Fix-point loci in $\mathrm{Hilb}^n(\mathbb{A}^2)$

Joint with Paul Johnson

$X_n = \mathrm{Hilb}^n(\mathbb{A}^2)$

Topology of X_n? E.g. Poincaré pol.

$$P_t(X_n) = \sum h^i(X_n) t^i$$

Thm (Ellingsrud–Strømme '87)

$$\sum_{n \geq 0} P_t(X_n) q^n = \prod_{m \geq 1} \frac{1}{(1 - t^{2m-2} q^m)}$$

Göttsche: General ver. $P_t(\mathrm{Hilb}^n(S))$

S smooth q.proj. surf.

Nakajima, Grojnowski: Rep. thy 'explanation' of formula

Bialynicki-Birula '73: Let $\mathbb{C}^* \circlearrowright X$

X smooth, q-proj. s.t. $\forall x \in X$ $\lim_{t \to 0} t \cdot x$ exists, $X^{\mathbb{C}^*} = \{x_1, \dots, x_n\}$

Then $X = \sqcup Z_i$, $Z_i := \{x \in X \mid \lim_{t\to 0} t\cdot x = x_i\}$

$Z_i \cong \mathbb{A}^{d_i^+}$

$\mathbb{C}^* \circlearrowright T_{X,x_i}$, decompose $T_{X,x_i} = T_{X,x_i}^+ \oplus T_{X,x_i}$

dim $\qquad d_i^+ \qquad d_i^-$

<u>Fact</u>: $H_c^*(X) \cong \bigoplus H_c^*(Z_i)$

$\rightsquigarrow$ $P_t(X) = \sum_{x_i \in X^{\mathbb{C}^*}} t^{2d_i^-}$.

Poincaré dual

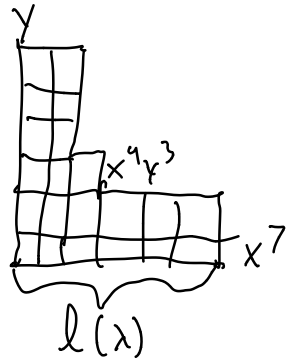
<u>ES's proof</u>: $\mathbb{C}^* \circlearrowright \mathbb{A}^2$ $\quad t\cdot(x,y) = (tx, t^N y)$

$N \gg 0$

$\rightsquigarrow \mathbb{C}^* \circlearrowright X_n \quad X_n^{\mathbb{C}^*} = \{\text{monomial ideals}\}$

$\Updownarrow$

$P_n = \{\text{Partitions of } n\}$



$Z_\lambda \in X_n^{\mathbb{C}^*}$

$d_\lambda^- = n - \ell(\lambda)$

$$\sum_{n\geq 0} P_t(X_n)\, q^n = \sum_{n\geq 0} \sum_{\lambda\in P_n} t^{2(n-\ell(\lambda))} q^n = \prod \left(1-t^{2m-2} q^m\right)^{-1}.$$

Fix point loci : Fix $k, r \in \mathbb{Z}_{\geq 1}$, $(k,r)=1$

Let $G=\mathbb{Z}/r$ act on $\mathbb{A}^2$ with weights $\left(\frac{1}{r}, \frac{k}{r}\right)$

$$i\cdot(x,y) = (\omega^i x,\ \omega^{ki} y) \qquad \omega^r = 1$$

$G \circlearrowright X_n$, consider $X_n^G \subset X_n$.

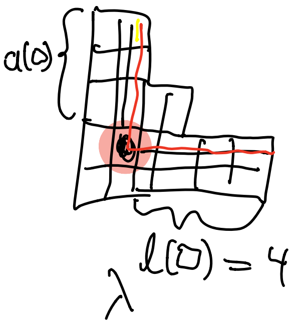
- X_n^G smooth
- X_n^G disconnected

$$X_n^G = \bigsqcup_{\alpha\in K^0(G)} X_\alpha \qquad Z\in X_\alpha \text{ if } [H^0(\mathcal{O}_Z)] = \alpha \text{ as a } G\text{-rep}$$

Thm (Maclagan-Smith '08): X_α are connected.

- $\dim X_\alpha$ varies

$P_t(X_n^G) = ?$ $\quad \mathbb{C}^*$-action on $\mathbb{A}^2$ commutes with G-action



$$\rightsquigarrow \mathbb{C}^* \circlearrowright X_n^G \qquad (X_n^G)^{\mathbb{C}^*} = X_n^{\mathbb{C}^*} = \{\text{mon. ideals}\}$$

$$T_{Z_\lambda, X_n^G} = T^+_{Z_\lambda, X_n^G} \oplus T^-_{Z_\lambda, X_n^G}$$

$$\begin{array}{cccc} \parallel & \dim & d_\lambda^+ & d_\lambda^- \\ T^G_{Z_\lambda, X_n} & & & \end{array}$$

$$d_\lambda^- = \left| \left\{ \square \in \lambda \;\middle|\; \begin{array}{l} \ell(\square) - k\,a(\square) \equiv -1 \pmod r \\ \& \ \ell(\square) > 0 \end{array} \right\} \right|$$

$$\sum P_t(X_n^G)\, q^n = \sum_n \sum_{\lambda \in P_n} q^n t^{2 d_\lambda^-}$$

SL-case (essentially known) $\quad G \subset SL_2(\mathbb{C}) \Updownarrow k = -1$

$$d_\lambda^- = \{ \square \in \lambda \mid h(\square) \equiv 0 \ (r),\ \ell(\square) > 0 \}$$

h = hook length
$h(\square) = a(\square) + \ell(\square) + 1$

Can analyse this using 'r-cores & r-quotients' (combinatorics)

$\Rightarrow$ Formula

$$\sum P_t(X_n^G)\, q^n = \prod_{m\geq 1} \frac{(1-q^{rm})^r}{(1-q^m)(1-t^{2m}q^{rm})^{r-1}(1-t^{2m-2}q^{rm})}$$

Anti-SL_2 case: $G \cap SL_2(\mathbb{C}) = \{e\} \iff (r, k+1) = 1$

E.g. $G = \mathbb{Z}/3 \quad r = 3 \quad k = 1$

$$i \cdot (x,y) = (\omega^i x, \omega^i y) \quad \omega^3 = 1.$$

Conj. (Gusein-Zade – Luengo – Melle-Hernandez '04)

$$\sum P_t(X_n^G)\, q^n = (1-q)^{-1}(1-t^2q^2)^{-1}(1-q^3)^{-1}$$
$$(1-t^2q^4)^{-1}(1-t^4q^5)^{-1}(1-t^2q^6)^{-1}\cdots$$

$$= \prod_{m\geq 0} \frac{1}{(1-t^{2m}q^{1+3m})(1-t^{2m+2}q^{2+3m})(1-t^{2m}q^{3m+3})}$$

$G = \mathbb{Z}/r, \ (1/r, k/r) \quad G \cap SL_2 = \{e\} \implies$

Thm (Johnson – R.) $\exists\, a_1^{(k)}, \dots, a_r^{(k)} \in \{0,1\}$ s.t.

$$\sum P_t(X_n^G)\, q^n = \prod_{i=1}^{r} \left(\prod_{m\geq 0} \left(1 - t^{2(a_i+m)} q^{rm+i}\right)\right)^{-1}$$

Failed approaches: – Refined DT thy.

– Combinatorial cuts (but see [Castro-Ross '18] $r=3$)

What works? Orbifold (Chen-Ruan) cohomology.

$G \circlearrowright X$, smooth, q-pr. var.

$I[X/G] = \{(g,x) \in G \times X \mid gx = x\}/G$

$i \in \mathbb{Q}$: $$H^i_{orb}([X/G]) = \bigoplus_{[Z] \in \pi_0(I[X/G])} H^{i+2age(Z)}(Z)$$

where $age(Z) \in \mathbb{Q}$ computed at $(g,x) \in Z$ via

$$g \circlearrowright T_{X,x} \cong \begin{pmatrix} e^{2\pi i \alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i \alpha_n} \end{pmatrix}$$

$$age(Z) = \sum_{i=1}^{n} \alpha_i - \lfloor \alpha_i \rfloor$$

Thm (Yasuda '05) If $\mathcal{X}$ & $\mathcal{Y}$ are birational, K-equivalent DM stacks, then $H^*_{orb}(\mathcal{X}) \cong H^*_{orb}(\mathcal{Y})$.

E.g. $\mathcal{X} = X_n$ $\mathcal{Y} = [(\mathbb{A}^2)^n/S_n]$

$$H^*(X_n) \overset{\vee}{=} H^*_{orb}(X_n) \cong H^*_{orb}(\mathcal{Y}).$$

$\rightsquigarrow$ new pf ES formula.

We take $\mathcal{X} = [\overline{X_n/G}]$, $\mathcal{Y} = [(\mathbb{A}^2)^n / S_n \times G]$.

$$H^*_{orb}(\mathcal{X}) \cong H^*_{orb}(\mathcal{Y}).$$

$$\begin{tikzcd} I\mathcal{X} \arrow[dr, "f_{\mathcal{X}}"'] & & I\mathcal{Y} \arrow[dl, "f_{\mathcal{Y}}"] \\ & G & \end{tikzcd}$$

$\rightsquigarrow$ decomposition

$$H^*_{orb}(\mathcal{X}) = \bigoplus_{g \in G} H^*_{orb}(\mathcal{X})_g$$

$$\begin{tikzcd} H^*_{orb}(\mathcal{X})_g \\ H^*(f_{\mathcal{X}}^{-1}(g)) \arrow[u, "\text{ungraded}"', "\shortparallel"] \end{tikzcd}$$

$$H^*_{orb}(\mathcal{Y}) = \bigoplus_{g \in G} H^*_{orb}(\mathcal{Y})_g$$

Yasuda $\cong$ preserves G-decomposition.

Anti-SL_2 case 'miracle': $\exists!$ generator $g_0 \in G$ such that age is constant on $f_{\mathcal{X}}^{-1}(g_0) \subset I[X/G]$

$\shortparallel$

A

$+2A$ $\quad$ $2A$

$$H^*_{orb}(\mathcal{X})_{g_0} = H^*(f_{\mathcal{X}}^{-1}(g_0)) = H^{*+\dots}(X_n^{g_0})$$

$\cong$ easily (!) gives formula. $\cong_{2A}$

$H^*_{orb}(\mathcal{Y})_{g_0} \rightsquigarrow$ $\qquad H^{*+}(X_n^G)$

$$\mathcal{Y} = [A^{2n}/S_n \times G]$$

Finding g_0: Write $\mathrm{frac}(\alpha) = \alpha - \lfloor \alpha \rfloor$.

$1 \in G = \mathbb{Z}/r$ acts on T_{Z_λ, X_n} with weights $\{\alpha_\square, \beta_\square\}_{\square \in \lambda}$,

$$\alpha_\square = \frac{l(\square) - k a(\square) + 1}{r}$$

$$\beta_\square = \frac{-l(\square) + k a(\square) + k}{r}$$

Set $g_0 = -(1+k)^{-1} \in \mathbb{Z}/r$, then g_0 acts with wts $\{g_0\alpha_\square, g_0\beta_\square\}$, have $\mathrm{frac}(g_0\alpha_\square + g_0\beta_\square) = 1 - \frac{1}{r}$.

Since $g_0\alpha_\square, g_0\beta_\square \in \frac{1}{r}\mathbb{Z}$, then $\mathrm{frac}(g_0\alpha_\square) + \mathrm{frac}(g_0\beta_\square) = 1 - \frac{1}{r}$.

So $\mathrm{age}(g_0, Z_\lambda) = \sum_{\square \in \lambda} \mathrm{frac}(g_0\alpha_\square) + \mathrm{frac}(g_0\beta_\square) = n(1 - \frac{1}{r})$, in particular independent of λ.

Alternatively choose automorphism of G so that G acts with weigths $(\frac{a}{r}, \frac{b}{r})$, $a + b = 1 - r$, then take $g_0 = 1 \in \mathbb{Z}/r$