Gromov–Witten theory of complete intersections

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Introduction

- An algorithm computing GW invariants of all smooth complete intersections of hypersurfaces in projective space.
- All GW classes of complete intersections in projective space are tautological elements in the cohomology of the moduli space of stable curves.
- Main technical tool: nodal GW theory, working with domain curves with prescribed nodes.

Argüz–B.–Pandharipande–Zvonkine, arxiv:2109.13323.

Plan

- GW invariants and complete intersections in projective space.
- The main issue: degeneration versus vanishing cycles.
- The main idea: trading vanishing cycles against nodes.
- Foundational results in nodal GW theory.
- GW classes are tautological.

Gromov–Witten invariants

- X: a smooth projective variety over $\mathbb{C}$
- **Gromov–Witten (GW) invariants of** X: numbers defined by intersection theory on the "moduli space of stable maps" to X.

Definition (Kontsevich, 1994)

An n-pointed genus g stable map to X of class β is a morphism

$$f : (C, x_1, \ldots, x_n) \longrightarrow X \,,$$

where

- C: nodal projective curve of arithmetic genus g.
- $x_1, \ldots, x_n$: n (ordered) smooth marked points on C.
- $f_*[C] = \beta \in H_2(X, \mathbb{Z})$.
- (stability) there are finitely automorphisms of $(C, x_1, \ldots, x_n)$ commuting with f.

- $\overline{\mathcal{M}}_{g,n,\beta}(X)$: moduli space of n-pointed genus g stable maps to X of class β. Proper Deligne-Mumford stack.
- $ev_i : \overline{\mathcal{M}}_{g,n,\beta}(X) \longrightarrow X$ evaluation at the i-th marked point;
  $$(f : (C, x_1, \dots, x_n) \to X) \mapsto f(x_i).$$
- Fix $g, n \in \mathbb{Z}_{\geq 0}\,, \quad \beta \in H_2(X, \mathbb{Z})\,, \quad \alpha_1, \dots, \alpha_n \in H^\star(X, \mathbb{Q})$

GW invariants of X:

$$\left\langle \prod_{i=1}^n \alpha_i \right\rangle_{g,n,\beta}^X := \deg \left(\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in \mathbb{Q}\,.$$

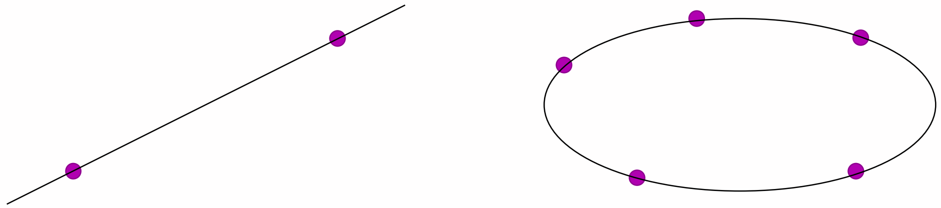
- (virtual) count of genus g curves in X of class β with n marked points passing through $\mathrm{PD}(\alpha_i)$.
- Deformation invariant!

Example: Genus zero GW invariants of $\mathbb{P}^2$

Example

- Rational curves in $\mathbb{P}^2$ of degree d, passing through $3d-1$ points $p_1, \dots, p_{3d-1}$.

d	1	2	3	4	5
$\langle p_1, \dots p_{3d-1} \rangle^{\mathbb{P}^2}_{0,3d-1,d}$	1	1	12	640	84000



Technical point: we need ψ-class insertions

- We need ψ-class insertions
  - they appear in the localization formula
- L_i: line bundle on $\overline{\mathcal{M}}_{g,n,\beta}(X)$, whose fiber over $(f : C, x_1 \dots, x_n \to X)$ is the cotangent line of C at the i-th marked point,

$$\psi_i := c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n,\beta}(X), \mathbb{Q}) \,.$$

- Fix

$$g, n \in \mathbb{Z}_{\geq 0} \,, \quad \beta \in H_2(X, \mathbb{Z}) \,, \quad \alpha_1, \dots, \alpha_n \in H^\star(X, \mathbb{Q})$$

$$\text{and} \quad k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$$

- GW invariants of X are:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\alpha_i) \right\rangle_{g,n,\beta}^{X} := \deg \left(\prod_{i=1}^{n} \psi_i^{k_i} \operatorname{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in \mathbb{Q} \,.$$

Gromov–Witten invariants

Problem

Given a smooth projective variety X, “compute" all GW invariants of X.

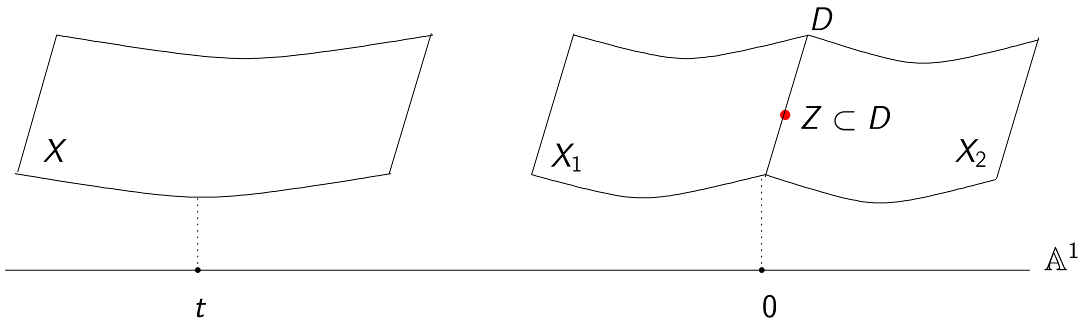
Known cases:

- X: point (Kontsevich, Witten's conjecture, 1992)
- X: projective space, or more generally an homogeneous variety (localization, Graber–Pandharipande, 1999)
- X: curve (Okounkov–Pandharipande, 2003)
- X: quintic 3-fold hypersurface in $\mathbb{P}^4$ (Maulik-Pandharipande, 2006)
- X: complete intersections in projective space (Argüz–B.–Pandharipande–Zvonkine, 2021).

- X: m-dim'l smooth complete intersection of r hypersurfaces in $\mathbb{P}^{m+r}$,

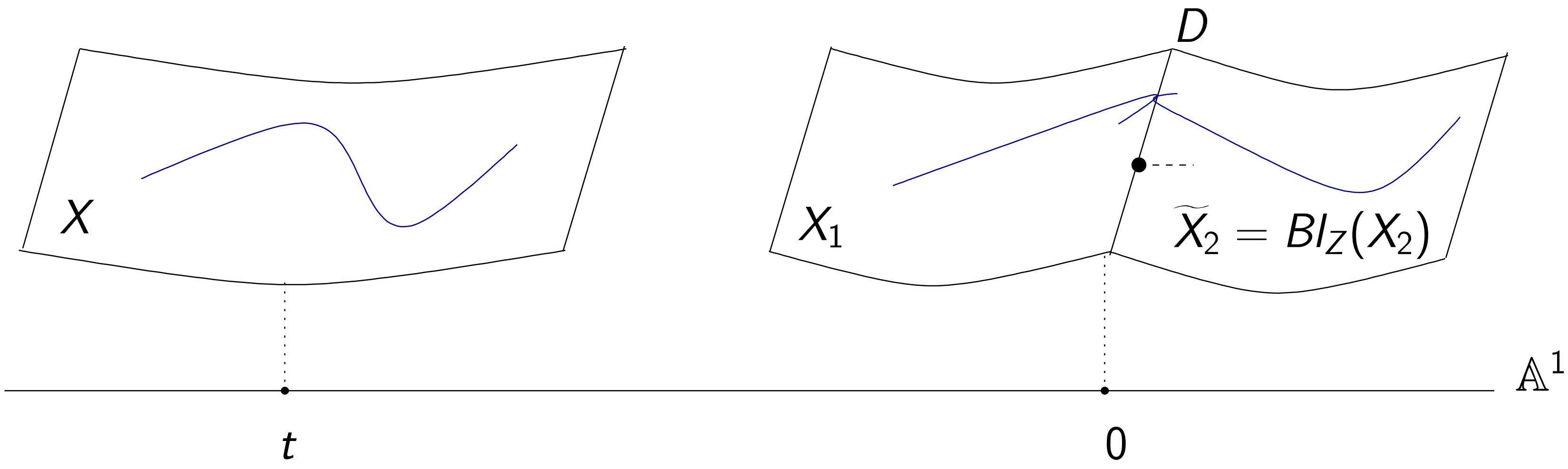
$$f_1 = \cdots = f_r = 0,$$

  of degrees $(d_1, \ldots, d_r)$.
- Study GW invariants of X using degeneration.
  - $d_r = d_{r,1} + d_{r,2}$, pick general $f_{r,1}$ and $f_{r,2}$ of degree $d_{r,1}$ and $d_{r,2}$.
  - $f_1 = \cdots = f_{r-1} = tf_r + f_{r,1}f_{r,2} = 0$: one-parameter family.



  - Total space singular along Z.
  - Blow-up X_2: new family $W \to \mathbb{A}^1$ with smooth total space. Components of the special fiber: X_1 and $\widetilde{X}_2 := Bl_Z(X_2)$.
- **GOAL**: $GW(X_1), GW(X_2), GW(D), GW(Z) \to GW(X)$

Jun Li's degeneration formula expresses $GW(X)$ in terms of "relative GW invariants" $GW(X_1, D)$ and $GW(\widetilde{X}_2, D)$, under restrictive assumptions.



Example: vanishing cycles

Jun Li's formula applies if the cohomology insertions α_i are in the image of the restriction map

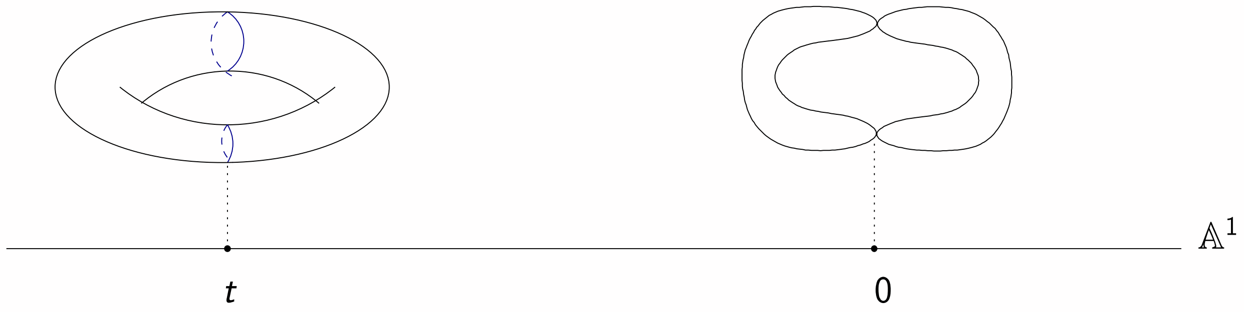
$$H^\star(W) \to H^\star(X)$$

- Not surjective in general!
  - Dually, $H_\star(X) \to H_\star(W)$ not injective (there exist vanishing cycles)

Example

Degeneration of a smooth elliptic curve E to a nodal elliptic curve E_0.

- $\dim H^1(E) = 2$, whereas $\dim H^1(E_0) = 1$.



Vanishing cycles / primitive cohomology

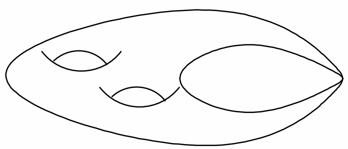
- X: complete intersection
- $H^\star(X,\mathbb{C}) = H^{simple} \oplus H^{prim}$
- $H^{simple} = \langle 1, H, H^2, \dots, H^m \rangle$
- Lefschetz: $H^{prim} \subset H^m(X,\mathbb{C})$
  - H^{prim} contains all vanishing cycles.
- We want to compute GW invariants with also primitive insertions.
  - Key idea: trade primitive insertions against nodes.
  - Compute nodal GW invariants with simple insertions using degenerations.

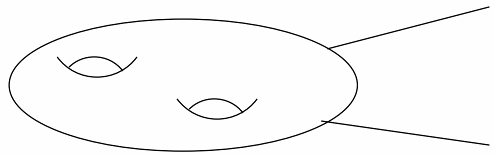
Can we recover all GW invariants of X, including primitive ones, by knowing only the data of simple nodal GW invariants of X?

Trading primitive insertions against nodes

Example

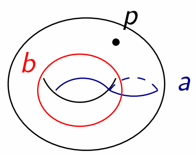
- X: elliptic curve E. Fix $g = 2$ and $n = 2$.
  - There are 4 invariants to compute: $\langle a, a \rangle$, $\langle b, b \rangle$, $\langle a, b \rangle$, $\langle b, a \rangle$





Simple nodal $GW(X)$ $\longleftrightarrow$ splitting formula

Insertion of the diagonal class $\Delta \subset E \times E$



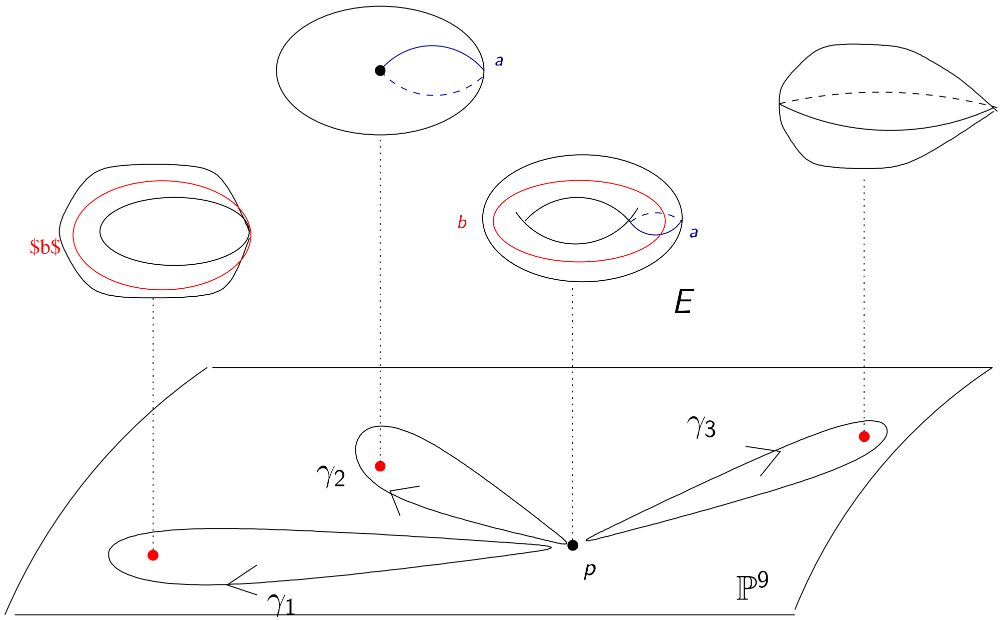
$\langle p, 1 \rangle + \langle 1, p \rangle + \langle a, b \rangle - \langle b, a \rangle$

$\langle a, a \rangle =?$ $\quad \langle b, b \rangle =?$ $\quad \langle a, b \rangle =?$ $\quad \langle b, a \rangle =?$

Deformation invariance $\longleftrightarrow$ $\langle a, a \rangle = 0$, $\langle b, b \rangle = 0$ $\quad \langle a, b \rangle = -\langle b, a \rangle$

How do we use deformation invariance in Gromov–Witten theory / what does it tell us?

- We consider a family of X given by varying the coefficients of $f_i's$.
  - Deformation invariance $\Longrightarrow$ monodromy invariance of $GW(X)$



- Around γ_1: $\langle a, b\rangle = \langle a+b, b\rangle = \langle a, b\rangle + \langle b, b\rangle \implies \langle b, b\rangle = 0$
- Around γ_2: $\langle a, b+a\rangle = \langle a, b+a\rangle = \langle a, b\rangle + \langle a, a\rangle \implies \langle a, a\rangle = 0$
- Around γ_3: $\langle a, b\rangle = \langle b, -a\rangle =$ - $\langle b, a\rangle$

Monodromy action

- X: complete intersection, $f_1 = \ldots = f_r = 0$
- Monodromy action on $H^\star(X)$:
  - $U = \{\text{coefficients of } f_i\}$,
  - $U_0 = \{X \text{ singular}\} \subset U$ closed subset,
  - $\pi_1(U \setminus U_0, p)$ acts on $H^\star(X)$

Theorem (Deligne)

Let G: Zariski closure of the image of $\pi_1(U \setminus U_0, p)$ in $GL(H^{prim})$. Then, $G = O(k)$ if m even ($k = \dim H^{prim}$), $G = Sp(2k)$ if m odd ($2k = \dim H^{prim}$), except if:

- X cubic surface, $G = W(E_6)$ (finite group),
- or X even dimensional complete intersection of two quadrics, $G = W(D_{m+3})$ (also finite group).

Main idea: trading primitive insertions against nodes

Theorem (Argüz-B.-Pandharipande-Zvonkine, 2021)

Let X be a complete intersection in projective space which is not a cubic surface or an even-dimensional complete intersection of two quadrics. Then, the GW invariants of X can be effectively reconstructed from the nodal GW invariants of X with only insertions of simple cohomology classes.

- Proof uses invariant theory of symplectic and orthogonal groups.
- The exceptional cases of the cubic surface and even-dimensional complete intersections of two quadrics are treated separately (the monodromy around the special fiber of the degeneration is in fact trivial in these cases: finite and unipotent (because semi-stable degeneration)).

Nodal Gromov–Witten invariants

- Γ: X-valued stable graph.
- Nodal Gromov–Witten invariants of X of type Γ are

$$\left\langle \prod_{i=1}^{n_\Gamma} \tau_{k_i}(\alpha_i) \prod_{h \in H_\Gamma \setminus L_\Gamma} \tau_{k_h} \right\rangle_\Gamma^X := \deg \left(\prod_{i=1}^{n_\Gamma} \psi_i^{k_i} \, \mathrm{ev}_i^*(\alpha_i) \prod_{h \in H_\Gamma \setminus L_\Gamma} \psi_h^{k_h} \cap [\overline{\mathcal{M}}_\Gamma(X)]^{\mathrm{virt}} \right)$$

Splitting formula

- Künneth decomposition of the class of the diagonal $\Delta \subset X \times X$ in $H^\star(X \times X) = H^\star(X) \otimes H^\star(X)$: for any basis $(\gamma_i)_i$ of $H^\star(X)$,

$$[\Delta] = \sum_i \gamma_i \otimes \gamma_i^\vee$$

where $(\gamma_i^\vee)$ is the Poincaré dual basis ($\int_X \gamma_i \cup \gamma_j^\vee = \delta_{ij}$).

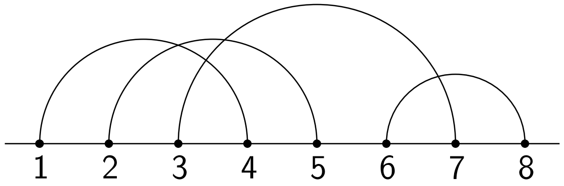
- Splitting formula in Gromov–Witten theory:

$$\left\langle \left(\prod_{i=1}^n \tau_{k_i}(\alpha_i) \right) \tau_{k_{h_1}} \tau_{k_{h_2}} \right\rangle_{\Gamma, g, n, \beta}^X$$

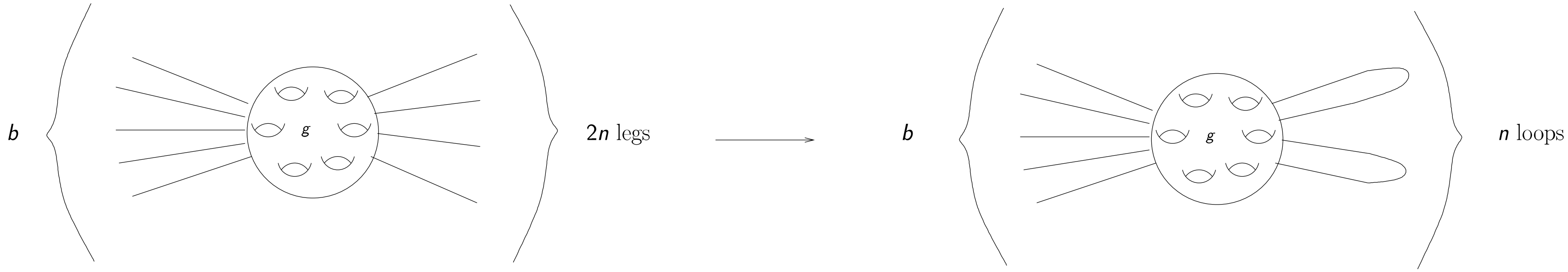
$$= \sum_j \left\langle \left(\prod_{i=1}^n \tau_{k_i}(\alpha_i) \right) \tau_{k_{h_1}}(\gamma_j) \tau_{k_{h_2}}(\gamma_j^\vee) \right\rangle_{g-1, n+2, \beta}^X$$

where Γ is the graph with one vertex and one loop imposing a self-intersecting node.

- $V := H^m(X, \mathbb{C})_{prim}$
- GW invariant with N primitive insertions: $GW \in ((V*)^{\otimes N})^{O,Sp}$.
- $-Id \in O, Sp$, so $GW = 0$ if N odd. Assume $N = 2n$.
- n-pairing on $2n$ objects:



- one equation for each n-pairing P:



- one invariant multilinear form for each n-paring P':

$$\alpha_P \colon V^{\otimes 8} \to \mathbb{C}$$

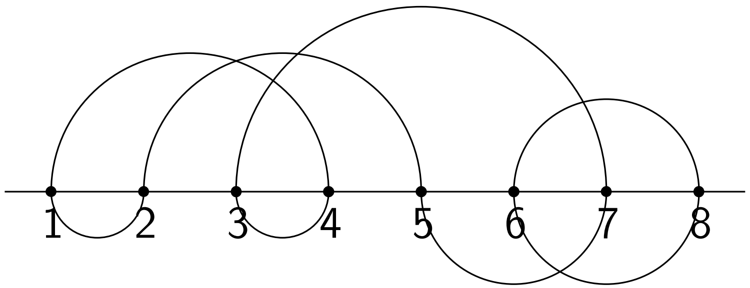
$$v_1 \otimes \cdots \otimes v_8 \mapsto (v_1, v_4)(v_2, v_5)(v_3, v_7)(v_6, v_8)\,.$$

Trading primitive insertions against nodes: proof

- Matrix of the system of equations from the splitting formula: $(2n-1)!! \times (2n-1)!!$ matrix

$$M(n,x)_{P,P'} = x^{L(P,P')}$$

- $L(P,P')$: loop number of the n-pairings P and P'.



- $x = \dim V$ when m even, $x = -\dim V$ when m odd.

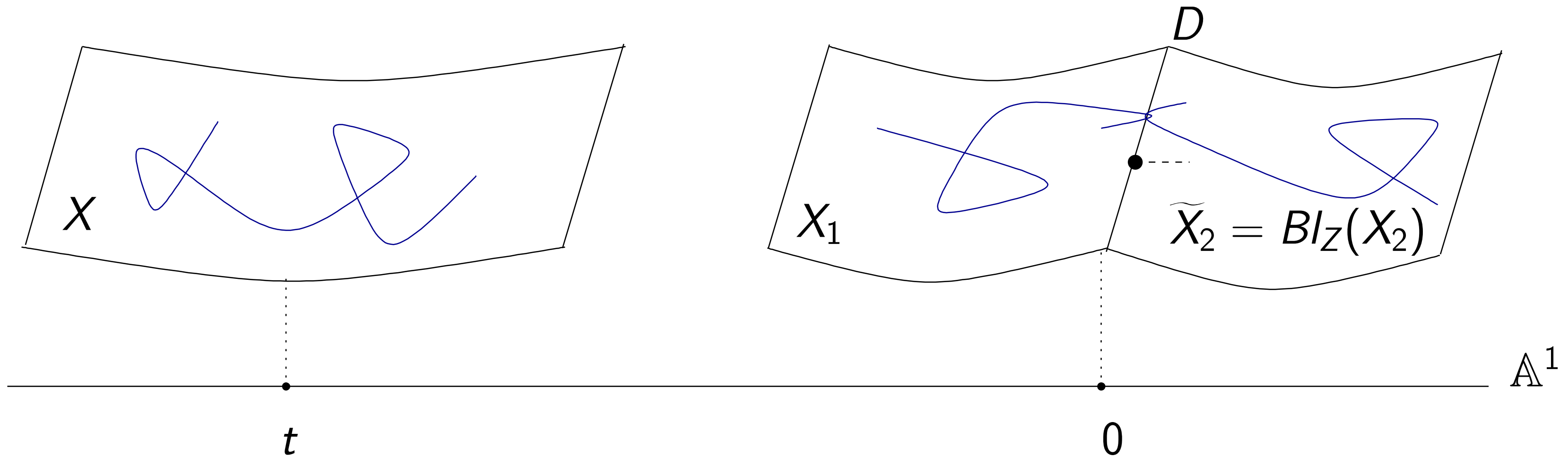
$$M(2,x) = \begin{pmatrix} x^2 & x & x \\ x & x^2 & x \\ x & x & x^2 \end{pmatrix},$$

- Subtlety: relations between the invariant forms α_P. Have to show that $M(n,x)$ has exactly the correct rank (see Macdonald book on symmetric functions, zonal symmetric polynomials).

How to compute simple nodal Gromov–Witten invariants?

Theorem (Argüz-B.-Pandharipande-Zvonkine, 2021)

- There is a nodal degeneration formula computing simple nodal GW invariants of X in terms of "nodal relative GW invariants" of (X_1, D) and $(\widetilde{X}_2, D)$.[1]



[1] This requires "carefully" defining **nodal relative GW invariants**!

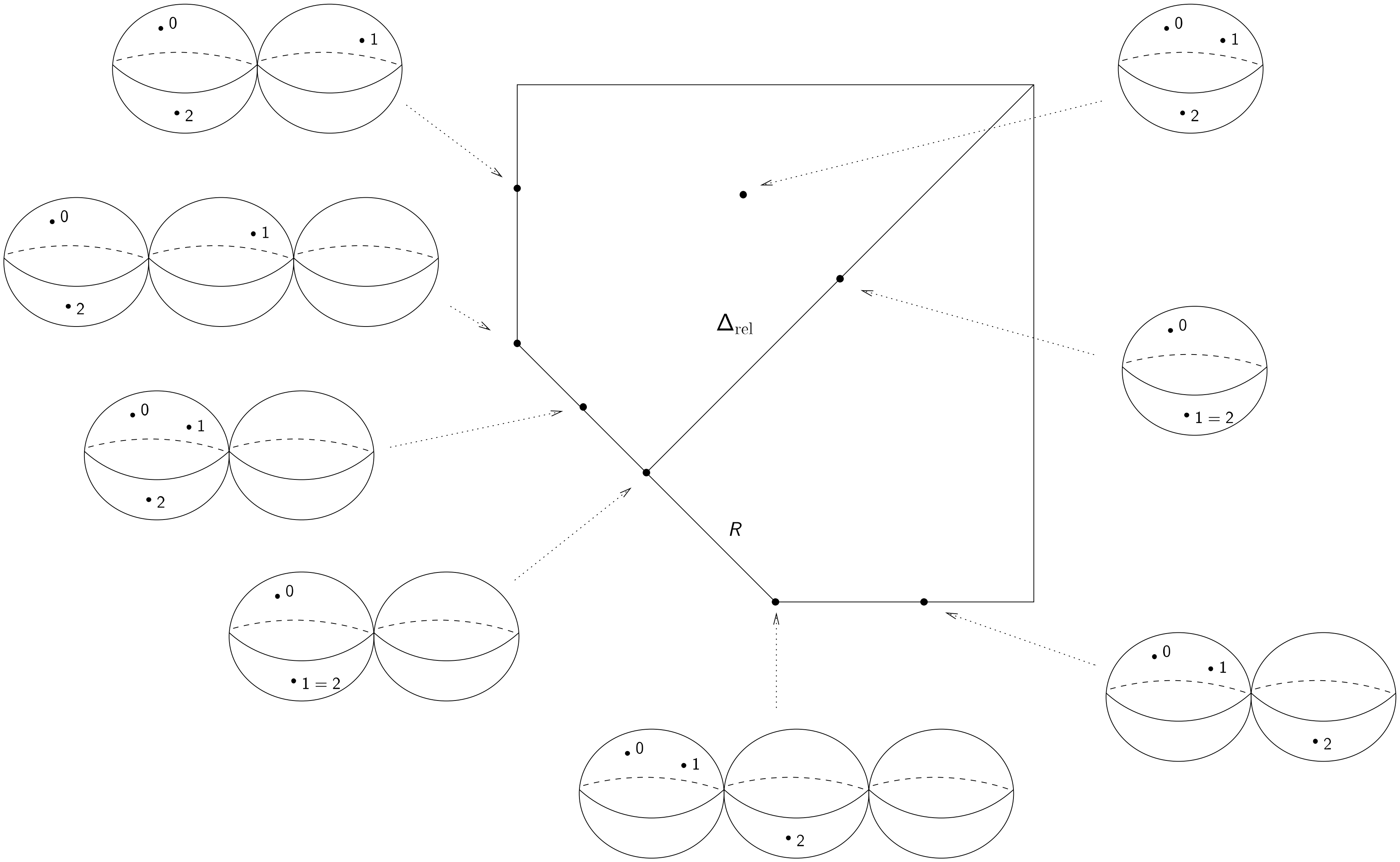
How to compute nodal relative GW invariants?

Theorem (Argüz-B.-Pandharipande-Zvonkine, 2021)

- There is a splitting formula for nodal relative invariants, computing nodal relative GW invariants of (X_1, D) and $(\widetilde{X}_2, D)$ in terms of relative GW invariants of (X_1, D), $(\widetilde{X}_2, D)$, and GW invariants of D.

- Compared to the usual splitting formula for absolute GW invariants, rubber correction term coming from nodes falling into D.

Splitting: $(\mathbb{P}^1, 0)^2$



Step by step

- Goal:

$$GW(X) \leftarrow GW(X_1), GW(X_2), GW(D), GW(Z),$$

where X_1, X_2, D, Z are complete intersections of either smaller degree or smaller dimension.

- Step 1: trade primitive insertions for nodes:

$$GW(X) \leftarrow sNGW(X)$$

- Step 2: apply the nodal degeneration formula to compute simple nodal GW invariants:

$$sNGW(X) \leftarrow NGW(X_1, D), NGW(\widetilde{X}_2, D)$$

- Step 3: apply the splitting formula to reduce nodal relative GW invariants to relative GW invariants

$$NGX(X_1, D), NGW(\widetilde{X}_2, D) \leftarrow GW(X_1, D), GW(\widetilde{X}_2, D)$$

- Step 4: apply previous results of Maulik-Pandharipande

$$GW(X_1, D), GW(\widetilde{X}_2, D) \leftarrow GW(X_1), GW(X_2), GW(D), GW(Z)$$

The main algorithm

Theorem (Argüz-B.-Pandharipande-Zvonkine, 2021)

Let X be an m-dimensional smooth complete intersection in $\mathbb{P}^{m+r}$ of degrees $(d_1, \ldots, d_r)$. Then, for every decomposition

$$d_r = d_{r,1} + d_{r,2} \quad \text{with} \quad d_{r,1}, d_{r,2} \in \mathbb{Z}_{\geq 1} \,,$$

then $GW(X)$ can be effectively reconstructed from:

(i) $GW(X_1)$, where $dim(X_1) = m$, degrees $(d_1, \ldots, d_{r-1}, d_{r,1})$.

(ii) $GW(X_2)$, where $dim(X_2) = m$, degrees $(d_1, \ldots, d_{r-1}, d_{r,2})$.

(iii) $GW(D)$, where $dim(D) = m - 1$, degrees $(d_1, \ldots, d_{r-1}, d_{r,1}, d_{r,2})$.

(iv) $GW(Z)$, where $dim(Z) = m - 2$, degrees $(d_1, \ldots, d_{r-1}, d_r, d_{r,1}, d_{r,2})$.

Upgrading to Gromov–Witten classes

- Forgetful morphism $\pi \colon \overline{\mathcal{M}}_{g,n,\beta}(X) \to \overline{\mathcal{M}}_{g,n}$.
- GW classes

$$\left[\prod_{i=1}^{n} \tau_{k_i}(\alpha_i)\right]^X_{g,n,\beta} := \pi_* \left(\prod_{i=1}^{n} \psi_i^{k_i} \operatorname{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in H^\star(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \,.$$

Conjecture

For every smooth projective variety X, the GW classes of X are tautological.

Tautological ring $RH^\star(\overline{\mathcal{M}}_{g,n}) \subset H^\star(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Smallest system of subrings containing 1 and preserved by pullback-pushforward along the natural maps $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$.

[1] Kontsevich–Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, **Communications in Mathematical Physics**, 1994

Gromov–Witten classes of complete intersections

Known cases when GW classes are tautological:

- X a projective space, or more generally an homogeneous variety (localization, Graber-Pandharipande, 1999)
- X a curve (Janda, 2013)

Theorem (Argüz-B.-Pandharipande-Zvonkine, 2021)

All GW classes of all complete intersections in projective space are tautological.

Thank you for your attention!