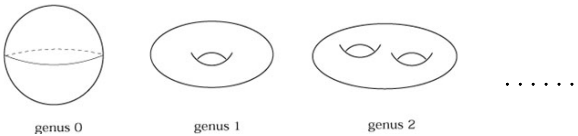

Relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$

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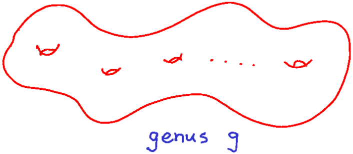
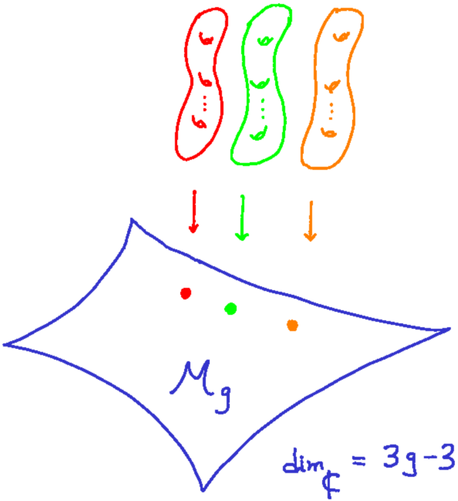
§I. **Nonsingular curves**

Let C be a complete, nonsingular, irreducible curve of genus $g \geq 2$:



The curve C has a complex structure which we can vary (while keeping the topology fixed).

Riemann studied the moduli space $\mathcal{M}_g$ of all genus g curves:



Riemann knew $\mathcal{M}_g$ was (essentially) a complex manifold of dimension 3g-3.

Theorie der *Abel*'schen Functionen.

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Riemann constructs the variations (via moving branch points), states the dimension, and coins the term moduli in a single sentence.

We are interested here in the cohomology of $\mathcal{M}_g$.

There are two basic questions:

(i) What is the cohomology $H^*(\mathcal{M}_g, \mathbb{Q})$ for fixed g?

(ii) What is the $\lim_{g\to\infty} H^*(\mathcal{M}_g, \mathbb{Q})$?

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(i) What is the cohomology $H^*(\mathcal{M}_g, \mathbb{Q})$ for fixed g?

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Both inspired by work of Mumford in the 70s and 80s following the previously developed Schubert calculus of the Grassmannian.

§II. **Grassmannian**

Let $\mathbb{C}^n$ be a n-dimensional complex vector space. The Grassmannian $\mathrm{Gr}(r, n)$ parameterizes all r-dimensional linear subspaces of $\mathbb{C}^n$.

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(ii) What is the $\lim_{n\to\infty} H^*(\mathrm{Gr}(r,n),\mathbb{Q})$?

The answers to (i) and (ii) are very well-known. The study has modern origins in Schubert's work. The rigorization of the Schubert calculus was Hilbert's 15^{th} problem.

Let $S \subset \mathbb{C}^n \times Gr(r, n)$ be the universal subbundle.

$$\begin{array}{ccc} S & \supset & V \\ \pi \downarrow & & \downarrow \\ Gr(r,n) & \ni & [V \subset \mathbb{C}^n] \end{array}$$

$$\dim_{\mathbb{C}} V = r$$

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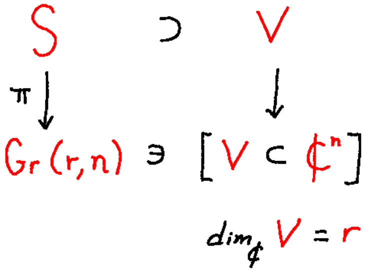

- $H^*(\mathrm{Gr}(r, n), \mathbb{Q})$ is generated by the Chern classes of S,

$$c_i(S) \in H^{2i}(\mathrm{Gr}(r, n), \mathbb{Q}) \ .$$

There are r Chern classes $c_1(S), \ldots, c_r(S)$.

Since S is a subbundle of the trivial rank n bundle over $\mathrm{Gr}(r, n)$, the quotient

$$0 \to \mathrm{S} \to \mathbb{C}^n \times \mathrm{Gr}(r, n) \to \mathrm{Q} \to 0$$

is a bundle Q of rank $n - r$. The Chern classes of Q are

$$c(\mathrm{Q}) = \sum_{i \geq 0} c_i(\mathrm{Q}) = \frac{1}{1 + c_1(\mathrm{S}) + \ldots + c_r(\mathrm{S})} .$$

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- The ideal of relations in $H^*(\mathrm{Gr}(r,n), \mathbb{Q})$ among the $c_i(S)$ is generated by the vanishing of the Chern classes of Q

$$c_{n-r+i}(Q) = \left[\frac{1}{1 + c_1(S) + \ldots + c_r(S)}\right]_{n-r+i} = 0$$

for $1 \leq i \leq r$.

The natural inclusion $\mathbb{C}^n \subset \mathbb{C}^{n+1}$, yields a natural inclusion

$$\mathrm{Gr}(r, n) \subset \mathrm{Gr}(r, n+1)$$

and natural limit $\lim_{n\to\infty} H^*(\mathrm{Gr}(r, n), \mathbb{Q})$.

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- Since the relations in $H^*(\mathrm{Gr}(r, n), \mathbb{Q})$ start in degree $n - r + 1$, the limit is free:

$$\lim_{n\to\infty} H^*(\mathrm{Gr}(r, n), \mathbb{Q}) = \mathbb{Q}[c_1(\mathrm{S}), \ldots, c_r(\mathrm{S})] \ .$$

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For the Grassmannian, we have very satisfactory answers to the two original questions in terms of tautological structures.

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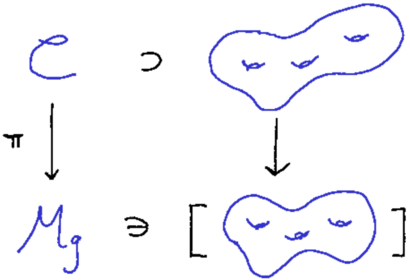
Answer: the universal curve $\mathcal{C}$:



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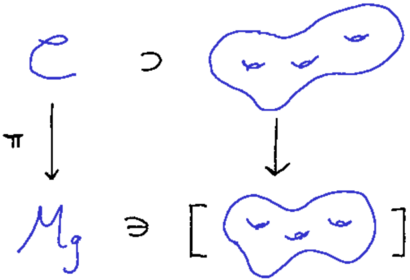
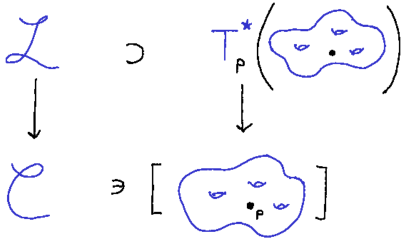


We can not directly take Chern classes of the universal curve since

$$\pi : \mathcal{C} \to \mathcal{M}_g$$

is not a vector bundle.

Let $\mathcal{L}$ be the cotangent line over the universal curve,



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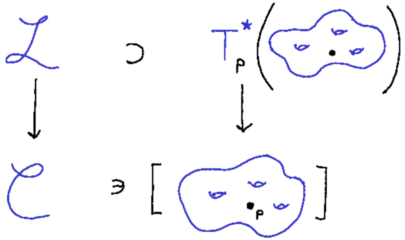

Since $\mathcal{L} \to \mathcal{C}$ is a line bundle, we can define

$$\psi = c_1(\mathcal{L}) \in H^2(\mathcal{C}, \mathbb{Q}) \ .$$

Via integration along the fiber of $\pi : \mathcal{C} \to \mathcal{M}_g$, we define

$$\kappa_i = \pi_*(\psi^{i+1}) \in H^{2i}(\mathcal{M}_g, \mathbb{Q}) \ .$$

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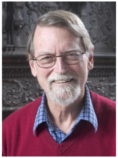
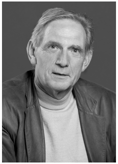
Question: Is $R^*(\mathcal{M}_g) = H^*(\mathcal{M}_g, \mathbb{Q})$?

Answer: No, but yes stably.

Mumford's Conjecture / Madsen-Weiss Theorem:

$$\lim_{g \to \infty} H^*(\mathcal{M}_g, \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \ .$$





For fixed genus g, we take Mumford's Conjecture as motivation to restriction our attention to the tautological subring

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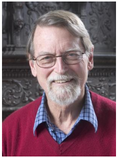
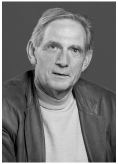
Other motivation comes from classical constructions in algebraic geometry: most interesting classes typically lie in $R^*(\mathcal{M}_g)$.

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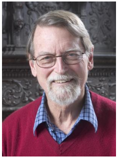 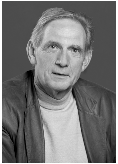 

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$$0 \to \mathcal{I}_g \to \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \to R^*(\mathcal{M}_g) \to 0 \ ?$$

§IV. Faber-Zagier **Conjecture**

Results by Looijenga and Faber determine the *lower end* of the tautological ring

$$R^{g-2}(\mathcal{M}_g) = \mathbb{Q} \ , \quad R^{>g-2}(\mathcal{M}_g) = 0 \ .$$

We use here the complex grading, so $R^{g-2}(\mathcal{M}_g) \subset H^{2(g-2)}(\mathcal{M}_g)$.

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Faber's method of construction involved the classical geometry of curves and Brill-Noether theory. The outcome in 2000 was the following Conjecture formulated with Zagier.

To write the Faber-Zagier relations, let the variable set

$$\mathbf{p} = \{\ p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \dots\ \}$$

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Define the series

$$\Psi(t, \mathbf{p}) = (1 + tp_3 + t^2 p_6 + t^3 p_9 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i$$

$$+ (p_1 + tp_4 + t^2 p_7 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i \ .$$

Since Ψ has constant term 1, we may take the logarithm.

Define the constants $C_r^{\mathrm{FZ}}(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathrm{FZ}}(\sigma)\ t^r \mathbf{p}^{\sigma}\ .$$

The sum is over all partitions σ of size $|\sigma|$ which avoid parts congruent to 2 modulo 3. To the partition

$$\sigma = 1^{n_1} 3^{n_3} 4^{n_4} \cdots,$$

we associate the monomial $\mathbf{p}^{\sigma} = p_1^{n_1} p_3^{n_3} p_4^{n_4} \cdots$.

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$$\gamma^{\mathsf{FZ}} = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ \kappa_r t^r \mathbf{p}^{\sigma}\ .$$

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For a series $\Theta \in \mathbb{Q}[\kappa][[t, \mathbf{p}]]$ in the variables κ_i, t, and p_j, let

$$[\Theta]_{t^r \mathbf{p}^{\sigma}}$$

denote the coefficient of $t^r \mathbf{p}^{\sigma}$ (which is a polynomial in the κ_i).

Theorem (P.-Pixton 2010)

In $R^r(\mathcal{M}_g)$, the Faber-Zagier relation

$$\left[\exp(-\gamma^{\mathsf{FZ}})\right]_{t^r \mathbf{p}^\sigma} = 0$$

holds when $g - 1 + |\sigma| < 3r$ and $g \equiv r + |\sigma| + 1 \mod 2$.

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For a given genus g and codimension r, the Theorem provides *finitely* many relations. The $\mathbb{Q}$-linear span of the Faber-Zagier relations determines an ideal

$$\mathcal{I}_g^{FZ} \subset \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \ .$$

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Answer: For $g \leq 23$, yes. For $g \geq 24$, the answer is unknown. Despite serious efforts using different methods (Faber, Q. Yin, Randal-Williams) no relation *not* in $\mathcal{I}_g^{FZ}$ has been found.

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Our construction of the Faber-Zagier relations uses the moduli space of stable quotients which mixes ingredients of Grothendieck's Quot scheme and the Deligne-Mumford compactification

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- The proof yields the following stronger boundary result. Under the hypotheses of the Theorem,

$$\left[\exp(-\gamma^{\mathsf{FZ}})\right]_{t^r \mathbf{p}^\sigma} \in R^*(\partial \overline{\mathcal{M}}_g) \ .$$

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§V. **Relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$**

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable pointed curves:

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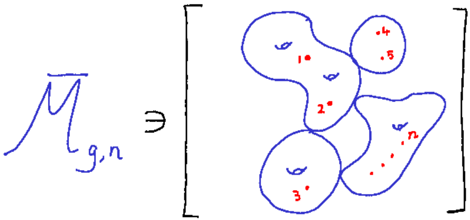
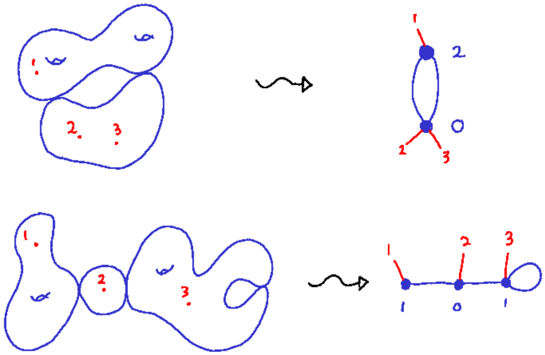


The boundary strata of the moduli $\overline{\mathcal{M}}_{g,n}$ of *fixed topological type* correspond to *stable graphs*.



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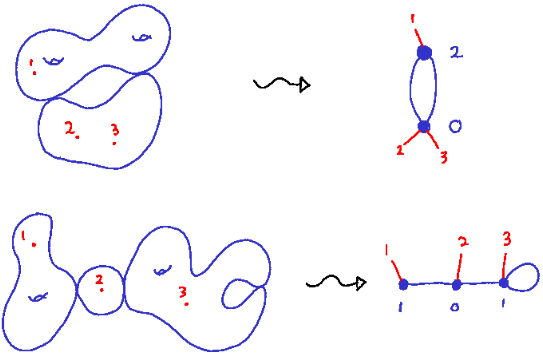

For such a graph Γ, let $[\Gamma] \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the class of the closure (with a multplicity related to symmetries of Γ).

Formally, a stable graph is the structure

$$\Gamma = (\mathrm{V}, \mathrm{E}, \mathrm{L}, \mathrm{g})$$

satisfying the following properties:

- V is the vertex set with a genus function $\mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}$,
- E is the edge set,
- L, the set of legs (corresponding to the set of markings),
- the pair (V, E) defines a *connected* graph,
- for each vertex v, the stability condition holds:

  $$2\mathrm{g}(v) - 2 + \mathrm{n}(v) > 0,$$

  where $\mathrm{n}(v)$ is the valence of Γ at v including both edges and legs.

The genus of a stable graph Γ is defined by:

$$\mathrm{g}(\Gamma) = \sum_{v \in V} \mathrm{g}(v) + h^1(\Gamma).$$

To each stable graph Γ, we associate the moduli space

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \mathrm{V}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}.$$

There is a canonical morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \ , \quad \xi_{\Gamma *}[\overline{\mathcal{M}}_\Gamma] = [\Gamma] \ .$$

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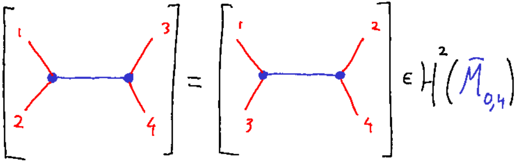

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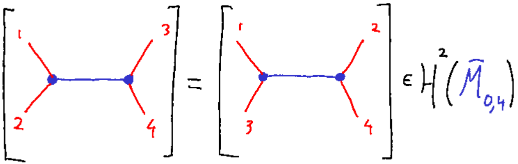

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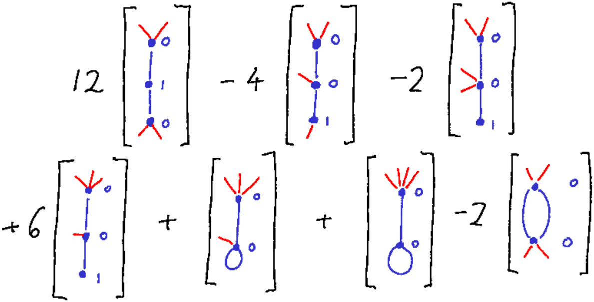
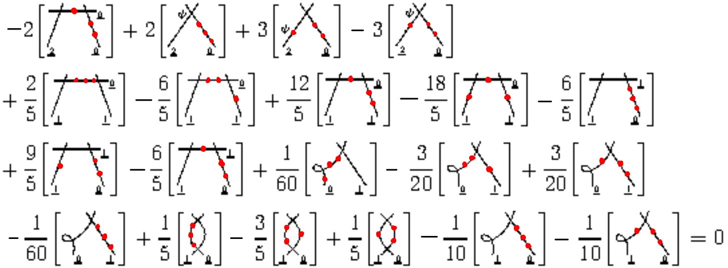
Getzler in 1996 found the first really interesting relation:



$$= 0 \in H^4(\overline{\mathcal{M}}_{1,4})$$

Of course there are more, but relations are not easy to find. The next interesting relation (Belorousski-P in 1998) occurs in genus 2:



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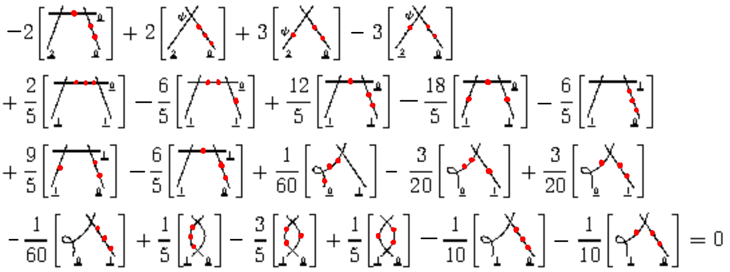

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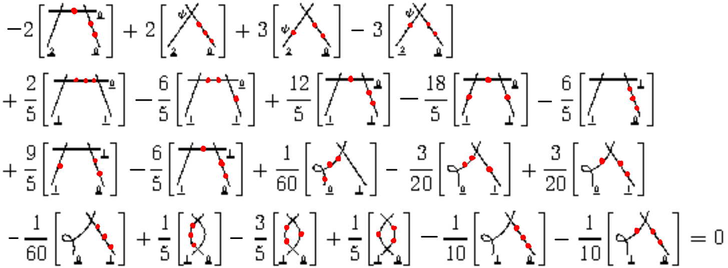

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Question: Is there any relationship to the Faber-Zagier relations?

§VI. Pixton's **relations on** $\overline{\mathcal{M}}_{g,n}$

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We define tautological classes $\mathcal{R}^d_{g,A}$ associated to the data

- $g, n \in \mathbb{Z}_{\geq 0}$ in the stable range $2g - 2 + n > 0$,
- $A = (a_1, \ldots, a_n), \quad a_i \in \{0, 1\}$,
- $d \in \mathbb{Z}_{\geq 0}$ satisfying $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$.

The elements $\mathcal{R}^d_{g,A}$ are expressed as sums over stable graphs of genus g with n legs. Pixton's relations then take the form

$$\mathcal{R}^d_{g,A} = 0 \quad \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

Before writing the formula for $\mathcal{R}^d_{g,A}$, a few definitions are required.

We have already seen the following two series:

$$B_0(T) = \sum_{m\geq 0} \frac{(6m)!}{(2m)!(3m)!}(-T)^m = 1 - 60\,T + 27720\,T^2 \cdots ,$$

$$B_1(T) = \sum_{m\geq 0} \frac{1+6m}{1-6m}\frac{(6m)!}{(2m)!(3m)!}(-T)^m = 1 + 84\,T - 32760\,T^2 \cdots .$$

These series control the original set of Faber-Zagier relations and continue to play a central role Pixton's relations.

Let $f(T)$ be a power series with vanishing constant and linear terms,

$$f(T) \in T^2\mathbb{Q}[[T]] \ .$$

For each $\overline{\mathcal{M}}_{g,n}$, we define

$$\kappa(f) = \sum_{m \geq 0} \frac{1}{m!} \ \pi_{m*}\Big(f(\psi_{n+1}) \cdots f(\psi_{n+m}) \Big) \ \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

where $\pi_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map. By the vanishing in degrees 0 and 1 of f, the sum is finite.

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For each vertex $v \in \mathrm{V}$ of a stable graph, we introduce an auxiliary variable ζ_v and impose the conditions

$$\zeta_v \zeta_{v'} = \zeta_{v'} \zeta_v \ , \quad \zeta_v^2 = 1 \ .$$

The variables ζ_v will be responsible for keeping track of a local parity condition at each vertex.

The formula for $\mathcal{R}^d_{g,A}$ is a sum over $\mathsf{G}_{g,n}$. The summand corresponding to $\Gamma \in \mathsf{G}_{g,n}$ is a product of vertex, leg, and edge factors:

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- For $\ell \in \mathrm{L}$, let $B_\ell = \zeta_{v(\ell)}^{a_\ell} B_{a_\ell}\big(\zeta_{v(\ell)}\psi_\ell\big)$, where $v(\ell) \in V$ is the vertex to which the leg is assigned.

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- For $e \in \mathrm{E}$, let

$$\Delta_e = \frac{\zeta' + \zeta'' - B_0(\zeta'\psi')\zeta'' B_1(\zeta''\psi'') - \zeta' B_1(\zeta'\psi')B_0(\zeta''\psi'')}{\psi' + \psi''}$$
$$= (60\zeta'\zeta'' - 84) + \left[32760(\zeta'\psi' + \zeta''\psi'') - 27720(\zeta'\psi'' + \zeta''\psi')\right] \cdots ,$$

where ζ', ζ'' are the ζ-variables assigned to the vertices adjacent to the edge e and ψ', ψ'' are the ψ-classes corresponding to the half-edges.

The numerator of Δ_e is divisible by the denominator due to the identity (discovered by Pixton)

$$B_0(T)B_1(-T) + B_0(-T)B_1(T) = 2.$$

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Definition (Pixton 2012)

Let $A = (a_1, \ldots, a_n) \in \{0,1\}^n$. We denote by $R^d_{g,A} \in H^{2d}(\overline{\mathcal{M}}_{g,n})$ the degree d component of the class

$$\sum_{\Gamma \in \mathrm{G}_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \left[\prod \kappa_v \prod B_\ell \prod \Delta_e\right]_{\prod_v \zeta_v^{\mathrm{g}(v)-1}}\right],$$

where the products are taken over all vertices, all legs, and all edges of the graph Γ.

The subscript $\prod_v \zeta_v^{\mathrm{g}(v)-1}$ indicates the coefficient of the monomial $\prod_v \zeta_v^{\mathrm{g}(v)-1}$ after the product inside the brackets is expanded.

Theorem (P.-Pixton-Zvonkine 2013)

For $2g - 2 + n > 0$, $a_i \in \{0,1\}$, *and* $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$, *Pixton's relations hold*

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