Cycles on moduli spaces:

K3 surfaces

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3d print of a Kummer K3

I. What is a K3 surface?

Projective Space $\mathbb{CP}^3$ has

homogeneous coordinates

$$[x_0, x_1, x_2, x_3] \in \mathbb{CP}^3.$$

The zero set of a

homogeneous polynomial

$$P_d \in \mathbb{C}[x_0, x_1, x_2, x_3]$$

of degree d defines an

algebraic hypersurface $S_d \subset \mathbb{CP}^3$.

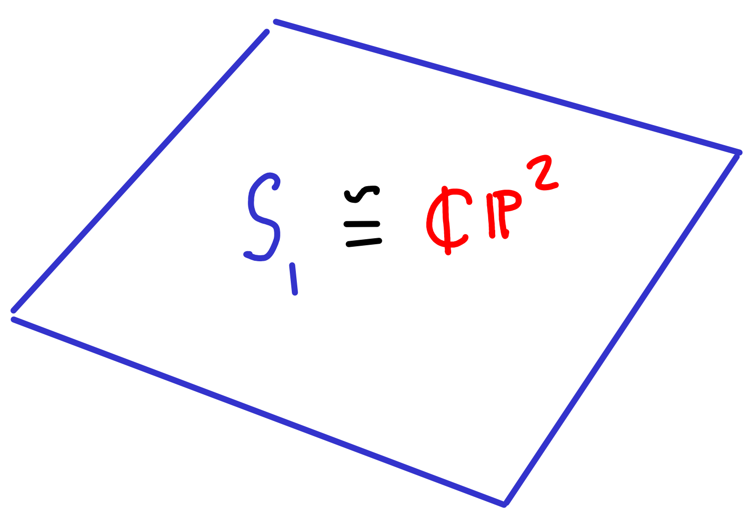
If the four degree $d-1$ polynomials

$$\frac{\partial P_d}{\partial x_0},\ \frac{\partial P_d}{\partial x_1},\ \frac{\partial P_d}{\partial x_2},\ \frac{\partial P_d}{\partial x_3}$$

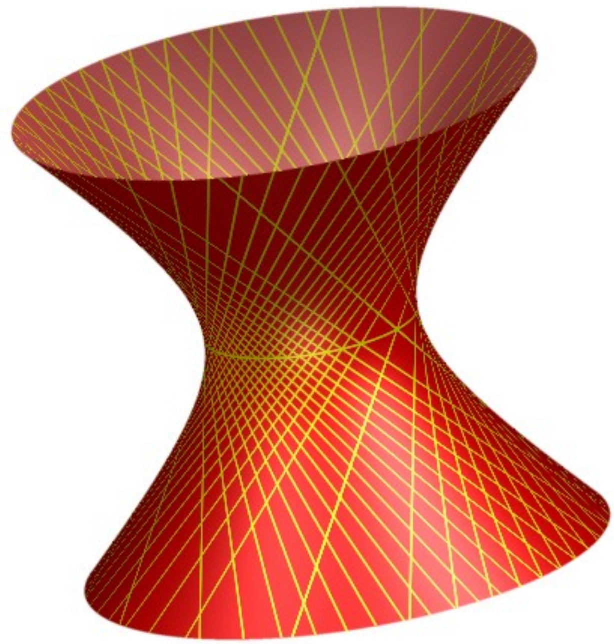
have no common zeros, then

$S_d \subset \mathbb{CP}^3$ is a nonsingular algebraic surface.

$d=1 \implies S_1 \subset \mathbb{CP}^3$ is linear.



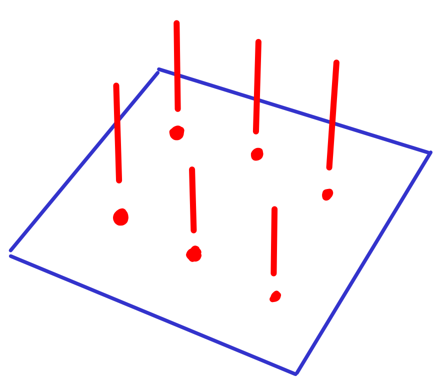
$d=2 \Rightarrow \quad S_2 \subset \mathbb{CP}^3$ is a quadric.



$S_2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$

$d=3 \Rightarrow \quad S_2 \subset \mathbb{CP}^3$ is a cubic surface

$S_3 \cong$ Blow-up of $\mathbb{CP}^2$ at 6 points.



Cubic surface with 27 lines by

Cayetano Ramirez Lopez

More interesting still,

$d=4 \Rightarrow$ S_4 is a K3 surface.

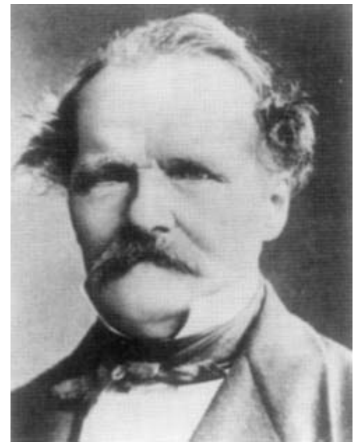
Example: The Fermat quartic

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

Definition I: a K3 surface is a 2 dimensional complex manifold S which has the underlying topology of the quartic S_4.

Definition II: a K3 surface is a simply connected 2 dimensional complex manifold S with $\Lambda^2 \Omega^1_S \cong \mathcal{O}_S$.

André Weil named K3 surfaces after



Kähler Kodaira Kummer

Et la belle montagne K2 au Cachemire :

Photo by Jules Jacot-Guillarmod 1902

II. Moduli of K3 surfaces

The moduli of all complex structures is 20 dimensional :

K3 surface

$$\text{First order Def} = H^1(S, \mathrm{Tan}_S)$$

$$= H^1(S, \Omega^1_S) \qquad \text{because } \Lambda^2\Omega^1_S \cong \mathcal{O}_S$$

dimension is the Hodge number $H^{1,1}(S)$

The Hodge diamond of S is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad\qquad \begin{array}{ccc} & H^{00} & \\ H^{0,2} & H^{1,1} & H^{2,0} \\ & H^{2,2} & \end{array}$$

- Most complex K3 surfaces are <u>not</u> algebraic.

- Moduli spaces of algebraic K3 surfaces are indexed by $g \geq 2$.

$\mathcal{F}_g$ = moduli of (S, L)

where $L \to S$ is a

big + Nef algebraic line bundle
primitive

and $2g-2 = \int_S c_1(L)^2$.

19 dimensional,

Constructed via the Torelli map and Hodge Theory

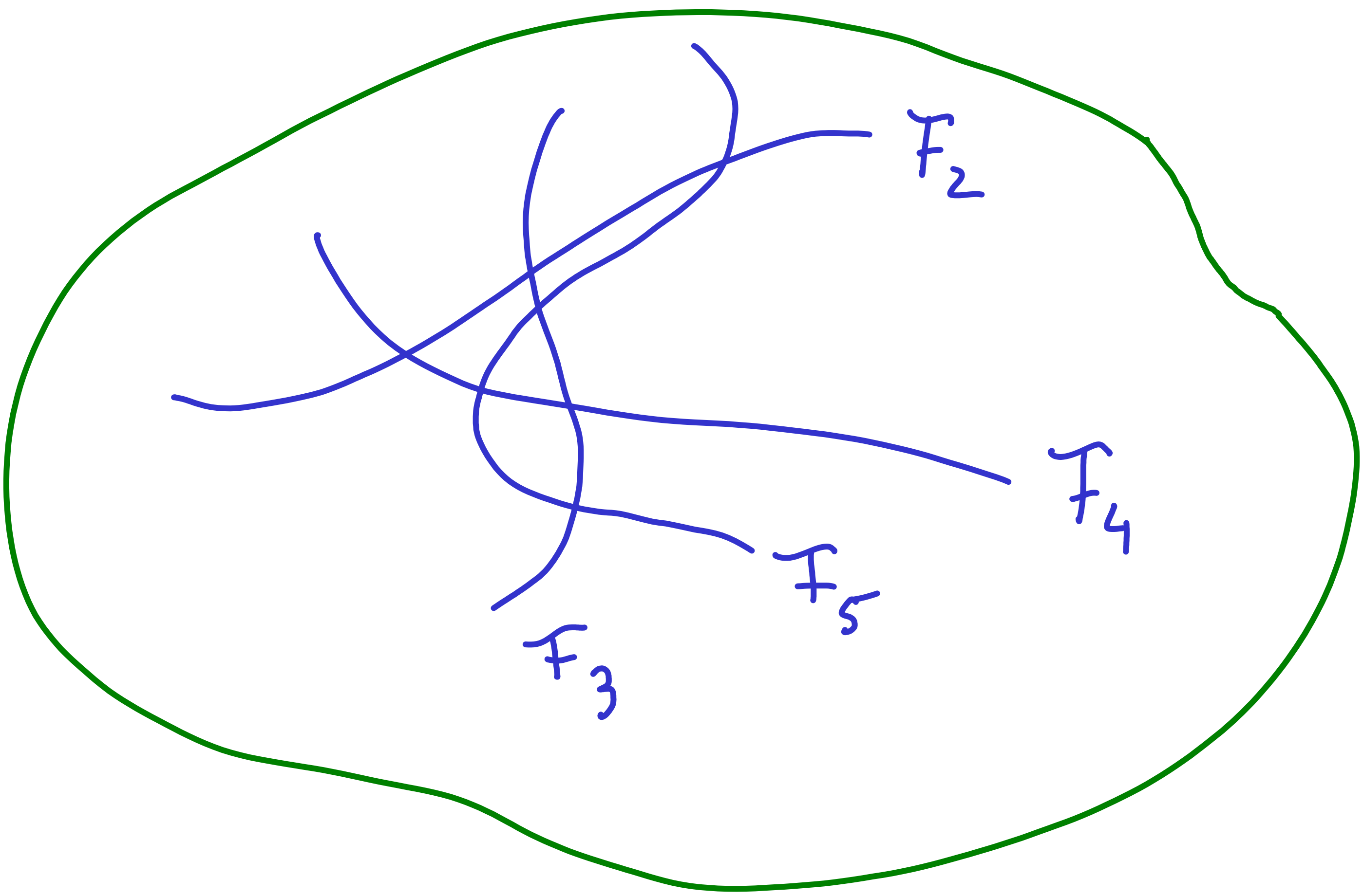
Piatetski-Shapiro, Shafarevitch (1971)

An excellent reference for the foundations is (2005):

Moduli spaces of K3 Surfaces and Complex Ball Quotients

Igor V. Dolgachev and Shigeyuki Kondō

- Moduli picture:



20 dimensional complex moduli

III. Curves on K3 surfaces

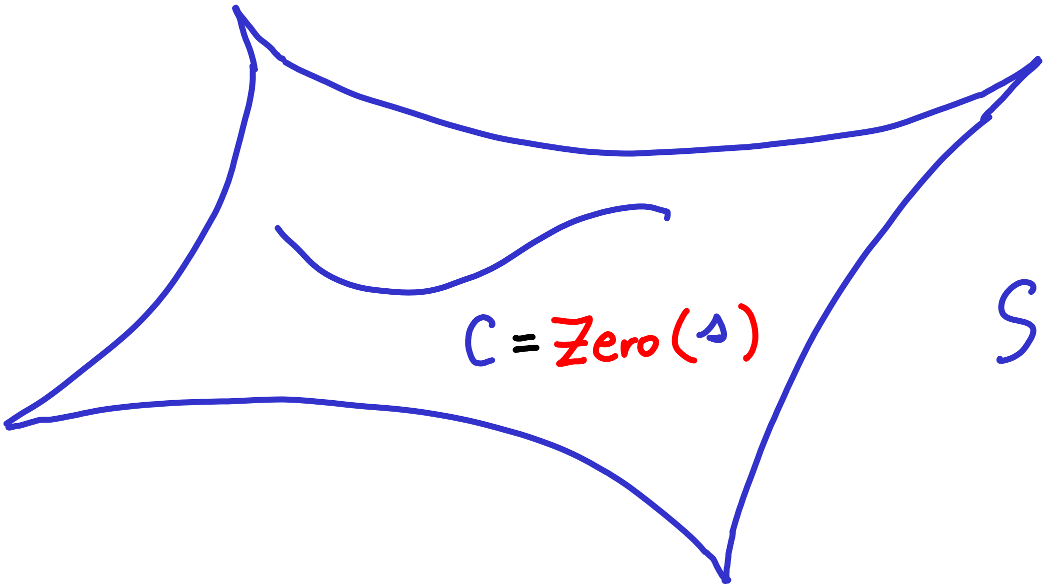
A K3 surface $(S, L) \in \mathcal{F}_g$ contains many algebraic curves.

The simplest to consider are zero sections of L:

For generic $(S, L) \in \mathcal{F}_g$ and

generic $s \in H^0(S, L)$, ← dim $g+1$

the zero locus is a nonsingular algebraic curve of genus g



We obtain a fundamental correspondence:

moduli of (S, L, s)

up to scale

dim $g+19$

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\alpha"'] \arrow[dr, dashed, "\beta"] & \\ \mathcal{F}_g & & \mathcal{M}_g \end{tikzcd}$$

dim 19

dim $3g-3$

Used by Mukai, Lazarsfeld, Voisin, Farkas - Popa, and others.

Compactifications

Deligne-Mumford

$$\begin{tikzcd}
\overline{\mathcal{A}}_g^{\text{Alexeev}} & & \overline{\mathcal{M}}_g \arrow[ll, bend right, "\text{Extended Torelli}"'] \\
\mathcal{A}_g \arrow[u, phantom, "\cup"] & & \mathcal{M}_g \arrow[ll, bend right, "\text{Torelli}"'] \arrow[u, phantom, "\subset"] \\
Sp(2g, \mathbb{Z}) \arrow[u, Rightarrow] & & Map(g) \arrow[ll, "\text{hom}"'] \arrow[u, Rightarrow] \\
& K3(g) \arrow[d, Rightarrow] & \\
& \mathcal{F}_g \arrow[uuur, bend right, "\text{Correspondence with linear sections as studied by Mukai and others}"'] & \\
& \text{various compactifications} \arrow[u, phantom, "\cap"] &
\end{tikzcd}$$

moduli of PPAVs

moduli of nonsingular curves

moduli spaces as classifying spaces

Groups

moduli of K3 surfaces

dim $g+1$

As $s \in H^0(S, L)$ varies, we find a g-dimensional family of genus g curves.

$$\Downarrow$$

Since a node imposes 1 condition, we expect finitely many curves of genus 0 in the family.

$$\sum_{g \geq 0} N_{0,g}\, q^{g-1} = q^{-1} + 24\, q^{0} + 324\, q^{1} + 3200\, q^{2} + \cdots$$

$$= \frac{1}{\Delta(q)} = \frac{1}{q \prod_{n \geq 1} (1-q^n)^{24}},$$

Yau-Zaslow Conjecture (1996), proofs by Bryan-Leung, Beauville, and others.

IV. Cycles on $\mathcal{F}_g$

The simplest cycle classes are divisors.

A geometric construction is via Noether-Lefschetz loci:

Let $(\Lambda, \langle , \rangle, \ell)$ be a polarized lattice of rank 2.

$\Lambda \cong \mathbb{Z}^2$

Symmetric bilinear form on Λ

$\ell \in \Lambda$

$\langle \ell, \ell \rangle = 2g-2$

Let $NL_\Lambda \in CH^1(\mathcal{F}_g)$ ← always with $\mathbb{Q}$-coefficients

be the class of the closure of the locus of $\mathcal{F}_g$ defined by:

$$(\mathrm{Pic}(S), \langle , \rangle_S, L) \cong (\Lambda, \langle , \rangle, \ell).$$

$$\boxed{\begin{array}{l}\underline{\text{Conjecture}}\ \text{(Maulik-P 2007)},\\[1em] \underline{\text{Theorem}}\ \text{(Bergeron, Zhiyuan Li, Millson, Moeglin 2014)}\\[1em] \text{The NL disivors generate } CH^1(\mathcal{F}_g).\end{array}}$$

Could take also $H^2(\mathcal{F}_g)$

The proof is difficult,

a lot of Shimura variety techniques.

By Borcherds $\Rightarrow$ dim $CH^1(\mathcal{F}_g)$ is computed.

The results open the study of tautological classes on $\mathcal{F}_g$.

But what is the definition in higher codimension?

- In the first two lectures, I discussed tautological classes on M_g and A_g.

- Is there a parallel definition for $\mathcal{F}_g$?

$$R^*(\mathcal{F}_g) \overset{\text{Def?}}{\subset} CH^*(\mathcal{F}_g)$$

we can also consider $H^*(\mathcal{F}_g)$

- The situation for K3 surfaces is different: we can't turn to cohomological stability,

$$\lim_{g \to \infty} H^*(\mathcal{F}_g) \quad \text{makes no sense.}$$

V. Tautological classes on $\mathcal{F}_g$

Idea A: We have classes of Noether-Lefschetz loci for lattices of all ranks.

Let $(\Lambda, \langle\, , \rangle, \ell)$ be a polarized lattice of rank r.

$\Lambda \cong \mathbb{Z}^r$

Symmetric bilinear form on Λ

$\ell \in \Lambda$

$\langle \ell, \ell \rangle = 2g-2$

Let $NL_\Lambda \in CH^{r-1}(\mathcal{F}_g)$ be the class of the closure of locus of $\mathcal{F}_g$ defined by:

$$(\mathrm{Pic}(S), \langle\, , \rangle_S, L) \cong (\Lambda, \langle\, , \rangle, \ell).$$

Definition A:

$$NL^*(\mathcal{F}_g) \subset CH^*(\mathcal{F}_g)$$

is the $\mathbb{Q}$-linear span of the classes NL_Λ of Noether-Lefschetz loci of all lattices Λ.

We can view the Noether-Lefschetz loci as analogous to the boundary of $\overline{M}_g$.

But there are also interior classes:

$$\kappa_i \in R^*(\overline{M}_g)$$

Interior classes are omitted in $NL^*(\mathcal{F}_g)$.

Idea B: Add interior classes analogous to the kappa class for curves.

How do we do that?

Let $(\Lambda, \langle,\rangle, \ell)$ be a polarized lattice of rank r

We have an associated morphism

$$\mathcal{F}_\Lambda \xrightarrow{i_\Lambda} \mathcal{F}_g$$

moduli of K3 surfaces S with a Λ marking

Consider the universal family of K3 surfaces:

$$\begin{array}{c} \mathcal{S} \\ \pi \downarrow \\ \mathcal{F}_\Lambda \end{array}$$

Carries universal line bundles associated to elements of Λ.

Subtle issue: universal line bundles are only defined up to twisting by pull backs from $\mathcal{F}_\Lambda$.

Solution: use a normalized line bundle defined by rational curves.

Definition B:

$$R^*(\mathcal{F}_g) \subset CH^*(\mathcal{F}_g)$$

is the $\mathbb{Q}$-linear span of the classes obtained from all push-forwards

$$i_{\Lambda\star}\pi_\star\left(c_1(\mathcal{L}_1)^{a_1}\cdots c_1(\mathcal{L}_k)^{a_k}\cdot c_1(\mathcal{T}_\pi)^{b_1}\cdot c_2(\mathcal{T}_\pi)^{b_2}\right)$$

$$\cap$$

$$CH^*(\mathcal{F}_g),$$

where $\mathcal{L}_i$ correspond to elements of Λ and $\mathcal{T}_\pi$ is the relative tangent bundle of

$$\begin{array}{c} \mathcal{S} \\ \pi \downarrow \\ \mathcal{F}_\Lambda \end{array}$$

from Marian - Oprea - P (2015)

Since Definition B includes all Noether-Lefschetz loci,

$$NL^*(\mathcal{F}_g) \subset R^*(\mathcal{F}_g).$$

Theorem (Qizheng Yin - P 2016)

$$NL^*(\mathcal{F}_g) = R^*(\mathcal{F}_g).$$

So either A or B can be taken to be the definition.

Theorem (Brunier-Raum 2014)

$$\dim_{\mathbb{Q}} NL^*(\mathcal{F}_g) < \infty .$$

To prove $NL^*(\mathcal{F}_g) = R^*(\mathcal{F}_g)$,

We use correspondences with the moduli of curves :

- Genus 0

moduli of π-relative stable maps in the class of the polarization

$$H = c_1(\mathcal{L})$$

4-fold fiber product

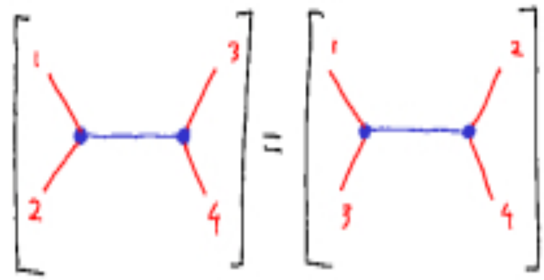
$$
\begin{tikzcd}
 & \overline{\mathcal{M}}_{0,4}(\pi, H) \arrow[dl, "EV"'] \arrow[d, "\varepsilon"] \\
S^4 \arrow[d, "\pi"'] & \overline{\mathcal{M}}_{0,4} \\
\mathcal{F}_g &
\end{tikzcd}
$$

All maps are morphisms,

$\overline{\mathcal{M}}_{0,4}(\pi, H)$ carries a virtual fundamental class $[\overline{\mathcal{M}}_{0,4}(\pi, H)]^{red}$.

How does this help?

WDVV



$$Ev_* \, \varepsilon^* (\mathrm{WDVV}) \cap \left[\overline{\mathcal{M}}_{0,4}(\pi, H) \right]^{\mathrm{red}} .$$

The result is a relation in $CH^5(\mathcal{Z}^4)$

which can be cut and pushed to $\mathcal{F}_g$.

To expand the relation, we use

the splitting axiom of GW theory,

Yau-Zaslow formula.

Not enough for the Theorem!

- <u>Genus 1</u>

moduli of π-relative stable maps in the class of the polarization

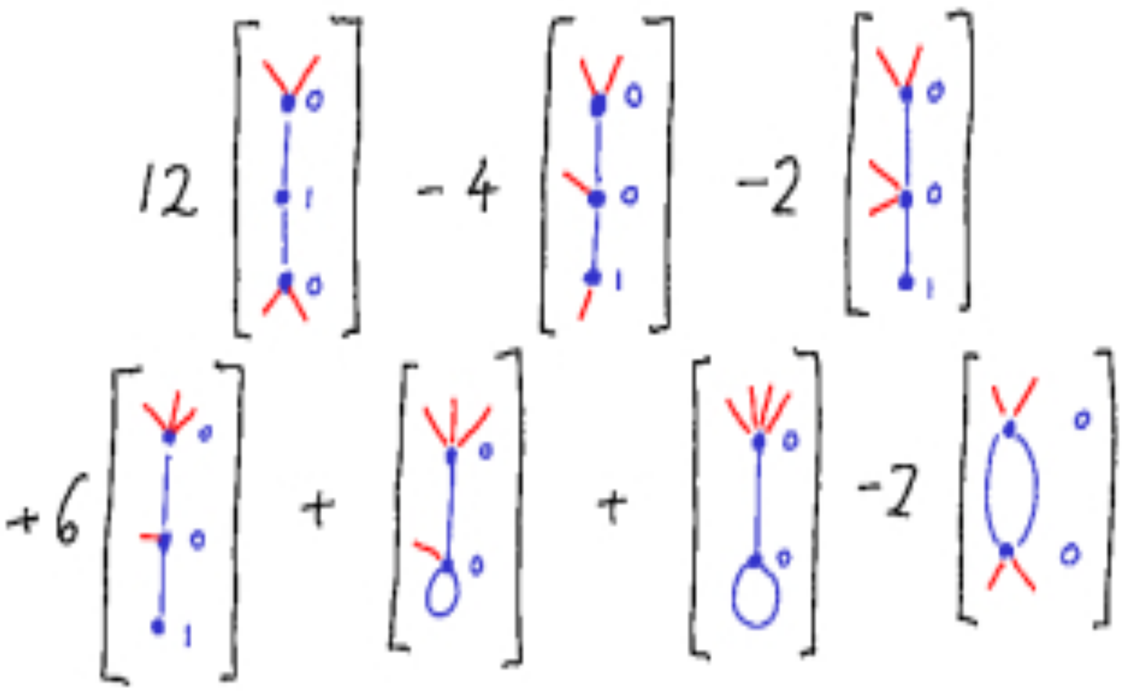
$$\begin{tikzcd} & \overline{\mathcal{M}}_{1,4}(\pi, H) \arrow[dl, "EV"'] \arrow[d, "\varepsilon"] \\ S^4 \arrow[d, "\pi"'] & \overline{\mathcal{M}}_{1,4} \\ \mathcal{F}_g & \end{tikzcd}$$

4-fold fiber product

We obtain a relation in $CH^5(S^4)$:

$$EV_\star \, \varepsilon^\star(\text{GETZLER}) \cap \left[\overline{\mathcal{M}}_{1,4}(\pi, H)\right]^{red}.$$

GETZLER



$$= 0 \in CH^2(\overline{\mathcal{M}}_{1,4})$$

We can push-forward along

$$\pi_{123*} : \mathcal{Z}^4 \longrightarrow \mathcal{Z}^3$$

to obtain relations in $CH^*(\mathcal{Z}^3)$.

$$\text{cut}$$

$$\pi_{123*}\left(H_4 \cdot \mathrm{Ev}_* \, \varepsilon^*(\mathrm{GETZLER}) \cap [\overline{\mathcal{M}}_{1,4}(\pi, H)]^{\mathrm{red}}\right)$$

Relation in $CH^4(\mathcal{S}^3)$ yields a universal Beauville-Voisin diagonal decomposition:

$$(2g-2)\,\Delta_{123} = H_1^2\,\Delta_{23} + H_2^2\,\Delta_{13} + H_3^2\,\Delta_{12}$$

$$- H_1^2\,\Delta_{12} - H_1^2\,\Delta_{13} - H_2^2\,\Delta_{23}$$

+ Corrections supported on Noether-Lefschetz divisors.

tautological classes

Conjecture (Qizheng Yin - P 2016)

Let $(S, L) \in \mathcal{F}_g$, $H = c_1(L)$

and consider

$$\overline{\mathcal{M}}_{h,n}(S,H) \xrightarrow{\text{Ev}} S^n$$

Then, $\text{Ev}_*\left[\overline{\mathcal{M}}_{h,n}(S,H)\right]^{red} \in BV^*(S^n)$

Beauville-Voisin subalgebra

$$BV^*(S^n) \subset CH^*(S^n)$$

is generated by all diagonals and all pullbacks of $CH^1(S)$ from all factors.

VI. An Example ($g = 2$)

For $\mathcal{F}_2$, the cycle calculations are complete.

For generic $(S, L) \in \mathcal{F}_2$:

S is a double cover of $\mathbb{P}^2$

$$S \xrightarrow{\mu} \mathbb{P}^2$$

branched along a sextic plane curve

and $L \cong \mu^*(\mathcal{O}_{\mathbb{P}^2}(1))$.

Shah (1980) constructs $\mathcal{F}_2$ as an explicit blow-up of a moduli space of sextics.

Theorem (Canning - Oprea - P 2023)

(i) $R^*(\mathcal{F}_2) = CH^*(\mathcal{F}_2)$

(ii) Betti Numbers of $R^*(\mathcal{F}_2)$ are

$$1 + 2q + 3q^2 + 5q^3 + 6q^4 + 8q^5$$
$$+ 10q^6 + 12q^7 + 13q^8 + 14q^9 + 12q^{10}$$
$$+ 10q^{11} + 8q^{12} + 6q^{13} + 5q^{14} + 3q^{15}$$
$$+ 2q^{16} + q^{17}$$

Related Cohomology Calculations by Kirwan Lee (1989)

(iii) Cycle class map is an isomorphism onto even cohomology.

Observations

(a) $\mathcal{R}^*(\mathcal{F}_2)$ is not generated by divisors.

(b) $\mathcal{R}^{18}(\mathcal{F}_2) = \mathcal{R}^{19}(\mathcal{F}_2) = 0$.

Conjecture (Oprea - P): $\mathcal{R}^{18}(\mathcal{F}_g) = \mathcal{R}^{19}(\mathcal{F}_g) = 0$

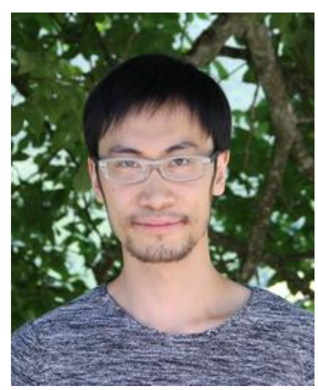
Petersen (2018) has proven both in cohomology.

(c) $\mathcal{R}^{17}(\mathcal{F}_2) = \mathbb{Q}$.

Conjecture (Oprea - P): $\mathcal{R}^{17}(\mathcal{F}_g) = \mathbb{Q}$

(d) Poincaré duality does not hold with respect to the pairing into $\mathcal{R}^{17}(\mathcal{F}_2) = \mathbb{Q}$.

What about $\mathcal{F}_3$?



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