to symmetries).

Formally, a stable graph is the structure

$$\Gamma = (\mathrm{V}, \mathrm{E}, \mathrm{L}, \mathrm{g})$$

satisfying the following properties:

- V is the vertex set with a genus function $\mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}$,
- E is the edge set,
- L, the set of legs (corresponding to the set of markings),
- the pair (V, E) defines a *connected* graph,
- for each vertex v, the stability condition holds:

$$2\mathrm{g}(v) - 2 + \mathrm{n}(v) > 0,$$

  where $\mathrm{n}(v)$ is the valence of Γ at v including both edges and legs.

The genus of a stable graph Γ is defined by:

$$\mathrm{g}(\Gamma) = \sum_{v \in \mathrm{V}} \mathrm{g}(v) + h^1(\Gamma).$$

To each stable graph Γ, we associate the moduli space

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \mathrm{V}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}.$$

There is a canonical morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \ , \quad \xi_{\Gamma *}[\overline{\mathcal{M}}_\Gamma] = [\Gamma] \ .$$

Question: Are there relations in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ among the $[\Gamma]$?

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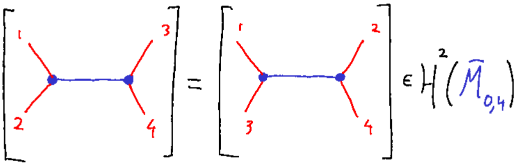
The first boundary relation is almost trivial:



Just an equivalence of two points in $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$.

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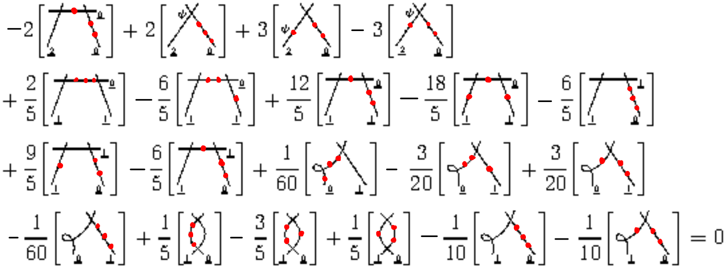

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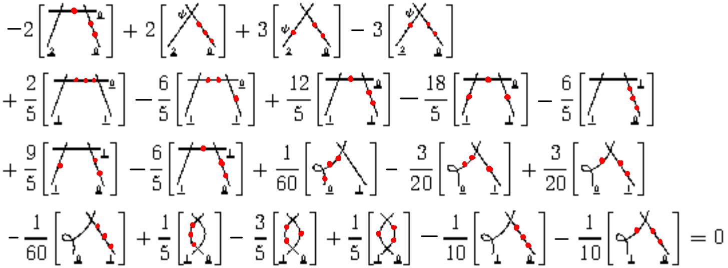

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Question: Is there any structure to these formulas?

§III. Pixton's **relations on** $\overline{\mathcal{M}}_{g,n}$

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We define classes $\mathrm{R}^d_{g,A}$ associated to the data

- $g, n \in \mathbb{Z}_{\geq 0}$ in the stable range $2g - 2 + n > 0$,
- $A = (a_1, \ldots, a_n), \quad a_i \in \{0, 1\}$,
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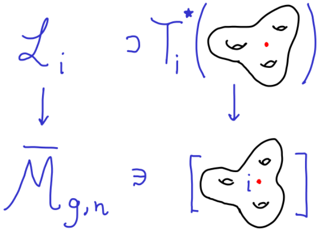
The elements $\mathsf{R}^d_{g,A}$ are expressed as sums over stable graphs of genus g with n legs. Pixton's relations then take the form

$$\mathsf{R}^d_{g,A} = 0 \quad \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

Before writing the formula for $\mathsf{R}^d_{g,A}$, a few definitions are required.

Let $\mathcal{L}_i$ be the cotangent line at the i^{th} marking:



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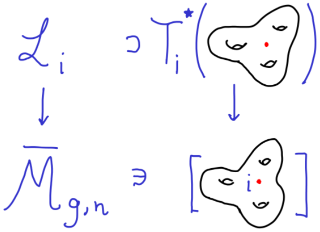

We can define the cotangent line class

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

Via the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, we define

$$\kappa_i = \pi_*(\psi_{n+1}^{i+1}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

$$B_0(T) = \sum_{m\geq 0} \frac{(6m)!}{(2m)!(3m)!}(-T)^m = 1 - 60\,T + 27720\,T^2 \cdots ,$$

$$B_1(T) = \sum_{m\geq 0} \frac{1+6m}{1-6m}\frac{(6m)!}{(2m)!(3m)!}(-T)^m = 1 + 84\,T - 32760\,T^2 \cdots .$$

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For a survey of the occurances of B_0 and B_1:

[Buryak, Janda, P. arXiv:1502.05150]

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$$\kappa(f) = \sum_{m \geq 0} \frac{1}{m!} \pi_{m*}\Big(f(\psi_{n+1}) \cdots f(\psi_{n+m}) \Big) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

where $\pi_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map, and

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By the vanishing in degrees 0 and 1 of f, the sum is finite.

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For each vertex $v \in \mathrm{V}$ of a stable graph, we introduce an auxiliary variable ζ_v and impose the conditions

$$\zeta_v \zeta_{v'} = \zeta_{v'} \zeta_v \ , \quad \zeta_v^2 = 1 \ .$$

The variables ζ_v will be responsible for keeping track of a local parity condition at each vertex.

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- For $\ell \in \mathrm{L}$, let $B_\ell = \zeta_{v(\ell)}^{a_\ell} B_{a_\ell}\big(\zeta_{v(\ell)}\psi_\ell\big)$, where $v(\ell) \in V$ is the vertex to which the leg is assigned.

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- For $e \in \mathrm{E}$, let

$$\Delta_e = \frac{\zeta' + \zeta'' - B_0(\zeta'\psi')\zeta'' B_1(\zeta''\psi'') - \zeta' B_1(\zeta'\psi') B_0(\zeta''\psi'')}{\psi' + \psi''}$$
$$= (60\zeta'\zeta'' - 84) + \big[32760(\zeta'\psi' + \zeta''\psi'') - 27720(\zeta'\psi'' + \zeta''\psi')\big] \cdots ,$$

where ζ', ζ'' are the ζ-variables assigned to the vertices adjacent to the edge e and ψ', ψ'' are the ψ-classes corresponding to the half-edges.

The numerator of Δ_e is divisible by the denominator due to the identity (discovered by Pixton)

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Definition (Pixton 2012)

Let $A = (a_1, \ldots, a_n) \in \{0,1\}^n$. We denote by $\mathrm{R}^d_{g,A} \in H^{2d}(\overline{\mathcal{M}}_{g,n})$ the degree d component of the class

$$\sum_{\Gamma \in \mathrm{G}_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \left[\prod \kappa_v \prod B_\ell \prod \Delta_e \right]_{\prod_v \zeta_v^{\mathrm{g}(v)-1}} \right],$$

where the products are taken over all vertices, all legs, and all edges of the graph Γ.

The subscript $\prod_v \zeta_v^{\mathrm{g}(v)-1}$ indicates the coefficient of the monomial $\prod_v \zeta_v^{\mathrm{g}(v)-1}$ after the product inside the brackets is expanded.

Theorem (P.-Pixton-Zvonkine 2013)

For $2g-2+n>0$, $a_i \in \{0,1\}$, and $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$, Pixton's relations hold

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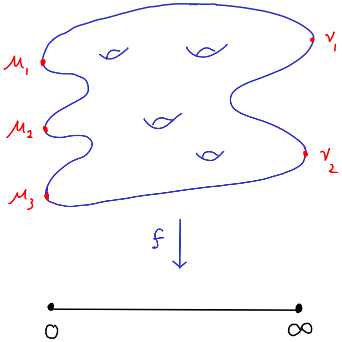

To define a cycle on $\overline{\mathcal{M}}_{g,n}$, the degenerate curves must be considered. The moduli space

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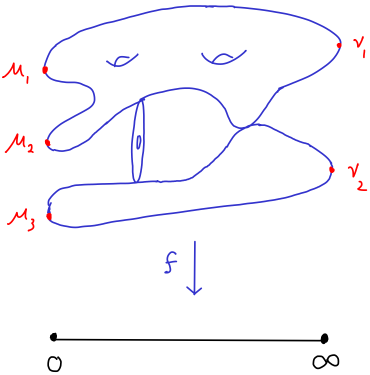

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The double ramification cycle is the push-forward of the virtual fundamental class,

$$\mathsf{DR}_{g,\mu,\nu} \;=\; \rho_* \left[\overline{\mathcal{M}}_g(\mathbf{CP}^1, \mu, \nu)^\sim \right]^{vir} \in \mathsf{A}^g(\overline{\mathcal{M}}_{g,\ell(\mu)+\ell(\nu)}) \ .$$

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Question [Eliashberg 2000]: Can we find a formula for $\mathrm{DR}_{g,\mu,\nu}$?

Best to place ramification data in a vector

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Pixton conjectured a beautiful formula for $\mathsf{DR}_{g,A}$ in 2014.

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which satisfies:

(i) $\forall h_i \in \mathrm{L}(\Gamma)$, $w(h_i) = a_i$,

(ii) $\forall e \in \mathrm{E}(\Gamma)$ consisting of the half-edges $h(e), h'(e) \in \mathsf{H}(\Gamma)$,

$$w(h) + w(h') = 0\,,$$

(iii) $\forall v \in \mathrm{V}(\Gamma)$, $\sum_{v(h)=v} w(h) = 0$.

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Let $\mathsf{W}_{\Gamma,r}$ be the set of admissible weightings mod r of Γ.

The set $\mathsf{W}_{\Gamma,r}$ is finite.

Definition (Pixton 2014)

Let r be a positive integer. We denote by $\mathcal{P}_{g,A}^{d,r} \in \mathrm{A}^d(\overline{\mathcal{M}}_{g,n})$ the degree d component of the class

$$\sum_{\Gamma \in \mathrm{G}_{g,n}} \sum_{w \in \mathrm{W}_{\Gamma,r}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*} \Bigg[\prod_{i=1}^{n} \exp({a_i}^2 \psi_{h_i}) \cdot \prod_{e=(h,h') \in \mathrm{V}(\Gamma)} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \Bigg].$$

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For fixed g, A, and d, the class

$$\mathcal{P}_{g,A}^{d,r} \in \mathrm{A}^d(\overline{\mathcal{M}}_{g,n})$$

is polynomial in r for sufficiently large r.

We denote by $\mathsf{P}^{d}_{g,A}$ the value at $r = 0$ of the polynomial, $\mathsf{P}^{d}_{g,A}$ is the *constant* term.

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$\mathrm{DR}_{g,A} = 2^{-g}\, \mathrm{P}^g_{g,A} \in \mathrm{A}^g(\overline{\mathcal{M}}_{g,n})$.

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We use the Gromov-Witten theory of the target $\mathbf{CP^1}$ with:

- orbifold $B\mathbb{Z}_r$-point at $0 \in \mathbf{CP^1}$,
- relative point $\infty \in \mathbf{CP^1}$.

So the proof uses orbifold GW theory, relative GW theory, virtual localization.

§VI. Chern characters of the Verlinde bundle

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Let G be a complex, simple, simply connected Lie group. For genus g and n irreducible representations

$$\mu_1, \ldots, \mu_n$$

of the Lie algebra $\widehat{\mathfrak{g}}$ at level ℓ, the Verlinde bundle

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The fiber of the Verlinde bundle over $[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$ is the space of non-abelian theta functions: global sections of (determinant line)$^\ell$ over the moduli of parabolic G-bundles on C.

The Verlinde formula calculates the rank

$$\operatorname{rk} \mathbb{E}_g(\mu_1, \ldots, \mu_n) = d_g(\mu_1, \ldots, \mu_n)$$

which is the constant term of the Chern character

$$\operatorname{ch} \mathbb{E}_g(\mu_1, \ldots, \mu_n) \in H^*(\overline{\mathcal{M}}_{g,n}) \ .$$

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Question: Can we find a formula for $\mathrm{ch}\,\mathbb{E}_g(\mu_1, \ldots, \mu_n)$?

Full solution found [Marian-Oprea-P.-Pixton-Zvonkine] for all G and ℓ using following geometric inputs

- the Chern character ch $\mathbb{E}$ defines CohFT,
- the genus 0 part is semisimple (the fusion algebra),
- the bundle $\mathbb{E}_g(\underline{\mu})$ is projectively flat over $\mathcal{M}_{g,n}$,
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Example of our formula in the first non-trival case:

$$G = \mathbb{SL}_2 \quad \text{and} \quad \ell = 1\,.$$

Theorem (Marian-Oprea-P.-Pixton-Zvonkine 2014)

Let $\square$ *be the standard representation of* $\mathbb{SL}_2$*. For* $\ell = 1$*,*

$$\mathrm{ch}\,\mathbb{E}_g(\square, \ldots, \square) =$$

$$e^{-\frac{\lambda}{2}} \sum_{\Gamma \in \mathsf{G}_{g,n}^{\mathrm{even}}} \frac{2^{g-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\, \xi_{\Gamma*}\left[\prod_{i=1}^{n} \exp\left(\frac{\psi_{h_i}}{4}\right) \cdot \prod_{e=(h,h')\in \mathrm{V}(\Gamma)} \frac{1-\exp(\frac{1}{4}(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \right],$$

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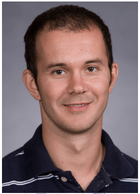

The End