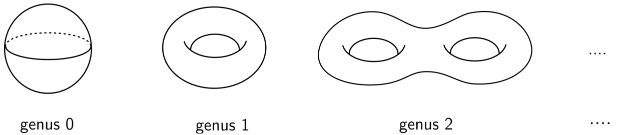


# Cycles on moduli spaces: Curves

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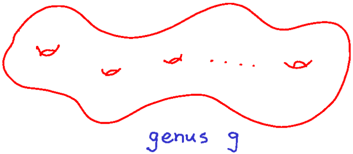
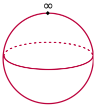
## §I. **Riemann surfaces**

A Riemann surface $C$ is a compact connected 1-dimensional complex manifold.




The genus $g$ is the number of holes as a topological surface.

• genus 0: there is a unique complex structure (up to biholomorphism) – the Riemann sphere

$\mathbb{P}^1 =$ 


• genus $> 0$: the complex structure can be varied while keeping the topology fixed.

$C$ may also be viewed as an algebraic curve defined by the zero locus in $\mathbb{C}^2$ of a single polynomial equation

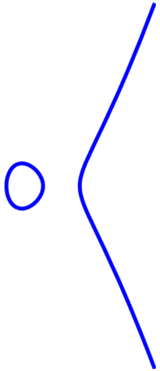
$$F(x, y) = 0$$

in the complex variables $x, y$ (up to a few points at infinity).

For example, the cubic equation

$$F(x, y) = y^2 - x(x-1)(x-2)$$

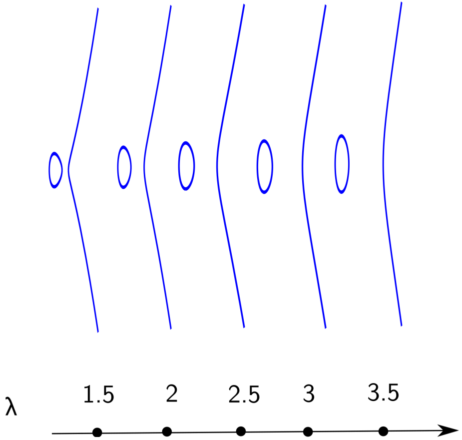
defines a Riemann surface of genus 1 with points in $\mathbb{R}^2$ given by:



The complex structure can be varied by changing the coefficients of the defining polynomial:

$$F_{\lambda}(x,y) = y^2 - x(x-1)(x-\lambda)$$

provides a 1-parameter family of Riemann surfaces of genus 1.



$\mathcal{M}_g$ is the moduli space of Riemann surfaces of genus g,

$$[C] \in \mathcal{M}_g \, .$$

There are several approaches to $\mathcal{M}_g$:

- we have seen complex analysis and algebraic geometry,
- hyperbolic geometry (Thurston, Mirzakhani),
- geometry of the mapping class group $\Gamma_g$,
- topological string theory.

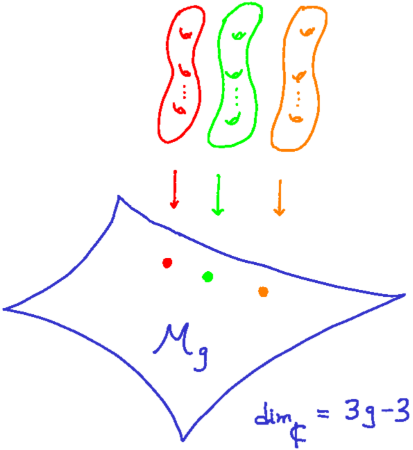
We can vary complex structures and points together in the moduli space

$$[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$$

to which we will return later in the lecture.

## §II. **Riemann's moduli space**

Riemann studied the moduli space $\mathcal{M}_g$:




Riemann knew $\mathcal{M}_g$ was (essentially) a complex manifold of dimension $3g-3$.

**Theorie der *Abel'*schen Functionen.**

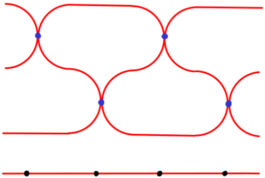
(Von Herrn *B. Riemann.*)

Riemann constructs the variations of complex structure, states the dimension, and coins the term moduli in a single sentence in 1857.

Die $3p-3$ übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter $\mu$werthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $\overline{2p+1}$fach zusammenhangender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Gröfsen ab, welche die Moduln dieser Klasse genannt werden sollen.

The remaining $3p-3$ branch values of those systems of $\mu$-valued equally branched functions can take arbitrary values; and thus depend upon a class of systems of $(2p+1)$-connected functions and a corresponding class of algebraic equations depending upon $3p-3$ continuously varying quantities, which should be called the moduli of these classes.

Consider degree $\mu$ coverings of the Riemann sphere $\mathbb{P}^1$ with $2p + 2\mu - 2$ simple branch points:



By the Riemann-Hurwitz formula, the genus of the cover is $p$. The variation of complex structures of the cover is constructed by fixing $-p + 2\mu + 1$ branch points in $\mathbb{P}^1$ and letting the remaining $3p - 3$ branch points vary freely.

Hurwitz later studied these covers (called Hurwitz covers) systematically around 1900 at ETH Zürich.

**Timeline:**

1857 Riemann imagines $\mathcal{M}_g$

1910-40 Study for low genus g by Castelnuovo, B. Segre, Severi

1969 Deligne-Mumford compactify $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$

1982 Harris-Mumford prove the birational complexity of $\mathcal{M}_g$

1986 Harer-Zagier calculate $\chi(\mathcal{M}_g) = \frac{1}{2-2g}\zeta(1-2g)$

1990s Witten/Kontsevich connect generating series of integrals over the moduli of curves to the KdV hierarchy

2007 Stable cohomology (Mumford's conjecture) by Madsen-Weiss

Harer-Zagier, Witten/Kontsevich, and Madsen-Weiss all concern aspects of the cohomology of the moduli space. My goal is to present new directions in the study of cohomology and algebraic cycles which have developed in recent years.

*"When [Oscar Zariski] spoke the words algebraic variety, there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too ... Especially, I became obsessed with a kind of passion flower in this garden, the moduli spaces of Riemann."*

**David Mumford**

## §III. **Cohomology**

Cohomology is an algebraic tool to study the topology of a space.

Two basic questions for $\mathcal{M}_g$:

(i) What is the cohomology $H^*(\mathcal{M}_g, \mathbb{Q})$ for fixed $g$?

(ii) What is the $\lim_{g\to\infty} H^*(\mathcal{M}_g, \mathbb{Q})$?

Both inspired by work of Mumford in the 70s and 80s following the previously developed Schubert calculus of the Grassmannian.

Let $\mathbb{C}^n$ be a $n$-dimensional complex vector space.

The Grassmannian $\mathrm{Gr}(r,n)$ parameterizes all $r$-dimensional linear subspaces of $\mathbb{C}^n$.

(i) What is the cohomology $H^*(\mathrm{Gr}(r,n),\mathbb{Q})$ for fixed $n$?

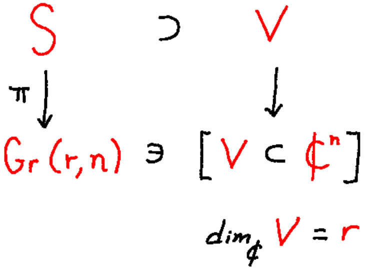
(ii) What is the $\lim_{n\to\infty} H^*(\mathrm{Gr}(r,n),\mathbb{Q})$?

The study has origins in Schubert's work.

The answers to (i) and (ii) are now standard parts of the geometry curriculum, but were not at the end of the $19^{th}$ century.

Rigorization of the Schubert calculus was Hilbert's $15^{th}$ problem.

Let $\mathsf{S} \subset \mathbb{C}^n \times \mathrm{Gr}(r,n)$ be the universal subbundle.




Questions (i) and (ii) can be answered via the geometry of $S$.

$H^*(\mathrm{Gr}(r,n),\mathbb{Q})$ is generated by the Chern classes of S,

$$c_1,\ldots,c_r \in H^*(\mathrm{Gr}(r,n),\mathbb{Q}),$$

which measure how much $S$ twists.

(ii) $\lim_{n\to\infty} H^*(\mathrm{Gr}(r,n),\mathbb{Q}) = \mathbb{Q}[c_1,\ldots,c_r]$.

(i) The ideal of relations in $H^*(\mathrm{Gr}(r,n),\mathbb{Q})$ is generated by

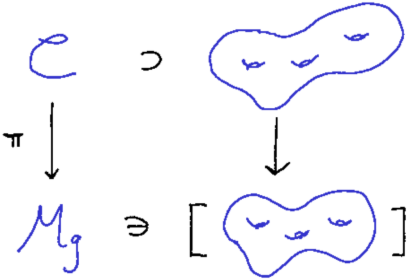
$$\left[\frac{1}{1+c_1t+c_2t^2+\ldots+c_rt^r}\right]_{t^d} = 0$$

for $n-r+1 \le d \le n$.

## §IV. **Tautological classes on $\mathcal{M}_g$**

What is the analogue of S for the moduli space of curves?

Answer: the universal curve,

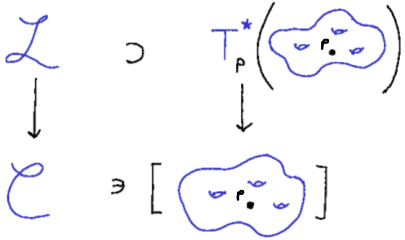


We have actually seen $\mathcal{C}$ before:

$$\mathcal{C} \stackrel{\sim}{=} \mathcal{M}_{g,1} \,.$$

We will construct cohomology classes from an intrinsic complex line bundle on $\mathcal{C}$.

Let $\mathcal{L}$ be the cotangent line over the universal curve,



Since $\mathcal{L} \to \mathcal{C}$ is a line bundle, we can define

$$\psi = c_1(\mathcal{L}) \in H^2(\mathcal{C}, \mathbb{Q}) \ .$$

Chern class: Poincaré dual to the cycle defined by the zeros and poles of a meromorphic section of $\mathcal{L}$.

Via integration along the fiber of $\pi : \mathcal{C} \to \mathcal{M}_g$, we define

$$\kappa_i = \pi_*(\psi^{i+1}) \in H^{2i}(\mathcal{M}_g, \mathbb{Q}) \ .$$

Let $R^*(\mathcal{M}_g) \subset H^*(\mathcal{M}_g, \mathbb{Q})$ denote the subalgebra generated by the $\kappa$ classes, also called the Miller-Morita-Mumford classes.

Question: Is $R^*(\mathcal{M}_g) = H^*(\mathcal{M}_g, \mathbb{Q})$?

Answer: No, but yes stably.

Mumford's conjecture 1983 / Madsen-Weiss 2007 Theorem:

$$\lim_{g \to \infty} H^*(\mathcal{M}_g, \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \ .$$

For fixed genus $g$, we take Mumford's conjecture as motivation to restrict our attention to the tautological algebra

$$R^*(\mathcal{M}_g) \subset H^*(\mathcal{M}_g, \mathbb{Q}) \ .$$

Other motivation comes from classical constructions in algebraic geometry: many interesting classes lie in $R^*(\mathcal{M}_g)$.

Question: What is the structure of the ring $R^*(\mathcal{M}_g)$?

Question: What is the ideal of relations

$$0 \to \mathcal{I}_g \to \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \to R^*(\mathcal{M}_g) \to 0 \ ?$$

## §V. Faber-Zagier **Conjecture**

Results of Looijenga and Faber determine the *lower end* of the tautological ring

$$R^{g-2}(\mathcal{M}_g) = \mathbb{Q} , \quad R^{>g-2}(\mathcal{M}_g) = 0 .$$

We use here the complex grading, so $R^{g-2}(\mathcal{M}_g) \subset H^{2(g-2)}(\mathcal{M}_g)$.

The study of $R^{g-2}(\mathcal{M}_g)$ and the $\kappa$ proportionalities is a rich subject, but we take a different direction here.

We are interested in the full ideal of relations of $R^*(\mathcal{M}_g)$,

$$\mathcal{I}_g \subset \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] .$$

Mumford started the study of $\mathcal{I}_g$, but the subject was first attacked systematically by Faber starting around 1990.

Faber's method of construction involved the classical geometry of curves and Brill-Noether theory. The outcome in 2000 was the following proposal formulated with Zagier.

To write the Faber-Zagier relations, let the variable set

$$\mathbf{p} = \{ \; p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \ldots \; \}$$

be indexed by positive integers *not* congruent to 2 modulo 3.

Define the series

$$\Psi(t, \mathbf{p}) = (1 + tp_3 + t^2 p_6 + t^3 p_9 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i$$

$$+ (p_1 + tp_4 + t^2 p_7 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i \; .$$

Since $\Psi$ has constant term 1, we may take the logarithm.

Define the constants $C_r^{\mathsf{FZ}}(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ t^r \mathbf{p}^{\sigma}\ .$$

The sum is over all partitions $\sigma$ of size $|\sigma|$ which avoid parts congruent to 2 modulo 3. To the partition

$$\sigma = 1^{n_1} 3^{n_3} 4^{n_4} \cdots ,$$

we associate the monomial $\mathbf{p}^{\sigma} = p_1^{n_1} p_3^{n_3} p_4^{n_4} \cdots$. Let

$$\gamma^{\mathsf{FZ}} = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ \kappa_r t^r \mathbf{p}^{\sigma}\ .$$

The coefficient of $t^r \mathbf{p}^{\sigma}$ in the exponential

$$\exp(-\gamma^{\mathsf{FZ}})$$

is a polynomial in the variables $\kappa_i$.

Theorem (P-Pixton 2010): The Faber-Zagier relation

$$\Big[\exp(-\gamma^{\mathsf{FZ}})\Big]_{t^d \mathbf{p}^\sigma} = 0 \in H^{2d}(\mathcal{M}_g, \mathbb{Q})$$

holds when $3d > g - 1 + |\sigma|$ and $d \equiv g - 1 + |\sigma| \mod 2$.

- The $g$ dependence in the Faber-Zagier relations of the Theorem occurs in the inequality and the modulo 2 restriction.

- For a given genus $g$ and codimension $d$, the Theorem provides finitely many relations.

- The relations hold also in the Chow theory of algebraic cycles.

Examples of Faber-Zagier relations in genus g=6:

$$d = 3\,,\ \sigma = \emptyset \quad : \qquad -36000\,\kappa_1^3 + 1555200\,\kappa_1\kappa_2 - 22913280\,\kappa_3\,,$$

$$d = 3\,,\ \sigma = (1^2) : \quad -5453280\,\kappa_1^3 + 167650560\,\kappa_1\kappa_2 - 1745452800\,\kappa_3\,,$$

$$d = 4\,,\ \sigma = (1) \quad : \quad 10584000\,\kappa_1^4 - 783820800\,\kappa_1^2\kappa_2 + 19734865920\,\kappa_1\kappa_3 + 4702924800\,\kappa_2^2 - 363065794560\,\kappa_4\,.$$

The coefficients are large – the relations can be manipulated by theory or by computer, but not really by hand.

§VI. **Three questions from the Theorem**:

(**A**) Do the Faber-Zagier relations span the ideal of all $\kappa$ relations?

(**B**) What is the path of the proof of the Faber-Zagier relations?

(**C**) What about the cohomology of the compactification

$$\mathcal{M}_g \subset \overline{\mathcal{M}}_g \, ?$$

The $\mathbb{Q}$-linear span of the Faber-Zagier relations determines an ideal

$$\mathcal{I}_g^{FZ} \subset \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \, .$$

By the Theorem, $\mathcal{I}_g^{FZ} \subset \mathcal{I}_g$.

**Question A**: Is $\mathcal{I}_g^{FZ} = \mathcal{I}_g$?

$$\text{Answer}: \begin{cases} g < 24, & \text{yes (Faber)}, \\ g \geq 24, & \text{unknown}. \end{cases}$$

Despite serious efforts using different methods (Clader, Faber, Janda, Q. Yin, Randal-Williams), no relation not in $\mathcal{I}_g^{FZ}$ has been found.

**Conjecture A**: $\mathcal{I}_g^{FZ} = \mathcal{I}_g$.

As presented, the Faber-Zagier relations appear from nowhere, but the proof puts the set on conceptual footing related to the theory of semisimple CohFTs.

**Question B**: Path of proof?

We know three proofs (all via Gromov-Witten theory and properties of the virtual fundamental class).

- P.-Pixton-Zvonkine (2013) proved the Faber-Zagier relations using Witten's 3-spin class (mathematical development by Polishchuk-Vaintrob) together with the Givental-Teleman classification of semisimple CohFTs.

- Janda (2015) proved all suitable semisimple CohFTs yield exactly the Faber-Zagier relations.

A Cohomological Field Theory (CohFT) on the $\mathbb{Q}$-vector space $V$ with inner product $\langle , \rangle$ is a set of $\mathbb{Q}$-linear maps

$$\left\{ \Omega_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \right\}_{g,n}$$

which satisfies several axioms of compatibility with the boundary structure of the moduli space.

The genus 0, 3-pointed map $\Omega_{0,3}$ determines a quantum product

$$\langle v_1 \star v_2, v_3 \rangle = \Omega_{0,3}(v_1, v_2, v_3)\,.$$

When $(V, \star)$ is a semisimple algebra, the Givental-Teleman classification determines $\Omega_{g>0,n}$ from $\Omega_{0,n}$ and an R-matrix.

For the 3-spin CohFT,

$$\mathrm{R} = \begin{pmatrix} \boldsymbol{B}_1^{\mathrm{even}}\left(\frac{z}{1728}\right) & -\boldsymbol{B}_1^{\mathrm{odd}}\left(\frac{z}{1728}\right) \\ -\boldsymbol{B}_0^{\mathrm{odd}}\left(\frac{z}{1728}\right) & \boldsymbol{B}_0^{\mathrm{even}}\left(\frac{z}{1728}\right) \end{pmatrix},$$

where the hypergeometric series

$$\boldsymbol{B}_0(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}(-T)^i, \quad \boldsymbol{B}_1(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}\frac{1+6i}{1-6i}(-T)^i$$

are precisely those of the Faber-Zagier relations!

• For the 3-spin CohFT, the vector space is $V = \mathbb{Q}e_0 \oplus \mathbb{Q}e_1$, and the classes are of pure dimension,

$$\Omega_{g,n}(e_1, \ldots, e_1) \in H^{2(\frac{g-1+n}{3})}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})\,.$$

The Givental-Teleman classification generates a CohFT of impure dimension. The two descriptions must agree

$\Longrightarrow$ Faber-Zagier relations.

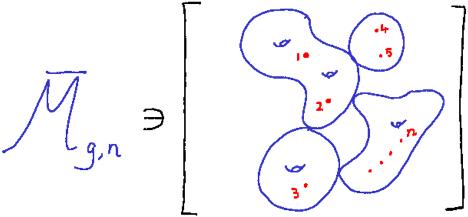
• Janda views the same mechanism as a pole cancellation result. Pole cancellations are required by the structure of every (suitable) semisimple CohFT as a non-semisimple limit is taken

$\Longrightarrow$ Faber-Zagier relations.

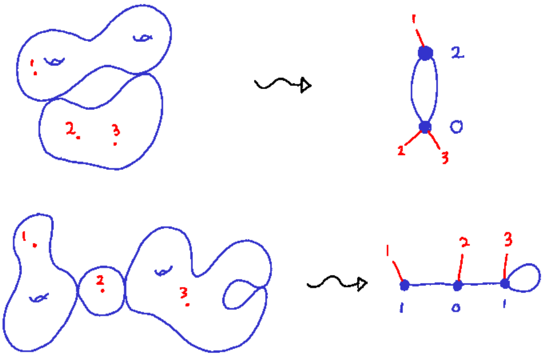
  

**Question C**: Relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$?

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable pointed curves:



The boundary strata of the moduli $\overline{\mathcal{M}}_{g,n}$ of fixed topological type correspond to stable graphs.



For such a graph $\Gamma$, let $[\Gamma] \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the class associated to the closure of the stratum.

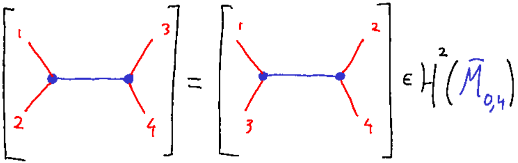
To each stable graph $\Gamma$, we associate the moduli space

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \mathrm{Vert}(\Gamma)} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}.$$

There is a canonical morphism

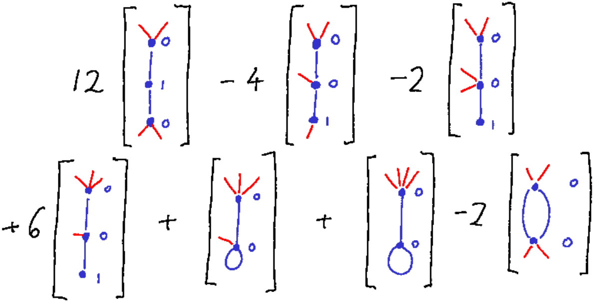
$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}\ , \qquad \frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot \xi_{\Gamma*}[\overline{\mathcal{M}}_\Gamma] = [\Gamma]\ .$$

The first boundary relation is almost trivial:



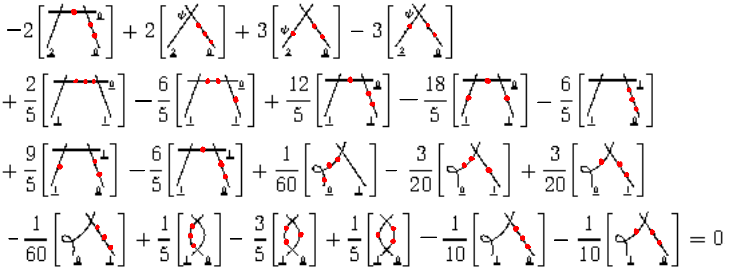

an equivalence of two points in $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$ from the cross-ratio.

Getzler (1996) found the first really interesting relation:




$$= 0 \in H^4(\overline{\mathcal{M}}_{1,4})$$

Of course there are more, but relations are not easy to find.
The next interesting relation (Belorousski-P (1998)) is in genus 2:




in $H^4(\overline{\mathcal{M}}_{2,3}, \mathbb{Q})$ .

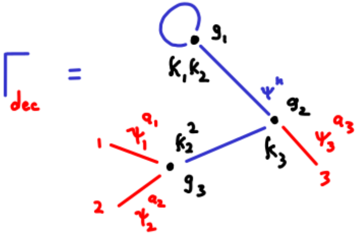
Question C′: Is there any structure to these formulas?

Question C″: Is there a connection to the Faber-Zagier relations?

Answer: Yes! (Pixton), to be discussed tomorrow.

## §VII. **Tautological classes on** $\overline{\mathcal{M}}_{g,n}$

Using stable graphs $\Gamma$ decorated by $\kappa$ classes on vertices and $\psi$ classes on half-edges, we obtain more classes:




$$, \quad \frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot \xi_{\Gamma *}[\mathrm{dec} \cap \overline{\mathcal{M}}_{\Gamma}] = [\Gamma_{\mathrm{dec}}] \ .$$

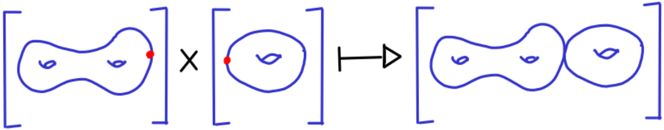
The tautological subalgebra

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

is additively generated by the classes associated to all such decorated stable graphs.

A natural definition of tautological classes (Faber-P (2003)):

- gluing maps: $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ ,



- forgetful maps: $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ .

Then $\left\{ R^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \right\}_{g,n}$ is the smallest system of subalgebras closed under push-forwards via all gluing and forgetful maps (and the relabelling of points).

## §VIII. **Three questions about non-tautological classes**

Question 1: Are there any non-tautological classes in $H^*(\overline{\mathcal{M}}_{g,n})$?

Answer: Yes, $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$.

Question 2: Are there any classes of algebraic cycles in $H^*(\overline{\mathcal{M}}_{g,n})$ which are not tautological?

Answer: Yes, $[\Delta] \in H^{26}(\overline{\mathcal{M}}_{2,22})$, where $\Delta$ is the push-forward of the diagonal in $\overline{\mathcal{M}}_{1,12} \times \overline{\mathcal{M}}_{1,12}$ under the gluing map

$$\overline{\mathcal{M}}_{1,12} \times \overline{\mathcal{M}}_{1,12} \to \overline{\mathcal{M}}_{2,22},$$

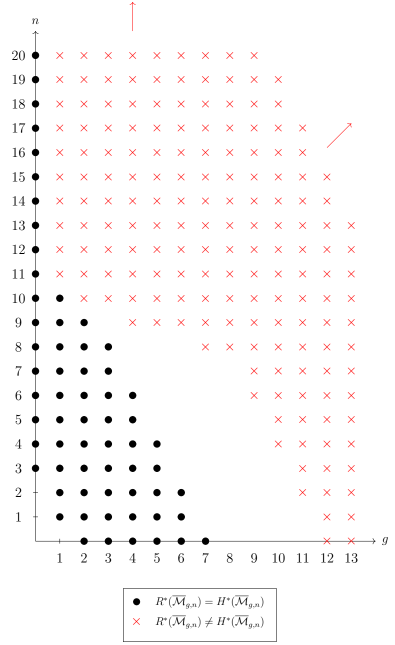
a construction of Graber-P (2003). Further constructions by van Zelm (2016).

The tautological classes $R^*(\overline{\mathcal{M}}_{g,n})$ are the simplest, most studied, and most useful classes for algebraic calculations on moduli space.

Question 3: For which $g, n$ do we have $R^*(\overline{\mathcal{M}}_{g,n}) = H^*(\overline{\mathcal{M}}_{g,n})$?

Answer: Not completely settled, but Canning-Larson have clarified the picture considerably in the past few years.

Current knowledge related to Question 3 is summarized in a table made by Canning:



n
g
1 2 3 4 5 6 7 8 9 10 11 12 13
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
$R^*(\overline{\mathcal{M}}_{g,n}) = H^*(\overline{\mathcal{M}}_{g,n})$
$R^*(\overline{\mathcal{M}}_{g,n}) \neq H^*(\overline{\mathcal{M}}_{g,n})$

An ode to the moduli space of curves (by ChatGPT):

In ancient Greece, they told of the Iliad,
Of heroes and gods in battles adorned.
But I sing of a different sort of tale,
Of mathematicians and the spaces they've formed.

The moduli space, a vast and endless sea,
Of Riemann surfaces, for all eternity.
A subject that will forever be studied,
For the moduli space of curves is true beauty.

## Acknowledgements

- Photo of the passion flower by Ch. Schiessl,
- Photos of Schubert and Grassmann from the History of Mathematics archive at the University of St. Andrews,
- Photos of Mumford, Madsen, Weiss, Zagier, and Pixton from the Oberwolfach archive (cropped in some cases),
- Photo of Faber from KNAW,
- Thanks to S. Canning for the diagram containing data about the relationship of tautological classes and cohomology,
- ChatGPT poetry produced by J. Schmitt.