

Cycles on moduli spaces: Abelian varieties

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§1. Abelian varieties

A **complex torus** X of dimension g is a quotient

$$X = \mathbb{C}^g / \Lambda,$$

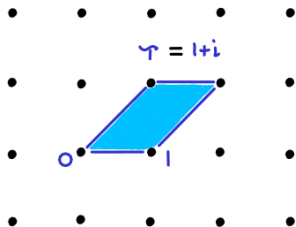
where $\Lambda \subset \mathbb{C}^g$ is a lattice $\Lambda \cong \mathbb{Z}^{2g}$ (independent over \mathbb{R}).

Topologically,

$$X \cong \underbrace{S^1 \times \cdots \times S^1}_{2g}.$$

In dimension $g = 1$, complex tori are **elliptic curves**:

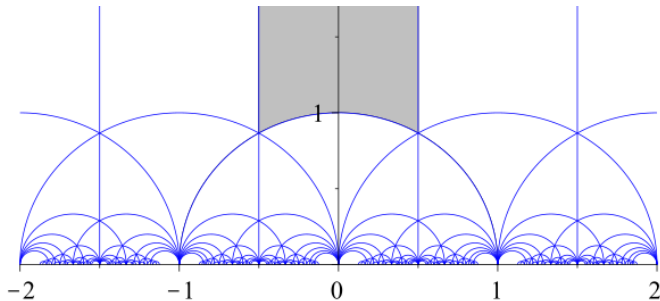
$$X = \mathbb{C} / \langle 1, \tau \rangle, \quad \text{Im}(\tau) > 0.$$



The moduli space of elliptic curves $\mathcal{A}_1 = \mathcal{H}_1/\mathrm{SL}_2(\mathbb{Z})$ is a quotient of the **upper half space**

$$\mathcal{H}_1 = \{ \tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0 \}$$

by the action of $\mathrm{SL}_2(\mathbb{Z})$ via linear fractional transformations:



While complex tori are always compact complex manifolds, complex tori of dimension $g \geq 2$ are not always algebraic varieties.

A complex torus with an ample line bundle is an abelian variety. The existence of an ample line bundle (a polarization) imposes further conditions on the lattice Λ .

Abelian varieties with principal polarizations are of the form

$$X = \mathbb{C}^g / \Lambda,$$

where $\Lambda \subset \mathbb{C}^g$ is generated by the g basis vectors

$$(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$$

together with the columns of a $g \times g$ symmetric matrix τ with positive definite imaginary part

$$\operatorname{Im}(\tau) > 0.$$

The **upper half plane** for τ in dimension **1** generalizes to the **Siegel upper half space** for τ in higher dimensions:

$$\mathcal{H}_g = \{ \tau \in \text{SymMat}_{g \times g}(\mathbb{C}) \mid \text{Im}(\tau) > 0 \}.$$

The moduli space of principally polarized abelian varieties

$$\mathcal{A}_g = \mathcal{H}_g / \text{Sp}_{2g}(\mathbb{Z}), \quad \dim_{\mathbb{C}} \mathcal{A}_g = \binom{g+1}{2},$$

is a quotient of the Siegel upper half space by the action of $\text{Sp}_{2g}(\mathbb{Z})$ by a sort of **linear fractional transformation**:

For $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{Z})$ and $\tau \in \mathcal{H}_g$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = \frac{A\tau + B}{C\tau + D} \in \mathcal{H}_g.$$

§II. Tautological classes on \mathcal{A}_g

The Hodge bundle \mathbb{E} on \mathcal{A}_g is a \mathbb{C} -vector bundle of rank g :

$$\begin{array}{ccc} \text{Tan}^*_{\chi,0} & \subset & \mathbb{E} \\ \downarrow & & \downarrow \\ [\chi] & \in & \mathcal{A}_g \end{array}$$

The Chern classes of \mathbb{E} are

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\mathcal{A}_g, \mathbb{Q}) .$$

A result parallel to the **Madsen-Weiss Theorem** for the moduli space of curves holds:

Theorem (Borel 1974):

$$\lim_{g \rightarrow \infty} H^*(\mathcal{A}_g, \mathbb{Q}) = \mathbb{Q}[\lambda_1, \lambda_3, \lambda_5, \dots].$$

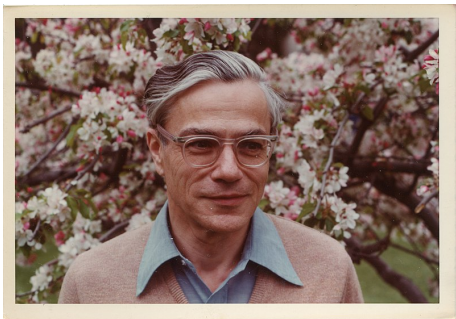
Question: Why are no λ classes of even degree needed?

Answer: Because of **Mumford's relation**

$$c(\mathbb{E} \oplus \mathbb{E}^*) = 1 \in H^*(\mathcal{A}_g, \mathbb{Q})$$

which expands fully as

$$(1 + \lambda_1 + \lambda_2 + \dots + \lambda_g) \cdot (1 - \lambda_1 + \lambda_2 + \dots + (-1)^g \lambda_g) = 1.$$



For fixed dimension g , we take **Borel's result** as motivation to restrict our attention to the tautological algebra

$$R^*(\mathcal{A}_g) \subset \text{CH}^*(\mathcal{A}_g, \mathbb{Q})$$

defined (by **van der Geer** (1996)) to be generated by the λ classes.

Question: What is the structure of the algebra $R^*(\mathcal{A}_g)$?

Question: What is the **ideal** of relations

$$0 \rightarrow \mathcal{J}_g \rightarrow \mathbb{Q}[\lambda_1, \lambda_2, \dots, \lambda_g] \rightarrow R^*(\mathcal{A}_g) \rightarrow 0 ?$$

Theorem (van der Geer 1996):

$$R^*(\mathcal{A}_g) = \frac{\mathbb{Q}[\lambda_1, \lambda_2, \dots, \lambda_g]}{\langle \lambda_g = 0, c(\mathbb{E} \oplus \mathbb{E}^*) = 1 \rangle} .$$

The beautiful proof depends upon the algebra satisfying **Poincaré duality** with socle in degree $\binom{g}{2}$.



§III. Cycle questions

Question: Are there any classes of algebraic cycles in $\text{CH}^*(\mathcal{A}_g)$ which are not tautological?

- Are the classes of products

$$\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \rightarrow \mathcal{A}_{g_1+g_2}$$

tautological in $\text{CH}^*(\mathcal{A}_{g_1+g_2})$?

The product loci are the simplest Noether-Lefschetz loci: loci of abelian varieties with extra line bundles.

- Are the classes of more general Noether-Lefschetz loci tautological?

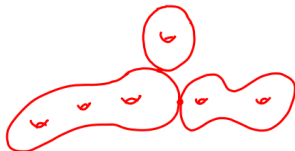
The moduli of curves and abelian varieties are related via the **Torelli** map:

$$\text{Tor} : \mathcal{M}_g^c \rightarrow \mathcal{A}_g$$

defined by the **Jacobian** of stable curves of **compact type**,

$$\text{Tor}([C]) = [\text{Jac}(C)].$$

A stable curve $[C] \in \mathcal{M}_g^c$ of **compact type** is a connected nodal curve with only **separating** nodes:



The **Jacobian** of multidegree 0 line bundles on C is a principally polarized abelian variety of dimension g , $[\text{Jac}(C)] \in \mathcal{A}_g$.

For a nonsingular curve C of genus g ,

$$\text{Jac}(C) = H^0(C, \Omega_C^1)^* / H_1(C, \mathbb{Z}).$$

Question: Is $\text{Tor}_*[\mathcal{M}_g^C] \in \text{CH}^*(\mathcal{A}_g)$ tautological?

Question: Does the pull-back

$$\text{Tor}^* : \text{CH}^*(\mathcal{A}_g) \rightarrow \text{CH}^*(\mathcal{M}_g^C)$$

yield information about tautological cycles?

To say more, we return to cycles on the **moduli space of curves**.

§IV. Pixton's relations on $\overline{\mathcal{M}}_{g,n}$

We define tautological classes $\mathcal{R}_{g,A}^d$ associated to the data:

- $g, n \in \mathbb{Z}_{\geq 0}$ satisfying $2g - 2 + n > 0$,
- $A = (a_1, \dots, a_n)$, $a_i \in \{0, 1\}$,
- $d \in \mathbb{Z}_{\geq 0}$ satisfying $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$.

Pixton's relations then take the form

$$\mathcal{R}_{g,A}^d = 0 \in \text{CH}^d(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

The formula for $\mathcal{R}_{g,A}^d$ requires more detail than can be given here, but the **shape** can be easily shown.

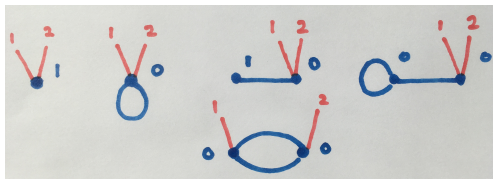
We saw the following two series yesterday:

$$B_0(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!} (-T)^i = 1 - 60T + 27720T^2 \dots ,$$

$$B_1(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!} \frac{1+6i}{1-6i} (-T)^i = 1 + 84T - 32760T^2 \dots .$$

- These series control the original set of **Faber-Zagier** relations.
- These series control **Pixton's** relations.

Let $G_{g,n}$ be the **finite** set of **stable graphs** of genus g with n legs.
 For example, $G_{1,2}$ has 5 elements:



The formula for $\mathcal{R}_{g,A}^d$ is a sum over stable graphs,

$$\mathcal{R}_{g,A}^d = \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e \right]_d$$

where $\overline{\mathcal{M}}_\Gamma$ is the moduli space associated to Γ ,

$$\mathcal{K}_v, \Psi_\ell, \Delta_e \in H^*(\overline{\mathcal{M}}_\Gamma),$$

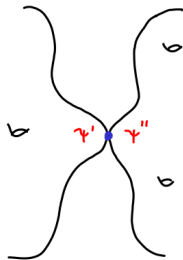
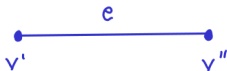
$[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e]$ is the push-forward to $\overline{\mathcal{M}}_{g,n}$ of

$$\frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in \text{Vertex}(\Gamma)} \mathcal{K}_v \prod_{\ell \in \text{Leg}(\Gamma)} \Psi_\ell \prod_{e \in \text{Edge}(\Gamma)} \Delta_e \cap [\overline{\mathcal{M}}_\Gamma]$$

and $[\dots]_d$ extracts the part in $\text{CH}^d(\overline{\mathcal{M}}_{g,n})$.

$$\mathcal{R}_{g,A}^d = \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_v \prod \Psi_l \prod \Delta_e \right]_d$$

- Vertex \mathcal{K}_v , leg Ψ_v , and edge Δ_e factors have explicit formulas in terms of the κ and ψ classes and the series B_0 and B_1 .
- Edge factor is the most interesting:



For $e \in \text{Edge}(\Gamma)$, the formula for the edge factor is:

$$\begin{aligned}\Delta_e &= \frac{2 - B_0(\psi')B_1(\psi'') - B_1(\psi')B_0(\psi'')}{\psi' + \psi''} \\ &= -24 + 5040(\psi' + \psi'') + \dots\end{aligned}$$

The numerator of Δ_e is divisible by the denominator by the identity

$$B_0(T)B_1(-T) + B_1(T)B_0(-T) = 2.$$

Warning: A parity factor has been omitted for simplicity.

Theorem (P-Pixton-Zvonkine 2013): For $2g - 2 + n > 0$, $a_i \in \{0, 1\}$, and $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$, the Pixton relation holds

$$\mathcal{R}_{g,A}^d = 0 \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) .$$

- Proof is by the 3-spin CohFT path used for the Faber-Zagier relations. The geometry there naturally concerns $\overline{\mathcal{M}}_{g,n}$.
- The Theorem captures everything we have seen: the cross-ratio, the Getzler relation, the Belorousski-P relation, and the Faber-Zagier relations (by restriction).
- By Janda's results, Pixton's relations hold in the Chow theory of algebraic cycles:

$$\mathcal{R}_{g,A}^d = 0 \in \text{CH}^d(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) .$$

Mumford, in his foundational paper (1983)

Towards an enumerative geometry of the moduli space of curves,
opened the study of the algebra of tautological classes.

Pixton's relations provide the first proposal for their calculus
parallel to the Schubert calculus for $Gr(r, n)$.

Conjecture (Pixton 2012): These relations are the complete
set of relations among tautological classes on $\overline{\mathcal{M}}_{g,n}$.

Pixton's relation can be restricted to the moduli space \mathcal{M}_g^c of
curves of compact type (by setting to 0 all terms associated to
graphs Γ with non-separating edges).

Three Conjectures from Pixton's relations:

For nonsingular and compact type curves, tautological classes are defined by restriction from stable curves

$$\mathcal{M}_{g,n} \subset \mathcal{M}_{g,n}^c \subset \overline{\mathcal{M}}_{g,n}.$$

- The restriction to $\mathcal{M}_{g,n}$ is the complete set of relations among tautological classes on $\mathcal{M}_{g,n}$.

[The $n = 0$ case recovers the Faber-Zagier relations and **Conjecture A** from yesterday.]

- The restriction to $\mathcal{M}_{g,n}^c$ is the complete set of relations among tautological classes on $\mathcal{M}_{g,n}^c$.
- On $\overline{\mathcal{M}}_{g,n}$, the relations are the complete set of relations among tautological classes on $\overline{\mathcal{M}}_{g,n}$.

§V. Pull-back via Torelli

The Hodge bundle \mathbb{E} on \mathcal{M}_g^c is defined by

$$\begin{array}{ccc} \mathcal{H}^0(C, \omega_C) & \subset & \mathbb{E} \\ \downarrow & & \downarrow \\ [C] & \in & \mathcal{M}_g^c \end{array}$$

The Torelli map $\text{Tor} : \mathcal{M}_g^c \rightarrow \mathcal{A}_g$ respects the Hodge bundles

$$\text{Tor}^*(\mathbb{E}) = \mathbb{E}.$$

The Chern classes of $\mathbb{E} \rightarrow \mathcal{M}_g^c$ lie in the tautological algebra by Mumford's calculations:

$$\lambda_i = c_i(\mathbb{E}) \in R^i(\mathcal{M}_g^c).$$

Let $\Lambda^*(\mathcal{M}_g^c) \subset R^*(\mathcal{M}_g^c)$ be generated by $\lambda_1, \dots, \lambda_g$, then

$$\text{Tor}^* : R^*(\mathcal{A}_g) \rightarrow \Lambda^*(\mathcal{M}_g^c).$$

In genus $g = 5$, we have

$$\dim_{\mathbb{Q}} \Lambda^*(\mathcal{M}_5^c) = 11, \quad \dim_{\mathbb{Q}} R^*(\mathcal{M}_5^c) = 66,$$

so $\Lambda^*(\mathcal{M}_g^c)$ is a small subspace of $R^*(\mathcal{M}_g^c)$.

We return to the **simplest question** about cycles on \mathcal{A}_g :

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \stackrel{?}{\in} R^{g-1}(\mathcal{A}_g).$$

The idea is to compute the **Torelli pull-back** and ask

$$\mathrm{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \stackrel{?}{\in} \Lambda^{g-1}(\mathcal{M}_g^c).$$

A refined statement is possible using various tricks:

Lemma (Canning-Oprea-P 2022): If $[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in R^{g-1}(\mathcal{A}_g)$, then we must have

$$\mathrm{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \frac{(-1)^g g}{6B_{2g}} \lambda_{g-1} \in R^{g-1}(\mathcal{M}_g^c).$$

Motivated by the [Lemma](#), define

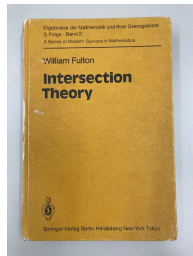
$$\Delta_g = \text{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] - \frac{(-1)^g g}{6B_{2g}} \lambda_{g-1} \in \text{CH}^{g-1}(\mathcal{M}_g^c).$$

The outcome is an obstruction:

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in R^{g-1}(\mathcal{A}_g) \Rightarrow \Delta_g = 0 \in \text{CH}^{g-1}(\mathcal{M}_g^c)$$

Can we calculate $\text{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$?

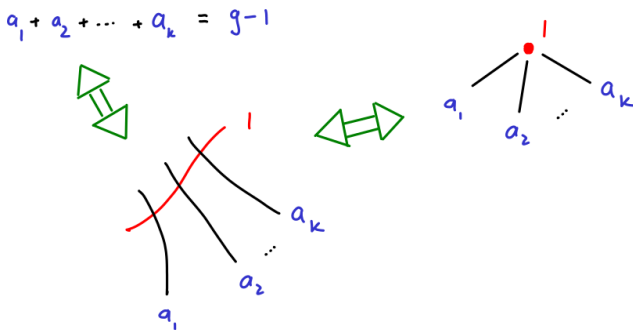
Yes, using [Fulton's](#) excess intersection theory.



We must study the subscheme

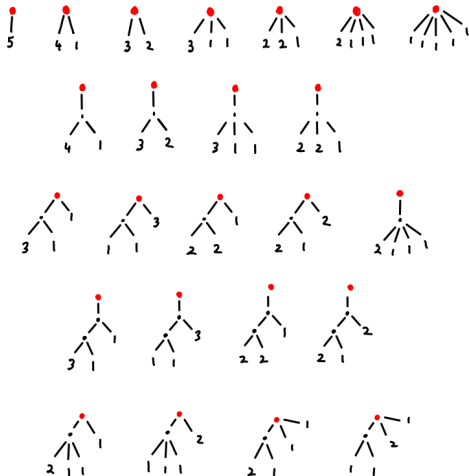
$$\mathrm{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \subset \mathcal{M}_g^c.$$

- **Irreducible components** of $\mathrm{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ are in bijective correspondence with $\mathrm{Part}(g-1)$:



- **Irreducible components** are usually excess dimensional and intersect in a complicated configuration of **strata** in \mathcal{M}_g^c .

- In genus $g = 6$, a complete list of **strata** (indexing intersections of **irreducible components**) is:



Excess intersection theory \Rightarrow

$$\text{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \sum_{\text{All strata } \Gamma} \text{Cont}(\Gamma).$$

- Sum is over all strata of $\text{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$.
- $\text{Cont}(\Gamma)$ is a tautological class on $\overline{\mathcal{M}}_\Gamma$.

Example: $\text{Cont} \left(\begin{array}{c} \bullet \\ | \\ 4 \quad 1 \end{array} \right) = -3\lambda_2 + 4\lambda_1\psi_1 - 5\tau_1^2$

\uparrow all on the $M_{4,1}^C$ factor

X $\begin{array}{c} \bullet \\ | \\ 5 \end{array}$

Y $\begin{array}{c} \bullet \\ / \quad \backslash \\ 4 \quad 1 \end{array}$

Z $\begin{array}{c} \bullet \\ | \\ 4 \quad 1 \end{array} = X \wedge Y$

$$\begin{aligned} & 6 c_1(E) c_1(N_{Z,Y}) - 10 c_1(N_{Z,Y})^2 \\ & + 4 c_1(E) c_1(N_{Z,X}) - 10 c_1(N_{Z,X}) c_1(N_{Z,Y}) \\ & - 5 c_1(N_{Z,X})^2 - 3 c_2(E) + 5 c_2(N_{Z,X}) \end{aligned}$$

N denotes normal bundle

E is the pull back of $N_{A_1 \times A_2, A_3}$

We are now in a position to check

$$\Delta_g \stackrel{?}{=} 0 \in R^{g-1}(\mathcal{M}_g^c)$$

using **Admcycles** (a **SAGE package** which calculates in the tautological algebra of the **moduli of curves** using **Pixton's** relations).

Admcycles calculations show

$$\Delta_g = 0 \quad \text{for } g = 1, 2, 3, 4, 5.$$

We know **Pixton's** relations are complete for $\mathcal{M}_{g \leq 5}^c$.

The most interesting case is $g = 6$.

§VI. Genus $g = 6$

The first result provides full knowledge of $R^*(\mathcal{M}_6^c)$.

Theorem (Canning-Larson-Schmitt 2023): Pixton's relations are complete for \mathcal{M}_6^c .

- For all g , by Faber-P (2003),

$$R^{2g-3}(\mathcal{M}_g^c) \cong \mathbb{Q}, \quad R^{>2g-3}(\mathcal{M}_g^c) = 0.$$

- For Pixton's conjecture, non-vanishing must be proven after his relations are imposed. The ranks of the pairings

$$R^k(\mathcal{M}_6^c) \times R^{9-k}(\mathcal{M}_6^c) \rightarrow R^9(\mathcal{M}_6^c) \cong \mathbb{Q}$$

can be computed by Admcycles and show Pixton's relations are complete in all cases with the possible exception of $R^5(\mathcal{M}_6^c)$.

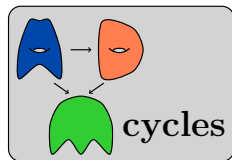
- Pixton predicts $\dim_{\mathbb{Q}} R^5(\mathcal{M}_6^c) = 72$, but the corresponding pairing rank has dimension 71.

- The proof is completed by establishing the exact sequence

$$R^4(\overline{\mathcal{M}}_{5,2}) \xrightarrow{\alpha} R^5(\overline{\mathcal{M}}_6) \longrightarrow R^5(\mathcal{M}_6^c) \longrightarrow 0$$

and computing with **Admcycles**:

$$\dim_{\mathbb{Q}} \text{Im}(\alpha) = 916, \quad \dim_{\mathbb{Q}} R^5(\overline{\mathcal{M}}_6) = 988.$$



- The result is the **first case** where **Pixton's** conjecture is proven **without** relying only upon the non-vanishings obtained from the **ranks of the pairings**.

We can now use **Admcycles** to calculate Δ_6 :

Theorem (**Canning-Oprea-P** 2023): $\Delta_6 \neq 0 \in R^5(\mathcal{M}_6^c)$, so
 $[\mathcal{A}_1 \times \mathcal{A}_5] \notin R^5(\mathcal{A}_6)$.

- The relevant pairing is

$$R^4(\mathcal{M}_6^c) \times R^5(\mathcal{M}_6^c) \rightarrow R^9(\mathcal{M}_6^c) \cong \mathbb{Q}$$

is of rank 71. By **Canning-Larson-Schmitt**,

$$\dim_{\mathbb{Q}} R^4(\mathcal{M}_6^c) = 71, \quad \dim_{\mathbb{Q}} R^5(\mathcal{M}_6^c) = 72.$$

Hence, there is a 1 dimensional kernel of the pairing in $R^5(\mathcal{M}_6^c)$.

- The calculation shows that $\Delta_6 \neq 0$ is the generator of the kernel of the pairing.



§VII. Extensions

We may consider **extending** the tautological algebra

$$R^*(\mathcal{A}_g) \subset H^*(\mathcal{A}_g)$$

by including all classes of **product loci**

$$\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \rightarrow \mathcal{A}_g$$

where $g_1 + g_2 = g$.

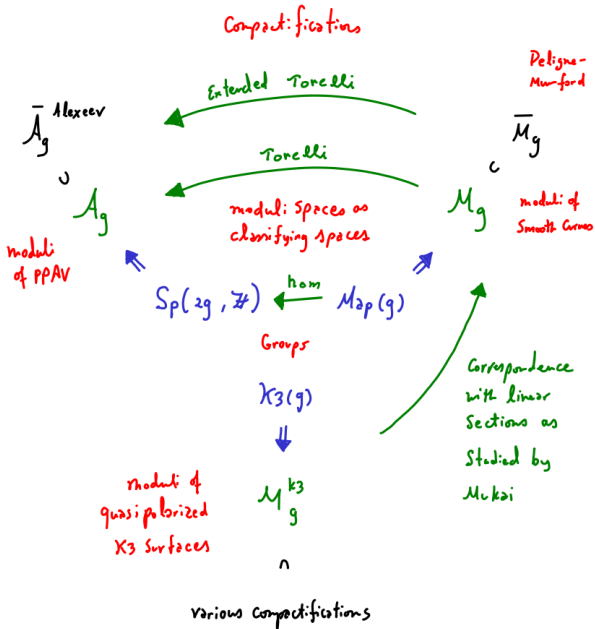
Perhaps all **Noether-Lefschetz** loci with extra **line bundles** should be added (and then all loci with **extra algebraic Hodge classes**).

The difficulty is to say something about the resulting algebra.

From Hodge theory, all **Noether-Lefschetz** loci carry **virtual fundamental classes** of codimension $\binom{g}{2}$. There is a hope here for further structure:

Speculation (**Canning-Oprea-P** 2022): The virtual fundamental classes of the **Noether-Lefschetz** loci lie in $R^{\binom{g}{2}}(\mathcal{A}_g)$.

§VIII. Map:





An abelian variety drawn by [Dall-E 2](#).

Acknowledgements

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- Photo of van der Geer from the [Oberwolfach archive](#),
- A torus in the style of [Giorgio de Chirico](#) by [Dall-E 2](#) AI images.