The symmetric power is a projective bundle over the Jacobian for $n > 2g-2$

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DEFINITION 1. A **projective bundle** of dimension r over a scheme X is a map $\pi : Y \to X$ such that for any point $p \in X$ there exists an open neighborhood $U \subset X$ of p in X with $Y \times_X U \simeq U \times \mathbb{P}^r$ as U-schemes, which agree on overlaps.

PROPOSITION 1 (Universal property of Proj). *Given a vector bundle $\mathscr{E}$ on a scheme X, Proj(Sym $\mathscr{E}^\vee$) $\to X$ is a projective bundle called the **projectivization of** $\mathscr{E}$. Commutative diagrams of maps of schemes of the form:*

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & \mathbb{P}\mathscr{E} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

are in natural $1:1$ *correspondence with with subbundles $\mathscr{L} \subset \rho^*\mathscr{E}$ which are line-bundles.*

Note that the points of $\mathbb{P}\mathscr{E}$ correspond to pairs (x, ξ) with $x \in X$ and $\xi \subset \mathscr{E}_x$ one-dimensional subspaces of the fiber. So the bundle $\pi^*\mathscr{E}$ comes equipped with a **tautological subbundle of rank 1** $\mathscr{O}_{\mathbb{P}\mathscr{E}}(-1)$ whose fiber at a point (x, ξ) is ξ. On trivialisations, it becomes the pullback of $\mathscr{O}_{\mathbb{P}^r}(-1)$ along $Y \times_X U \simeq U \times \mathbb{P}^r$. We get a surjection $\pi^*\mathscr{E}^\vee \twoheadrightarrow \mathscr{O}_{\mathbb{P}\mathscr{E}}(1) := \mathscr{O}_{\mathbb{P}\mathscr{E}}(-1)^\vee$.

We will also need two lemmas of sheaf cohomology to prove our statement.

THEOREM 1 (Base change). *Let $\pi : X \to Y$ be a projective morphism of varieties and $\mathscr{F}$ a coherent sheaf on X flat over Y. If the dimension of $H^0(\mathscr{F}|_{\pi^{-1}(b)})$ is independent on the closed point $b \in B$, then $\pi_*\mathscr{F}$ is a vector bundle of rank* $\dim_{\mathbb{C}} H^0(\mathscr{F}|_{\pi^{-1}(b)})$.

THEOREM 2 (Proper base change). *Given a proper morphism of schemes $f : X \to Y$ and coherent sheaf $\mathscr{F}$ flat over Y, then $\chi(X_y, \mathscr{F}_y)$ is locally constant.*

THEOREM 3. *Given a smooth morphism of projective schemes $\pi : Y \to X$ whose fibers are all isomorphic to $\mathbb{P}^r$ and there exists a Cartier divisor $D \subset Y$ intersecting a general fiber $Y_x \simeq \mathbb{P}^r$ of π in a hyperplane. Then $Y \simeq \mathbb{P}\pi_*\mathscr{O}_Y(D)$ as X-schemes.*

PROOF. Note that it follows that π is also projective and thus our base-change theorems become applicable. By 2, the Euler characteristic of a sheaf in a flat family is constant, the restriction to any fiber, $\chi(\mathscr{O}_Y(D)_x) = \chi(\mathscr{O}_{\mathbb{P}^r}(1))$. Any line bundle on $\mathbb{P}^r$ is isomorphic to $\mathscr{O}_{\mathbb{P}^r}(m)$ for some $m \in \mathbb{Z}$, $\chi(\mathscr{O}_{\mathbb{P}^r}(m)) = \binom{m+n}{n}$, so the Euler characteristic is a complete invariant. It follows, that $\mathscr{O}_Y(D)_x \simeq \mathscr{O}_{\mathbb{P}^r}(1)$ for any point. It follows by theorem 1, that $\mathscr{E} = \pi_*\mathscr{O}_Y(D)$ is a vector bundle with fiber $H^0(\mathscr{O}_{\mathbb{P}^r}(1))$ at p.

Defining a morphism $\alpha : Y \to \mathbb{P}\mathscr{E}$ is by the universal property 1 equivalent to giving a line bundle that is a subbundle of $\pi^*\mathscr{E}$.

By adjunction, we get a map $\beta : \pi^* \pi_* \mathscr{O}_Y(D) \to \mathscr{O}_Y(D)$. We can check that β is surjective on stalks; restricted to a fiber, by the discussion before, this becomes the surjection $\mathscr{E}_p \otimes \mathscr{O}_{\mathbb{P}\mathscr{E}_p} \to \mathscr{O}_{\mathbb{P}(\mathscr{E}_p)}(1)$. We define α as the corresponding morphism to the dual of β.

Now we need to show α is an isomorphism. We show it is a set-theoretic isomorphism and étale. We have $\mathscr{L}_{\pi^{-1}(p)} \simeq \mathscr{O}_{\mathbb{P}^r}(1)$. So the map α is an isomorphism as it restricts to the map $\pi^{-1}(p) \simeq \mathbb{P}^r \simeq \mathbb{P}^r$ given by the complete linear series $|\mathscr{O}_{\mathbb{P}^r}(1)|$.

To show it is étale, we only need to show that it induces an isomorphism on completions. By smoothness, the completions of the stalks on both sides are isomorphic to $\widehat{\mathscr{O}_{X,x}}[\![z_1, ..., z_n]\!]$. Since α commutes with the projections, it induced the identity modulo the maximal ideal. Hence it gives an isomorphism and the proposition follows. □

THEOREM 4. *For a smooth projective curve of genus g, given a point $x_0 \in C$ and number $n > 2g-2$. Then the Abel-Jacobi map $\pi := AJ(\sigma - nx_0) : Sym^{(n)}C \to Jac(C)$ exhibits the symmetric power as a projective bundle over $Jac(C)$.*

PROOF. We seek to apply 3 to show it is a projective bundle.

First, we compute the fibers. From the lecture, we know that $\pi^{-1}(\mathscr{L}) = |\mathscr{L}|$, with Riemann-Roch and Serre Duality;

$$H^0(C, \mathscr{O}(p_1 + ... + p_n)) - H^0(C, K(-p_1 - ... - p_n)) = \deg(\mathscr{O}(p_1 + ... + p_n)) - g + 1 = d - g + 1$$

Furthermore $\deg(K(-p_1 - ... - p_n)) = 2g - 2 - n < 0$, so $H^0(C, K(-p_1 - ... - p_n)) = 0$. Now $H^0(C, \mathscr{O}(p_1 + ... + p_n)) = n - g + 1$. $|\mathscr{L}| = \mathbb{P}^{d-g}$.

Now we have to find a suitable Cartier divisor. Take D the divisor consisting of effective divisors containing x_0. Thus D This intersects each fiber in a hyperplane automatically.

We conclude $Sym^{(n)}C \simeq \mathbb{P}\pi_* \mathscr{O}_{Sym^{(n)}C}(D)$.

□