Quasimodular forms from Betti numbers

Pierrick Bousseau

ETH-ITS Zurich

3 June 2020

ETH Zoominar

Introduction

- Based on arXiv:1909.02985, arXiv:1909.02992, and, joint with Honglu Fan, Shuai Guo, Longting Wu, arXiv:2001.05347.

Introduction

- Based on arXiv:1909.02985, arXiv:1909.02992, and, joint with Honglu Fan, Shuai Guo, Longting Wu, arXiv:2001.05347.
- Main result: quasimodularity result for generating series constructed from Betti numbers of moduli spaces of one-dimensional semistable sheaves on $\mathbb{P}^2$.

Introduction

- Based on arXiv:1909.02985, arXiv:1909.02992, and, joint with Honglu Fan, Shuai Guo, Longting Wu, arXiv:2001.05347.
- Main result: quasimodularity result for generating series constructed from Betti numbers of moduli spaces of one-dimensional semistable sheaves on $\mathbb{P}^2$.
- Refined Donaldson-Thomas invariants for one-dimensional sheaves on $K_{\mathbb{P}^2}$ ('local $\mathbb{P}^2$': non-compact Calabi-Yau 3-fold).

Introduction

- Based on arXiv:1909.02985, arXiv:1909.02992, and, joint with Honglu Fan, Shuai Guo, Longting Wu, arXiv:2001.05347.
- Main result: quasimodularity result for generating series constructed from Betti numbers of moduli spaces of one-dimensional semistable sheaves on $\mathbb{P}^2$.
- Refined Donaldson-Thomas invariants for one-dimensional sheaves on $K_{\mathbb{P}^2}$ ('local $\mathbb{P}^2$': non-compact Calabi-Yau 3-fold).
- Refined genus 0 Gopakumar-Vafa invariants of $K_{\mathbb{P}^2}$.

Introduction

- Based on arXiv:1909.02985, arXiv:1909.02992, and, joint with Honglu Fan, Shuai Guo, Longting Wu, arXiv:2001.05347.
- Main result: quasimodularity result for generating series constructed from Betti numbers of moduli spaces of one-dimensional semistable sheaves on $\mathbb{P}^2$.
- Refined Donaldson-Thomas invariants for one-dimensional sheaves on $K_{\mathbb{P}^2}$ ('local $\mathbb{P}^2$': non-compact Calabi-Yau 3-fold).
- Refined genus 0 Gopakumar-Vafa invariants of $K_{\mathbb{P}^2}$.
- Conjecture of Huang-Klemm (around 2010) on the Nekrasov-Shatashvili limit of refined topological string theory on $K_{\mathbb{P}^2}$.

Modularity

- (a_n) a sequence of numbers (of geometric, number theoretic,... interest)

Modularity

- (a_n) a sequence of numbers (of geometric, number theoretic,... interest)
- Form a generating series

$$\sum_n a_n q^n \,,$$

formal power series in a formal variable q.

Modularity

Often, a miracle happens, $\sum_n a_n q^n$ is the q-series expansion of a modular function, that is:

Modularity

Often, a miracle happens, $\sum_n a_n q^n$ is the q-series expansion of a modular function, that is:

- Writing $q = e^{2i\pi\tau}$, $f(\tau) := \sum_n a_n q^n$ is a holomorphic function on the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\tau > 0\}$

Modularity

Often, a miracle happens, $\sum_n a_n q^n$ is the q-series expansion of a modular function, that is:

- Writing $q = e^{2i\pi\tau}$, $f(\tau) := \sum_n a_n q^n$ is a holomorphic function on the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\tau > 0\}$
- Symmetry property of $f(\tau)$ with respect to the natural action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$: $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. More precisely, $f(\tau)$ is modular of weight k for $SL(2,\mathbb{Z})$ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)\,,$$

for every

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Modularity

Variants:

- For Γ a subgroup of finite index in $SL(2,\mathbb{Z})$, define modularity for Γ by restrcting to elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

- The group $\Gamma := \Gamma_1(3)$ will appear later:

$$\Gamma_1(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 3 \right\},$$

Modularity

Variants:

Modularity

Variants:

- $f(\tau)$ is quasimodular of weight k for Γ if there exists finitely many non-zero holomorphic functions $f_j(\tau)$ such that

$$(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{j\geq 0} \left(\frac{c}{c\tau+d}\right)^j f_j(\tau)$$

  for every

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

Modularity

Variants:

- $f(\tau)$ is quasimodular of weight k for Γ if there exists finitely many non-zero holomorphic functions $f_j(\tau)$ such that

$$(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{j\geq 0} \left(\frac{c}{c\tau+d}\right)^j f_j(\tau)$$

for every

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

- Example:

$$E_2(\tau) := 1 - 24 \sum_{n\geq 1} \frac{nq^n}{1-q^n}$$

is quasimodular of weight 2 for $SL(2,\mathbb{Z})$ (not modular).

Modularity

- Quasimodular form: quasimodular function holomorphic at the cusps.

Modularity

- Quasimodular form: quasimodular function holomorphic at the cusps.
- $$A(\tau) := \left(\frac{\eta(\tau)^9}{\eta(3\tau)^3} + 27 \frac{\eta(3\tau)^9}{\eta(\tau)^3} \right)^{\frac{1}{3}}, \quad B(\tau) := \frac{1}{4}\big(E_2(\tau) + 3E_2(3\tau)\big),$$
$$C(\tau) := \frac{\eta(\tau)^9}{\eta(3\tau)^3},$$
where
$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$
is the Dedekind eta function.

Modularity

- Quasimodular form: quasimodular function holomorphic at the cusps.
- $$A(\tau) := \left(\frac{\eta(\tau)^9}{\eta(3\tau)^3} + 27 \frac{\eta(3\tau)^9}{\eta(\tau)^3} \right)^{\frac{1}{3}}, \quad B(\tau) := \frac{1}{4} \left(E_2(\tau) + 3E_2(3\tau) \right),$$
$$C(\tau) := \frac{\eta(\tau)^9}{\eta(3\tau)^3},$$
where
$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$
is the Dedekind eta function.
- The functions A, B, and C are quasimodular forms for $\Gamma_1(3)$. More precisely, A and C are modular respectively of weight 1 and 3, and B is quasimodular of weight 2. In fact, A, B, and C freely generate the ring of quasimodular forms of $\Gamma_1(3)$:
$$\mathrm{QMod}(\Gamma_1(3)) = \mathbb{C}[A, B, C].$$

Moduli spaces

- Complex projective plane $\mathbb{P}^2$

Moduli spaces

- Complex projective plane $\mathbb{P}^2$
- Linear system $|\mathcal{O}(d)|$ of degree d curve in $\mathbb{P}^2$

Moduli spaces

- Complex projective plane $\mathbb{P}^2$
- Linear system $|\mathcal{O}(d)|$ of degree d curve in $\mathbb{P}^2$
- General curve in $|\mathcal{O}(d)|$ is a smooth projective curve of genus $\frac{(d-1)(d-2)}{2}$.

Moduli spaces

- Complex projective plane $\mathbb{P}^2$
- Linear system $|\mathcal{O}(d)|$ of degree d curve in $\mathbb{P}^2$
- General curve in $|\mathcal{O}(d)|$ is a smooth projective curve of genus $\frac{(d-1)(d-2)}{2}$.
- C smooth projective curve of genus g, $\mathrm{Pic}^n(C)$: moduli space of lines bundles L on C with $\chi(L) = n$, abelian variety of dimension g.

Moduli spaces

- Complex projective plane $\mathbb{P}^2$
- Linear system $|\mathcal{O}(d)|$ of degree d curve in $\mathbb{P}^2$
- General curve in $|\mathcal{O}(d)|$ is a smooth projective curve of genus $\frac{(d-1)(d-2)}{2}$.
- C smooth projective curve of genus g, $\mathrm{Pic}^n(C)$: moduli space of lines bundles L on C with $\chi(L) = n$, abelian variety of dimension g.
- Relative version of Pic^n over the open locus in $|\mathcal{O}(d)|$ of smooth projective curves.

Moduli spaces

- Complex projective plane $\mathbb{P}^2$
- Linear system $|\mathcal{O}(d)|$ of degree d curve in $\mathbb{P}^2$
- General curve in $|\mathcal{O}(d)|$ is a smooth projective curve of genus $\frac{(d-1)(d-2)}{2}$.
- C smooth projective curve of genus g, $\mathrm{Pic}^n(C)$: moduli space of lines bundles L on C with $\chi(L) = n$, abelian variety of dimension g.
- Relative version of Pic^n over the open locus in $|\mathcal{O}(d)|$ of smooth projective curves.
- Compactification over $|\mathcal{O}(d)|$?

Moduli spaces

- Consider coherent sheaves on $\mathbb{P}^2$ with one-dimensional support.

Moduli spaces

- Consider coherent sheaves on $\mathbb{P}^2$ with one-dimensional support.
- Numerical invariants: degree $d(F)$, Euler characteristic $\chi(F)$.

Moduli spaces

- Consider coherent sheaves on $\mathbb{P}^2$ with one-dimensional support.
- Numerical invariants: degree $d(F)$, Euler characteristic $\chi(F)$.
- F coherent sheaf on $\mathbb{P}^2$ with one-dimensional support is called Gieseker semistable (resp. stable) if F is pure (every non-zero subsheaf of F has one-dimensional support) and, for every non-zero strict subsheaf F' of F, we have $\frac{\chi(F')}{d(F')} \leq \frac{\chi(F)}{d(F)}$ (resp. $\frac{\chi(F')}{d(F')} < \frac{\chi(F)}{d(F)}$).

Moduli spaces

- Consider coherent sheaves on $\mathbb{P}^2$ with one-dimensional support.
- Numerical invariants: degree $d(F)$, Euler characteristic $\chi(F)$.
- F coherent sheaf on $\mathbb{P}^2$ with one-dimensional support is called Gieseker semistable (resp. stable) if F is pure (every non-zero subsheaf of F has one-dimensional support) and, for every non-zero strict subsheaf F' of F, we have $\frac{\chi(F')}{d(F')} \leq \frac{\chi(F)}{d(F)}$ (resp. $\frac{\chi(F')}{d(F')} < \frac{\chi(F)}{d(F)}$).
- Moduli space (good moduli space for the Artin stack of Gieseker semistable sheaves):

$$M_{d,n} = \{\text{S-equivalence classes of Gieseker semistable coherent sheaves}$$

$$F \text{ on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

Moduli spaces

$$M_{d,n} = \{\text{S-equivalence classes of Gieseker semistable coherent sheaves}$$
$$F \text{ on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

Moduli spaces

$$M_{d,n} = \{\text{S-equivalence classes of Gieseker semistable coherent sheaves}$$

$$F \text{ on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

- (Simpson, Le Potier, around 1990) $M_{d,n}$ irreducible algebraic projective variety of dimension $d^2 + 1$, smooth if $\gcd(d,n) = 1$, singular in general.

Moduli spaces

$M_{d,n} = \{$S-equivalence classes of Gieseker semistable coherent sheaves

$$F \text{ on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

- (Simpson, Le Potier, around 1990) $M_{d,n}$ irreducible algebraic projective variety of dimension $d^2 + 1$, smooth if $\gcd(d,n) = 1$, singular in general.
- $\pi \colon M_{d,n} \to |\mathcal{O}(d)|$, $F \mapsto \mathrm{supp}(F)$.

Moduli spaces

$M_{d,n} = \{$S-equivalence classes of Gieseker semistable coherent sheaves

$$F \text{ on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

- (Simpson, Le Potier, around 1990) $M_{d,n}$ irreducible algebraic projective variety of dimension $d^2 + 1$, smooth if $\gcd(d,n) = 1$, singular in general.
- $\pi \colon M_{d,n} \to |\mathcal{O}(d)|$, $F \mapsto \text{supp}(F)$.
- $\pi^{-1}(C) = \text{Pic}^n(C)$ if C is smooth.

Moduli spaces

$M_{d,n} = \{$S-equivalence classes of Gieseker semistable coherent sheaves

$$F \text{ on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

- (Simpson, Le Potier, around 1990) $M_{d,n}$ irreducible algebraic projective variety of dimension $d^2 + 1$, smooth if $\gcd(d,n) = 1$, singular in general.
- $\pi\colon M_{d,n} \to |\mathcal{O}(d)|$, $F \mapsto \mathrm{supp}(F)$.
- $\pi^{-1}(C) = \mathrm{Pic}^n(C)$ if C is smooth.
- Fiber $\pi^{-1}(C)$ complicated if C is singular.

Moduli spaces

$M_{d,n} = \{$S-equivalence classes of Gieseker semistable coherent sheaves

$$F \text{ on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

- (Simpson, Le Potier, around 1990) $M_{d,n}$ irreducible algebraic projective variety of dimension $d^2 + 1$, smooth if $\gcd(d, n) = 1$, singular in general.
- $\pi \colon M_{d,n} \to |\mathcal{O}(d)|$, $F \mapsto \mathrm{supp}(F)$.
- $\pi^{-1}(C) = \mathrm{Pic}^n(C)$ if C is smooth.
- Fiber $\pi^{-1}(C)$ complicated if C is singular.
- The global topology of $M_{d,n}$ is non-trivial.
- Betti numbers $b_j(M_{d,n})$ (for the intersection cohomology if $M_{d,n}$ is singular). It is known that $b_j(M_{d,n})$ only depends on $n \mod d$. Conjecturally, $b_j(M_{d,n})$ is independent of n.

Moduli spaces

$$M_{d,n} = \{F \text{ Gieseker semistable coherent sheaf on } \mathbb{P}^2$$
$$\text{with } d(F) = d, \chi(F) = n\},$$

- Betti numbers $b_j(M_{d,n})$ (for the intersection cohomology if $M_{d,n}$ is singular).

Moduli spaces

$$M_{d,n} = \{F \text{ Gieseker semistable coherent sheaf on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

- Betti numbers $b_j(M_{d,n})$ (for the intersection cohomology if $M_{d,n}$ is singular).
- $F \mapsto F \otimes \mathcal{O}(1)$ induces isomorphisms $M_{d,n} \simeq M_{d,n+d}$, so the Betti numbers $b_j(M_{d,n})$ only depends on $n \mod d$.

Moduli spaces

$$M_{d,n} = \{F \text{ Gieseker semistable coherent sheaf on } \mathbb{P}^2$$
$$\text{with } d(F) = d, \chi(F) = n\},$$

- Betti numbers $b_j(M_{d,n})$ (for the intersection cohomology if $M_{d,n}$ is singular).
- $F \mapsto F \otimes \mathcal{O}(1)$ induces isomorphisms $M_{d,n} \simeq M_{d,n+d}$, so the Betti numbers $b_j(M_{d,n})$ only depends on $n \mod d$.
- Conjecturally, $b_j(M_{d,n})$ is independent of n.

Moduli spaces

$$M_{d,n} = \{F \text{ Gieseker semistable coherent sheaf on } \mathbb{P}^2$$
$$\text{with } d(F) = d, \chi(F) = n\},$$

- Betti numbers $b_j(M_{d,n})$ (for the intersection cohomology if $M_{d,n}$ is singular).
- $F \mapsto F \otimes \mathcal{O}(1)$ induces isomorphisms $M_{d,n} \simeq M_{d,n+d}$, so the Betti numbers $b_j(M_{d,n})$ only depends on $n \mod d$.
- Conjecturally, $b_j(M_{d,n})$ is independent of n.
- Define
$$b_j(M_d) := \frac{1}{d} \sum_{n \mod d} b_j(M_{d,n}).$$

Examples

- $\sum_j b_j(M_1) y^{\frac{j}{2}} = 1 + y + y^2$

Examples

- $\sum_j b_j(M_1) y^{\frac{j}{2}} = 1 + y + y^2$
- $\sum_j b_j(M_2) y^{\frac{j}{2}} = 1 + y + y^2 + y^3 + y^4 + y^5$

Examples

- $\sum_j b_j(M_1) y^{\frac{j}{2}} = 1 + y + y^2$
- $\sum_j b_j(M_2) y^{\frac{j}{2}} = 1 + y + y^2 + y^3 + y^4 + y^5$
- $\sum_j b_j(M_3) y^{\frac{j}{2}} = 1 + 2y + 3y^2 + 3y^3 + 3y^4 + 3y^5 + 3y^6 + 3y^7 + 3y^8 + 2y^9 + y^{10}$

Examples

- $\sum_j b_j(M_1) y^{\frac{j}{2}} = 1 + y + y^2$
- $\sum_j b_j(M_2) y^{\frac{j}{2}} = 1 + y + y^2 + y^3 + y^4 + y^5$
- $\sum_j b_j(M_3) y^{\frac{j}{2}} = 1+2y+3y^2+3y^3+3y^4+3y^5+3y^6+3y^7+3y^8+2y^9+y^{10}$
- $\sum_j b_j(M_4) y^{\frac{j}{2}} = 1 + 2y + 6y^2 + 10y^3 + 14y^4 + 15y^5 + 16y^6 + 16y^7 + 16y^8 + 16y^9 + 16y^{10} + 16y^{11} + 15y^{12} + 14y^{13} + 10y^{14} + 6y^{15} + 2y^{16} + y^{17}$

DT invariants

- Donaldson-Thomas invariants: sheaf counting on Calabi-Yau 3-folds.

DT invariants

- Donaldson-Thomas invariants: sheaf counting on Calabi-Yau 3-folds.
- Calabi-Yau 3-fold ? $K_{\mathbb{P}^2}$, total space of the canonical line bundle $\mathcal{O}(-3)$ of $\mathbb{P}^2$.

DT invariants

- Donaldson-Thomas invariants: sheaf counting on Calabi-Yau 3-folds.
- Calabi-Yau 3-fold ? $K_{\mathbb{P}^2}$, total space of the canonical line bundle $\mathcal{O}(-3)$ of $\mathbb{P}^2$.
- Negativity of $\mathcal{O}(-3)$ implies that Gieseker semistable one-dimensional sheaves on $K_{\mathbb{P}^2}$ are scheme-theoretically supported on $\mathbb{P}^2$.

DT invariants

- Donaldson-Thomas invariants: sheaf counting on Calabi-Yau 3-folds.
- Calabi-Yau 3-fold ? $K_{\mathbb{P}^2}$, total space of the canonical line bundle $\mathcal{O}(-3)$ of $\mathbb{P}^2$.
- Negativity of $\mathcal{O}(-3)$ implies that Gieseker semistable one-dimensional sheaves on $K_{\mathbb{P}^2}$ are scheme-theoretically supported on $\mathbb{P}^2$.
- $M_{d,n}$ is a moduli space of coherent sheaves on $K_{\mathbb{P}^2}$.

DT invariants

- Donaldson-Thomas invariants: sheaf counting on Calabi-Yau 3-folds.
- Calabi-Yau 3-fold ? $K_{\mathbb{P}^2}$, total space of the canonical line bundle $\mathcal{O}(-3)$ of $\mathbb{P}^2$.
- Negativity of $\mathcal{O}(-3)$ implies that Gieseker semistable one-dimensional sheaves on $K_{\mathbb{P}^2}$ are scheme-theoretically supported on $\mathbb{P}^2$.
- $M_{d,n}$ is a moduli space of coherent sheaves on $K_{\mathbb{P}^2}$.
- $\mathrm{Ext}^2(E,E) = 0$ if E Gieseker semistable with one-dimensional support.

DT invariants

- Donaldson-Thomas invariants: sheaf counting on Calabi-Yau 3-folds.
- Calabi-Yau 3-fold ? $K_{\mathbb{P}^2}$, total space of the canonical line bundle $\mathcal{O}(-3)$ of $\mathbb{P}^2$.
- Negativity of $\mathcal{O}(-3)$ implies that Gieseker semistable one-dimensional sheaves on $K_{\mathbb{P}^2}$ are scheme-theoretically supported on $\mathbb{P}^2$.
- $M_{d,n}$ is a moduli space of coherent sheaves on $K_{\mathbb{P}^2}$.
- $\mathrm{Ext}^2(E,E) = 0$ if E Gieseker semistable with one-dimensional support.
- Connection between intersection cohomology and DT invariants under the Ext^2-vanishing assumption: Meinhardt-Reineke, Meinhardt.

DT invariants

- Donaldson-Thomas invariants: sheaf counting on Calabi-Yau 3-folds.
- Calabi-Yau 3-fold ? $K_{\mathbb{P}^2}$, total space of the canonical line bundle $\mathcal{O}(-3)$ of $\mathbb{P}^2$.
- Negativity of $\mathcal{O}(-3)$ implies that Gieseker semistable one-dimensional sheaves on $K_{\mathbb{P}^2}$ are scheme-theoretically supported on $\mathbb{P}^2$.
- $M_{d,n}$ is a moduli space of coherent sheaves on $K_{\mathbb{P}^2}$.
- $\mathrm{Ext}^2(E, E) = 0$ if E Gieseker semistable with one-dimensional support.
- Connection between intersection cohomology and DT invariants under the Ext^2-vanishing assumption: Meinhardt-Reineke, Meinhardt.
- $b_j(M_{d,n})$ are refined DT invariants of the non-compact Calabi-Yau 3-fold $K_{\mathbb{P}^2}$.

GV invariants

- Unrefined limit: replace Betti numbers by Euler characteristic.

$$e(M_d) := \sum_j b_j(M_d)(-1)^j$$

GV invariants

- Unrefined limit: replace Betti numbers by Euler characteristic.

$$e(M_d) := \sum_j b_j(M_d)(-1)^j$$

- $e(M_d) = (-1)^{d-1} n_{0,d}^{K_{\mathbb{P}^2}}$
- $n_{0,d}^{K_{\mathbb{P}^2}}$ genus 0 Gopakumar-Vafa of $K_{\mathbb{P}^2}$, encoding genus 0 Gromov-Witten theory of $K_{\mathbb{P}^2}$.

GV invariants

- Unrefined limit: replace Betti numbers by Euler characteristic.

$$e(M_d) := \sum_j b_j(M_d)(-1)^j$$

- $e(M_d) = (-1)^{d-1} n_{0,d}^{K_{\mathbb{P}^2}}$
- $n_{0,d}^{K_{\mathbb{P}^2}}$ genus 0 Gopakumar-Vafa of $K_{\mathbb{P}^2}$, encoding genus 0 Gromov-Witten theory of $K_{\mathbb{P}^2}$.
- Katz's conjecture, known for $K_{\mathbb{P}^2}$ by MNOP+Toda+Konishi.

GV invariants

- Unrefined limit: replace Betti numbers by Euler characteristic.

$$e(M_d) := \sum_j b_j(M_d)(-1)^j$$

- $e(M_d) = (-1)^{d-1} n_{0,d}^{K_{\mathbb{P}^2}}$
- $n_{0,d}^{K_{\mathbb{P}^2}}$ genus 0 Gopakumar-Vafa of $K_{\mathbb{P}^2}$, encoding genus 0 Gromov-Witten theory of $K_{\mathbb{P}^2}$.
- Katz's conjecture, known for $K_{\mathbb{P}^2}$ by MNOP+Toda+Konishi.
- $n_{0,1}^{K_{\mathbb{P}^2}} = 3$, $n_{0,2}^{K_{\mathbb{P}^2}} = -6$, $n_{0,3}^{K_{\mathbb{P}^2}} = 27$, $n_{0,4}^{K_{\mathbb{P}^2}} = -192$.

GV invariants

- Unrefined limit: replace Betti numbers by Euler characteristic.

$$e(M_d) := \sum_j b_j(M_d)(-1)^j$$

- $e(M_d) = (-1)^{d-1} n_{0,d}^{K_{\mathbb{P}^2}}$
- $n_{0,d}^{K_{\mathbb{P}^2}}$ genus 0 Gopakumar-Vafa of $K_{\mathbb{P}^2}$, encoding genus 0 Gromov-Witten theory of $K_{\mathbb{P}^2}$.
- Katz's conjecture, known for $K_{\mathbb{P}^2}$ by MNOP+Toda+Konishi.
- $n_{0,1}^{K_{\mathbb{P}^2}} = 3$, $n_{0,2}^{K_{\mathbb{P}^2}} = -6$, $n_{0,3}^{K_{\mathbb{P}^2}} = 27$, $n_{0,4}^{K_{\mathbb{P}^2}} = -192$.
- Think about $\sum_j b_j(M_d) y^{\frac{j}{2}}$ as a refined genus 0 Gopakumar-Vafa invariant.

Moduli spaces

$$M_{d,n} = \{F \text{ Gieseker semistable coherent sheaf on } \mathbb{P}^2 \text{ with } d(F) = d, \chi(F) = n\},$$

- Betti numbers $b_j(M_{d,n})$ (for the intersection cohomology if $M_{d,n}$ is singular).
- $F \mapsto F \otimes \mathcal{O}(1)$ induces isomorphisms $M_{d,n} \simeq M_{d,n+d}$, so the Betti numbers $b_j(M_{d,n})$ only depends on $n \mod d$.
- Conjecturally, $b_j(M_{d,n})$ is independent of n.
- Define

$$b_j(M_d) := \frac{1}{d} \sum_{n \mod d} b_j(M_{d,n}).$$

Generating series

- 'Obvious' generating series

$$\sum_{d\geq 1}\sum_{j} b_j(M_d) y^{\frac{j}{2}} Q^d \,.$$

Generating series

- ‘Obvious’ generating series

$$\sum_{d \geq 1} \sum_{j} b_j(M_d) y^{\frac{j}{2}} Q^d .$$

- ‘Almost obvious’ generating series (from the DT point of view)

$$i \sum_{d \geq 1} \sum_{\ell \geq 1} \frac{(-1)^{d-1}}{\ell} \frac{y^{-\frac{\ell}{2}(d^2+1)} \sum_j b_j(M_d) y^{\frac{\ell j}{2}}}{y^{\frac{\ell}{2}} - y^{-\frac{\ell}{2}}} Q^{\ell d}$$

Generating series

- 'Obvious' generating series

$$\sum_{d\geq 1} \sum_j b_j(M_d) y^{\frac{j}{2}} Q^d \,.$$

- 'Almost obvious' generating series (from the DT point of view)

$$i \sum_{d\geq 1} \sum_{\ell \geq 1} \frac{(-1)^{d-1}}{\ell} \frac{y^{-\frac{\ell}{2}(d^2+1)} \sum_j b_j(M_d) y^{\frac{\ell j}{2}}}{y^{\frac{\ell}{2}} - y^{-\frac{\ell}{2}}} Q^{\ell d}$$

- Not obvious step at all (string theory prediction of Huang and Klemm): write $y = e^{i\hbar}$ and expand in powers of $\hbar$.

Theorem: Quasimodularity

Define series $F_g^{NS}(Q) \in \mathbb{Q}[[Q]]$ by the change of variables $y = e^{i\hbar} = \sum_{n\geq 0} \frac{(i\hbar)^n}{n!}$:

$$i \sum_{d\geq 1} \sum_{\ell \geq 1} \frac{(-1)^{d-1}}{\ell} \frac{y^{-\frac{\ell}{2}(d^2+1)} \sum_j b_j(M_d) y^{\frac{\ell j}{2}}}{y^{\frac{\ell}{2}} - y^{-\frac{\ell}{2}}} Q^{\ell d}$$

$$= \sum_{g\geq 0} F_g^{NS}(Q)(-1)^g \hbar^{2g-1} .$$

Theorem: Quasimodularity

Define series $F_g^{NS}(Q) \in \mathbb{Q}[[Q]]$ by the change of variables $y = e^{i\hbar} = \sum_{n\geq 0} \frac{(i\hbar)^n}{n!}$:

$$i \sum_{d\geq 1} \sum_{\ell \geq 1} \frac{(-1)^{d-1}}{\ell} \frac{y^{-\frac{\ell}{2}(d^2+1)} \sum_j b_j(M_d) y^{\frac{\ell j}{2}}}{y^{\frac{\ell}{2}} - y^{-\frac{\ell}{2}}} Q^{\ell d}$$

$$= \sum_{g\geq 0} F_g^{NS}(Q)(-1)^g \hbar^{2g-1} .$$

Theorem [B.,Fan,Guo,Wu, 2020]

- F_0^{NS} and F_1^{NS} can be 'explicitly' computed.
- There exists an explicit change of variables $Q \mapsto q = e^{2i\pi\tau}$ such that, for every $g \geq 2$, $F_g^{NS}(\tau)$ is a weight 0 quasimodular function for $\Gamma_1(3)$.

Theorem: Quasimodularity

Theorem [B.,Fan,Guo,Wu, 2020]

More precisely, for every $g \geq 2$, we have

$$F_g^{NS} \in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6} \,.$$

Moreover, we have $\deg_B F_g^{NS} \leq 2g - 3$.

Theorem: Quasimodularity

Theorem [B.,Fan,Guo,Wu, 2020]

More precisely, for every $g \geq 2$, we have

$$F_g^{NS} \in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6} \,.$$

Moreover, we have $\deg_B F_g^{NS} \leq 2g-3$.

Example:

$$F_2^{NS} = \frac{1}{11520\, C^2} \left(-37A^6 + 5A^4 B + 48A^3 C - 16C^2\right).$$

Theorem: Holomorphic anomaly equation

Theorem [B.,Fan,Guo,Wu, 2020]

For every $g \geq 2$, we have

$$2\frac{\partial}{\partial B} F_g^{NS} = \frac{1}{2} \sum_{j=1}^{g-1} \left(Q \frac{d}{dQ} F_j^{NS} \right) \left(Q \frac{d}{dQ} F_{g-j}^{NS} \right) .$$

Context

- Previous results solve a special case (the 'refined genus 0 case' = 'Nekrasov-Shatashvili limit') of physics conjectures about the refined topological string theory of $K_{\mathbb{P}^2}$ (Huang-Klemm, 2010).

Context

- Previous results solve a special case (the 'refined genus 0 case' = 'Nekrasov-Shatashvili limit') of physics conjectures about the refined topological string theory of $K_{\mathbb{P}^2}$ (Huang-Klemm, 2010).
- First mathematical result in the 'refined' direction.

Context

- Previous results solve a special case (the 'refined genus 0 case' = 'Nekrasov-Shatashvili limit') of physics conjectures about the refined topological string theory of $K_{\mathbb{P}^2}$ (Huang-Klemm, 2010).
- First mathematical result in the 'refined' direction.
- Unrefined topological string: higher genus Gromov-Witten theory of $K_{\mathbb{P}^2}$. Generating series of genus g Gromov-Witten invariants of $K_{\mathbb{P}^2}$:

$$F_g^{GW}(Q) := \sum_{d \geq 1} N_{g,d}^{GW, K_{\mathbb{P}^2}} Q^d \, .$$

Context

Theorem (Lho-Pandharipande, Coates-Iritani, 2017-2018)

- F_0^{GW} and F_1^{GW} explicitly known.

Context

Theorem (Lho-Pandharipande, Coates-Iritani, 2017-2018)

- F_0^{GW} and F_1^{GW} explicitly known.
- There exists an explicit change of variables $Q \mapsto q = e^{2i\pi\tau}$ such that, for every $g \geq 2$, $F_g^{GW}(\tau)$ is a weight 0 quasimodular function for $\Gamma_1(3)$: $F_g^{GW} \in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6}$.

Context

Theorem (Lho-Pandharipande, Coates-Iritani, 2017-2018)

- F_0^{GW} and F_1^{GW} explicitly known.
- There exists an explicit change of variables $Q \mapsto q = e^{2i\pi\tau}$ such that, for every $g \geq 2$, $F_g^{GW}(\tau)$ is a weight 0 quasimodular function for $\Gamma_1(3)$: $F_g^{GW} \in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6}$.
- For every $g \geq 2$, we have

$$2\frac{\partial}{\partial B} F_g^{GW} = \frac{1}{2}\sum_{j=1}^{g-1}\left(Q\frac{d}{dQ}F_j^{GW}\right)\left(Q\frac{d}{dQ}F_{g-j}^{GW}\right) + \frac{1}{2}\left(Q\frac{d}{dQ}\right)^2 F_{g-1}^{GW}\,.$$

Context

Theorem (Lho-Pandharipande, Coates-Iritani, 2017-2018)

- F_0^{GW} and F_1^{GW} explicitly known.
- There exists an explicit change of variables $Q \mapsto q = e^{2i\pi\tau}$ such that, for every $g \geq 2$, $F_g^{GW}(\tau)$ is a weight 0 quasimodular function for $\Gamma_1(3)$: $F_g^{GW} \in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6}$.
- For every $g \geq 2$, we have

$$2\frac{\partial}{\partial B} F_g^{GW} = \frac{1}{2}\sum_{j=1}^{g-1}\left(Q\frac{d}{dQ}F_j^{GW}\right)\left(Q\frac{d}{dQ}F_{g-j}^{GW}\right) + \frac{1}{2}\left(Q\frac{d}{dQ}\right)^2 F_{g-1}^{GW}.$$

Example:

$$F_2^{GW} = \frac{1}{8640C^2}(-8A^6 + 30A^4B - 45A^2B^2 + 25B^3 + 2A^3C - 4C^2)$$

Context: refinement

- Gromov–Witten/stable pairs correspondence (MNOP), the series $\bar{F}_g^{K_{\mathbb{P}^2}}$ can be described in terms of the stable pairs invariants $P_{d,n}$ of $K_{\mathbb{P}^2}$:

$$1 + \sum_{d \geq 1} \sum_{n \in \mathbb{Z}} P_{d,n} (-x)^n Q^d = \exp \left(\sum_{g \geq 0} F_g^{GW} u^{2g-2} \right)$$

where $x = e^{iu}$.

Context: refinement

- Gromov–Witten/stable pairs correspondence (MNOP), the series $\bar{F}_g^{K_{\mathbb{P}^2}}$ can be described in terms of the stable pairs invariants $P_{d,n}$ of $K_{\mathbb{P}^2}$:

$$1 + \sum_{d \geq 1} \sum_{n \in \mathbb{Z}} P_{d,n} (-x)^n Q^d = \exp \left(\sum_{g \geq 0} F_g^{GW} u^{2g-2} \right)$$

where $x = e^{iu}$.

- The stable pairs invariants $P_{d,n}$ are expected to admit a refinement $P_{d,n,j}$ (various approaches: cohomological, K-theoretic...) The refined topological string free energies $F_{g_1,g_2}^{K_{\mathbb{P}^2},\text{ref}}$ are then defined by the expansion

$$1 + \sum_{d \geq 1} \sum_{n,j \in \mathbb{Z}} P_{d,n,j} y^j (-x)^n Q^d = \exp \left(\sum_{g \geq 0} F_{g_1,g_2}^{\text{ref}} (\epsilon_1 + \epsilon_2)^{2g_1} (-\epsilon_1 \epsilon_2)^{g_2 - 1} \right) \tag{1}$$

where $x = e^{i \frac{\epsilon_1 - \epsilon_2}{2}}$ and $y = e^{i \frac{\epsilon_1 + \epsilon_2}{2}}$.

Context: refinement

- Unrefined limit: Gromov-Witten theory, $F^{\mathrm{ref}}_{0,g} = F^{GW}_{g}$.

Context: refinement

- Unrefined limit: Gromov-Witten theory, $F^{\mathrm{ref}}_{0,g} = F^{GW}_g$.
- Genus-0/Nekrasov-Shatashvili limit: (conjectural) description in terms of moduli spaces of one-dimensional sheaves, $F^{\mathrm{ref}}_{g,0} = F^{NS}_g$.

Context: refinement

- Unrefined limit: Gromov-Witten theory, $F^{\mathrm{ref}}_{0,g} = F^{GW}_g$.
- Genus-0/Nekrasov-Shatashvili limit: (conjectural) description in terms of moduli spaces of one-dimensional sheaves, $F^{\mathrm{ref}}_{g,0} = F^{NS}_g$.
- Remark: $F^{\mathrm{ref}}_{0,0} = F^{GW}_0 = F^{NS}_0$.

Context: refinement

Conjecture (Huang-Klemm, 2010)

- After the change of variabes $Q \mapsto q = e^{2i\pi\tau}$, for every g_1, g_2 with $g_1 + g_2 \geq 2$, $F^{\text{ref}}_{g_1,g_2}(\tau)$ is a weight 0 quasimodular function for $\Gamma_1(3)$: $F^{\text{ref}}_{g_1,g_2} \in C^{-(2(g_1+g_2)-2)} \cdot \mathbb{Q}[A,B,C]_{6(g_1+g_2)-6}$.

Context: refinement

Conjecture (Huang-Klemm, 2010)

- After the change of variabes $Q \mapsto q = e^{2i\pi\tau}$, for every g_1, g_2 with $g_1 + g_2 \geq 2$, $F^{\text{ref}}_{g_1,g_2}(\tau)$ is a weight 0 quasimodular function for $\Gamma_1(3)$: $F^{\text{ref}}_{g_1,g_2} \in C^{-(2(g_1+g_2)-2)} \cdot \mathbb{Q}[A, B, C]_{6(g_1+g_2)-6}$.
- For every g_1, g_2 with $g_1 + g_2 \geq 2$, we have

$$2\frac{\partial}{\partial B} F^{\text{ref}}_{g_1,g_2} = \frac{1}{2} \sum_{\substack{0 \leq j_1 \leq g_1 \\ 0 \leq j_2 \leq g_2 \\ (j_1,j_2) \neq (0,0) \\ (j_1,j_2) \neq (g_1,g_2)}}^{g-1} \left(Q\frac{d}{dQ} F^{\text{ref}}_{j_1,j_2} \right) \left(Q\frac{d}{dQ} F^{\text{ref}}_{(g_1-j_1,g_2-j_2)} \right)$$

$$+\frac{1}{2} \left(Q\frac{d}{dQ} \right)^2 F^{GW}_{g_1,g_2-1} \,.$$

Why does it seem difficult ?

- The proof of quasimodularity and holomorphic anomaly equation for $F_{0,g}^{\mathrm{ref}} = F_g^{GW}$ (Lho-Pandharipande, Coates-Iritani) uses the Gromov-Witten side, where the parameter g has a clear geometric meaning as genus parameter. No known proof starting from the sheaf side.

Why does it seem difficult ?

- The proof of quasimodularity and holomorphic anomaly equation for $F_{0,g}^{\text{ref}} = F_g^{GW}$ (Lho-Pandharipande, Coates-Iritani) uses the Gromov-Witten side, where the parameter g has a clear geometric meaning as genus parameter. No known proof starting from the sheaf side.
- In general, $F_{(g_1,g_2)}^{\text{ref}}$ is defined via the sheaf side and exponential changes of variables. The geometric interpretation of the parameters g_1 and g_2 is unclear.

Why does it seem difficult ?

- The proof of quasimodularity and holomorphic anomaly equation for $F^{\mathrm{ref}}_{0,g} = F^{GW}_g$ (Lho-Pandharipande, Coates-Iritani) uses the Gromov-Witten side, where the parameter g has a clear geometric meaning as genus parameter. No known proof starting from the sheaf side.
- In general, $F^{\mathrm{ref}}_{(g_1,g_2)}$ is defined via the sheaf side and exponential changes of variables. The geometric interpretation of the parameters g_1 and g_2 is unclear.
- It would be useful to have a Gromov-Witten-like interpretation of the series $F^{\mathrm{ref}}_{(g_1,g_2)}$. "No known worldsheet definition of the refined topological string".

Why does it seem difficult ?

- Key point of the story: we can find a Gromov-Witten interpretation of the series $F_g^{NS} = F_{g,0}^{\mathrm{ref}}$.

Why does it seem difficult ?

- Key point of the story: we can find a Gromov-Witten interpretation of the series $F_g^{NS} = F_{g,0}^{\mathrm{ref}}$.
- We don't know how to do that for $F_{g_1,g_2}^{\mathrm{ref}}$ with $(g_1, g_2) \neq 0$

Why does it seem difficult ?

- Key point of the story: we can find a Gromov-Witten interpretation of the series $F_g^{NS} = F_{g,0}^{\text{ref}}$.
- We don't know how to do that for F_{g_1,g_2}^{ref} with $(g_1, g_2) \neq 0$
- How to find a Gromov-Witten definition of F_g^{NS}? We know that it is not Gromov-Witten theory of $K_{\mathbb{P}^2}$: $F_g^{NS} \neq F_g^{GW}$. Need to look at Gromov-Witten theory of a different geometry.

New geometry

- New geometry: fix E a smooth cubic curve in $\mathbb{P}^2$.

New geometry

- New geometry: fix E a smooth cubic curve in $\mathbb{P}^2$.
- $N_{g,d}$: Gromov-Witten invariant for genus g degree d curves in $\mathbb{P}^2$ intersecting E in a single point, viewed in the relative Calabi-Yau 3-fold $\mathbb{P}^2 \times \AA^1/E \times \mathbb{A}^1$.

New geometry

- New geometry: fix E a smooth cubic curve in $\mathbb{P}^2$.
- $N_{g,d}$: Gromov-Witten invariant for genus g degree d curves in $\mathbb{P}^2$ intersecting E in a single point, viewed in the relative Calabi-Yau 3-fold $\mathbb{P}^2 \times \mathring{A}^1/E \times \mathbb{A}^1$.
- $$N_{g,d} := \int_{[\overline{M}_g(\mathbb{P}^2/E,d)]^{vir}} (-1)^g \lambda_g \,.$$

New geometry

- New geometry: fix E a smooth cubic curve in $\mathbb{P}^2$.
- $N_{g,d}$: Gromov-Witten invariant for genus g degree d curves in $\mathbb{P}^2$ intersecting E in a single point, viewed in the relative Calabi-Yau 3-fold $\mathbb{P}^2 \times \AA^1/E \times \mathbb{A}^1$.
- $$N_{g,d} := \int_{[\overline{M}_g(\mathbb{P}^2/E,d)]^{vir}} (-1)^g \lambda_g \,.$$

Theorem (B, 2019)

For every $g \geq 0$, we have

$$F_g^{NS} = \sum_{d \geq 1} \frac{(-1)^{d-1}}{3d} N_{g,d} Q^d \,.$$

New geometry

- New geometry: fix E a smooth cubic curve in $\mathbb{P}^2$.
- $N_{g,d}$: Gromov-Witten invariant for genus g degree d curves in $\mathbb{P}^2$ intersecting E in a single point, viewed in the relative Calabi-Yau 3-fold $\mathbb{P}^2 \times Å^1/E \times \mathbb{A}^1$.
- $$N_{g,d} := \int_{[\overline{M}_g(\mathbb{P}^2/E,d)]^{vir}} (-1)^g \lambda_g \,.$$

Theorem (B, 2019)

For every $g \geq 0$, we have

$$F_g^{NS} = \sum_{d \geq 1} \frac{(-1)^{d-1}}{3d} N_{g,d} Q^d \,.$$

- Correspondence between refined DT invariants and higher genus GW invariants of two different geometries (different from previously known GW/DT correspondence).

Proof

- How to prove such sheaf/Gromov-Witten correspondence?

Proof

- How to prove such sheaf/Gromov-Witten correspondence?
- No direct geometric connection. Use a combinatorial/algebraic intermediate step: scattering diagram.

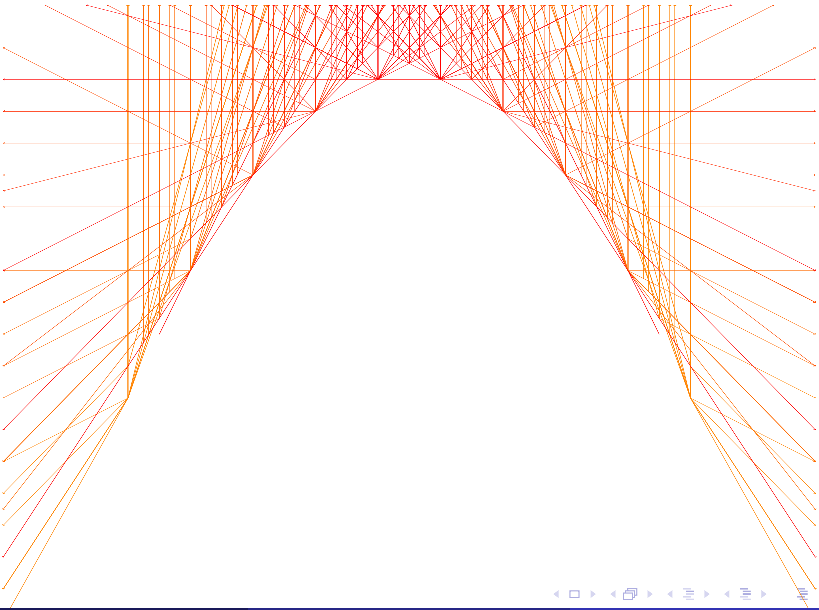
Proof

- How to prove such sheaf/Gromov-Witten correspondence?
- No direct geometric connection. Use a combinatorial/algebraic intermediate step: scattering diagram.
- Scattering diagram: collections of rays decorated with generated functions, algorithmically produced from initial rays.

Proof

- How to prove such sheaf/Gromov-Witten correspondence?
- No direct geometric connection. Use a combinatorial/algebraic intermediate step: scattering diagram.
- Scattering diagram: collections of rays decorated with generated functions, algorithmically produced from initial rays.
- The same algorithm compute the sheaf and the Gromov-Witten sides.

Scattering diagram



Algorithm on the sheaf side

- Compute the Betti numbers $b_j(M_d)$ by moving in the space of Bridgeland stability conditions on $D^b \operatorname{Coh}(\mathbb{P}^2)$ and applying the Kontsevich-Soibelman formula (natural from the DT Calabi-Yau 3-dimensional point of view).

Algorithm on the sheaf side

- Compute the Betti numbers $b_j(M_d)$ by moving in the space of Bridgeland stability conditions on $D^b\,\mathrm{Coh}(\mathbb{P}^2)$ and applying the Kontsevich-Soibelman formula (natural from the DT Calabi-Yau 3-dimensional point of view).
- Need to consider arbitrary classes of coherent sheaves on $\mathbb{P}^2$, not just one-dimensional sheaves, but also positive ranks sheaves.

Algorithm on the sheaf side

- Compute the Betti numbers $b_j(M_d)$ by moving in the space of Bridgeland stability conditions on $D^b \operatorname{Coh}(\mathbb{P}^2)$ and applying the Kontsevich-Soibelman formula (natural from the DT Calabi-Yau 3-dimensional point of view).
- Need to consider arbitrary classes of coherent sheaves on $\mathbb{P}^2$, not just one-dimensional sheaves, but also positive ranks sheaves.
- Initial data: line bundles $\mathcal{O}(n)$, generating the derived category $D^b \operatorname{Coh}(\mathbb{P}^2)$.

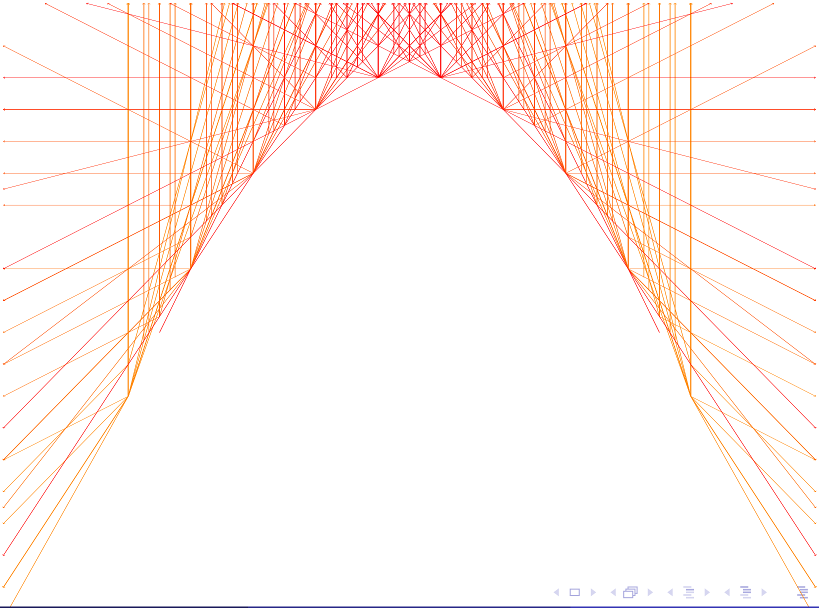
Algorithm on the sheaf side

- Compute the Betti numbers $b_j(M_d)$ by moving in the space of Bridgeland stability conditions on $D^b \operatorname{Coh}(\mathbb{P}^2)$ and applying the Kontsevich-Soibelman formula (natural from the DT Calabi-Yau 3-dimensional point of view).
- Need to consider arbitrary classes of coherent sheaves on $\mathbb{P}^2$, not just one-dimensional sheaves, but also positive ranks sheaves.
- Initial data: line bundles $\mathcal{O}(n)$, generating the derived category $D^b \operatorname{Coh}(\mathbb{P}^2)$.
- Scattering diagram: organization of moves in the space of stability conditions.

Algorithm on the Gromov-Witten side

- Compute the Gromov-Witten $N_{g,d}$ using tropical geometry (combinatorial description of degenerations). Holomorphic curves degenerate to tropical curves.
- Correspondence theorem between counts of holomorphic maps and counts of tropical maps (Mikhalkin, Nishinou-Siebert, Gabele for $g = 0$, B. for $g > 0$).
- Scattering diagram: organization of the tropical computation.

Scattering diagram



Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Degeneration argument. Degeneration of $\mathbb{P}^2$ to the normal cone of E. Line bundle defined by the family of divisors E. General fiber: $K_{\mathbb{P}^2} = \mathcal{O}(-E)$. Special fiber: $\mathbb{P}^2 \times \mathbb{A}^1$, glued along $E \times \AA^1$ to a non-trivial line bundle over $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$.

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Degeneration argument. Degeneration of $\mathbb{P}^2$ to the normal cone of E. Line bundle defined by the family of divisors E. General fiber: $K_{\mathbb{P}^2} = \mathcal{O}(-E)$. Special fiber: $\mathbb{P}^2 \times \mathbb{A}^1$, glued along $E \times \mathring{A}^1$ to a non-trivial line bundle over $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$.
- Localization on the bubble $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$: reduction to equivariant Gromov-Witten theory of $N_{E|\mathbb{P}^2} \oplus N^\vee_{E|\mathbb{P}^2} \to E$ with stationary descendent insertions.

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Degeneration argument. Degeneration of $\mathbb{P}^2$ to the normal cone of E. Line bundle defined by the family of divisors E. General fiber: $K_{\mathbb{P}^2} = \mathcal{O}(-E)$. Special fiber: $\mathbb{P}^2 \times \mathbb{A}^1$, glued along $E \times \mathring{A}^1$ to a non-trivial line bundle over $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$.
- Localization on the bubble $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$: reduction to equivariant Gromov-Witten theory of $N_{E|\mathbb{P}^2} \oplus N^\vee_{E|\mathbb{P}^2} \to E$ with stationary descendent insertions.
- Use Grothendieck-Riemann-Roch (in Coates-Givental form) to reduce to Gromov-Witten theory of E with stationary descendent insertions.

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Upshot: formula computing Gromov-Witten invariants $N_{g,d}$ of $(\mathbb{P}^2, E)$ in terms of Gromov-Witten invariants of $K_{\mathbb{P}^2}$ and the elliptic curve E (Higher-genus version of the log/local correspondence of van Garrel-Graber-Ruddat for a smooth divisor).

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Upshot: formula computing Gromov-Witten invariants $N_{g,d}$ of $(\mathbb{P}^2, E)$ in terms of Gromov-Witten invariants of $K_{\mathbb{P}^2}$ and the elliptic curve E (Higher-genus version of the log/local correspondence of van Garrel-Graber-Ruddat for a smooth divisor).
- $$F_g^{GW} = (-1)^g F_g^{NS} + \sum_{n\geq 0} \sum_{\substack{g=h+g_1+\cdots+g_n, \\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0} \\ (a_j,g_j)\neq(0,0),\, \sum_{j=1}^n a_j=2h-2}} \frac{(-1)^{h-1} F^E_{h,\mathbf{a}}}{|\mathrm{Aut}(\mathbf{a},\mathbf{g})|} \prod_{j=1}^n (-1)^{g_j-1} D^{a_j+2} F_{g_j}^{NS} .$$
- $F^E_{h,\mathbf{a}}$: Gromov-Witten theory of E with stationary descendent insertions.

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Use quasimodularity (Okounkov-Pandharipande, 2003) and holomorphic anomaly equation (Oberdieck-Pixton 2017) for Gromov-Witten invariants of the elliptic curve

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Use quasimodularity (Okounkov-Pandharipande, 2003) and holomorphic anomaly equation (Oberdieck-Pixton 2017) for Gromov-Witten invariants of the elliptic curve
- Use quasimodularity and holomorphic anomaly equation for Gromov-Witten invariants of $K_{\mathbb{P}^2}$ (Lho-Pandharipande, Coates-Iritani, 2018).

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Use quasimodularity (Okounkov-Pandharipande, 2003) and holomorphic anomaly equation (Oberdieck-Pixton 2017) for Gromov-Witten invariants of the elliptic curve
- Use quasimodularity and holomorphic anomaly equation for Gromov-Witten invariants of $K_{\mathbb{P}^2}$ (Lho-Pandharipande, Coates-Iritani, 2018).
- Slightly miraculous combination of these modularity results gives the desired result.

Thank you for your attention !