

Geometry of the moduli of curves

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$\Sl.$ Genus 0

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All the biholomorphisms of \mathbb{CP}^1 are given by linear fractional transformations



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The moduli space $\mathcal{M}_{0,n}$ parameterizes *n* distinct points on \mathbb{CP}^1 up to biholomorphism,

 $[\mathbb{CP}^1, p_1, \ldots, p_n] \in \mathcal{M}_{0,n}.$

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$$[\mathbb{CP}^1, p_1, \ldots, p_n] \in \mathcal{M}_{0,n}$$
 .

• Let $p_1, p_2, p_3 \in \mathbb{CP}^1$ be three distinct points.

There exists a unique linear fractional transformation



satisfying $f(p_1) = 0$, $f(p_2) = 1$, $f(p_3) = \infty$.

 $\implies \mathcal{M}_{0,3}$ is a single point.

• Given four distinct points $p_1, p_2, p_3, p_4 \in \mathbb{CP}^1$, the first three can be moved via linear fractional transformation to $0, 1, \infty \in \mathbb{CP}^1$.

$$\implies \qquad \mathcal{M}_{0,4} \stackrel{\simeq}{=} \mathbb{CP}^1 \setminus \{0,1,\infty\}.$$

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The statement may also be approached via the classical cross-ratio (which goes back to Pappus of Alexandria 300 AD).

• We can always fix the first three points to be $0, 1, \infty \in \mathbb{CP}^1$.

$$\implies \qquad \mathcal{M}_{0,n} \; \stackrel{\simeq}{=} \; \left(\mathbb{CP}^1 \setminus \{0,1,\infty\} \right)^{n-3} \, \setminus \, \mathsf{Diagonals} \, .$$

While there are open questions about $\mathcal{M}_{0,n}$, we will go immediately to higher genus.

§II. Higher genus

A Riemann surface C is a compact connected 1-dimensional complex manifold.



The genus g is the number of holes as a topological surface.

- genus 0: there is a unique complex structure (up to biholomorphism) the Riemann sphere.
- genus > 0: the complex structure can be varied while keeping the topology fixed.

C may also be viewed as an algebraic curve defined by the zero locus in \mathbb{C}^2 of a single polynomial equation

$$F(x,y)=0$$

in the complex variables x, y (up to a few points at infinity).

For example, the cubic equation

$$F(x, y) = y^2 - x(x - 1)(x - 2)$$

defines a Riemann surface of genus 1 with points in \mathbb{R}^2 given by:



The complex structure can be varied by changing the coefficients of the defining polynomial:

$$F_{\lambda}(x,y) = y^2 - x(x-1)(x-\lambda)$$

provides a 1-parameter family of Riemann surfaces of genus 1.



 \mathcal{M}_g is the moduli space of Riemann surfaces of genus g,

 $[C] \in \mathcal{M}_g$.

There are several approaches to \mathcal{M}_g :

- we have seen complex analysis and algebraic geometry,
- hyperbolic geometry (Mirzakhani),
- geometry of the mapping class group Γ_g ,
- more recently, topological string theory.

We can vary complex structures and points together in the moduli space

$$[\mathcal{C}, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$$

to which we will return later in the lecture.

§III. Riemann

Riemann studied the moduli space \mathcal{M}_g :



Riemann knew \mathcal{M}_g was (essentially) a complex manifold of dimension 3g - 3.

Theorie der Abel'schen Functionen.

(Von Herrn B. Riemann.)

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Die 3p-3 übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter μ werthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $\overline{2p+1}$ fach zusammenhangender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von 3p-3 stetig veränderlichen Größen ab, welche die Moduln dieser Klasse genannt werden sollen.

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Os restantes 3p - 3 valores de ramificação nesses

sistemas de funções com μ -valores e igualmente ramificadas podem tomar valores arbitrários; e assim uma classe de sistemas de funções $\overline{2p+1}$ -vezes conexas e a correspondente classe de equações algébricas dependem continuamente de 3p-3quantidades, as quais deverão ser chamadas moduli desta classe.

Timeline:

- 1857 Riemann imagines \mathcal{M}_g
- 1910-40 Study for low genus g by Castelnuovo, B. Segre, Severi 1969 Deligne-Mumford compactify $\mathcal{M}_{g} \subset \overline{\mathcal{M}}_{g}$
- 1982 Harris-Mumford prove the birational complexity of \mathcal{M}_g
- 1986 Harer-Zagier calculate $\chi(\mathcal{M}_g) = \frac{1}{2-2g}\zeta(1-2g)$
- 1990s Witten/Kontsevich connect generating series of integrals over the moduli of curves to the KdV hierarchy
- 2007 Stable cohomology (Mumford's conjecture) by Madsen-Weiss

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Harer-Zagier, Witten/Kontsevich, and Madsen-Weiss all concern aspects of the cohomology of the moduli space. My goal here is to present a new direction in the cohomological study which has developed in the past few years. "When [Oscar Zariski] spoke the words algebraic variety, there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too ... Especially, I became obsessed with a kind of passion flower in this garden, the moduli spaces of Riemann."

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§IV. Cohomology

Cohomology is an algebraic tool to study the topology of a space.

Two basic questions for \mathcal{M}_g :

(i) What is the cohomology $H^*(\mathcal{M}_g, \mathbb{Q})$ for fixed g?

(ii) What is the $\lim_{g\to\infty} H^*(\mathcal{M}_g,\mathbb{Q})$?

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Both inspired by work of Mumford in the 70s and 80s following the previously developed Schubert calculus of the Grassmannian.





Let \mathbb{C}^n be a *n*-dimensional complex vector space.

The Grassmannian Gr(r, n) parameterizes all *r*-dimensional linear subspaces of \mathbb{C}^n .

- (i) What is the cohomology $H^*(Gr(r, n), \mathbb{Q})$ for fixed *n*?
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The study has origins in Schubert's work.

The answers to (i) and (ii) are now standard parts of the geometry curriculum, but were not at the end of the 19^{th} century.

Rigorization of the Schubert calculus was Hilbert's 15th problem.

Let $S \subset \mathbb{C}^n \times Gr(r, n)$ be the universal subbundle.



Questions (i) and (ii) can be answered via the geometry of S.

 $H^*(Gr(r, n), \mathbb{Q})$ is generated by the Chern classes of S,

 $c_1,\ldots,c_r\in H^*(\mathrm{Gr}(r,n),\mathbb{Q}),$

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(ii)
$$\lim_{n\to\infty} H^*(\operatorname{Gr}(r,n),\mathbb{Q}) = \mathbb{Q}[c_1,\ldots,c_r].$$

(i) The ideal of relations in $H^*(Gr(r, n), \mathbb{Q})$ is generated by

$$\left[\frac{1}{1+c_{1}t+c_{2}t^{2}+\ldots+c_{r}t^{r}}\right]_{t^{d}}=0$$

for $n - r + 1 \leq d \leq n$.

 $\S V$. Tautological classes on \mathcal{M}_g

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What is the analogue of S for the moduli space of curves? Answer: the universal curve,



We have actually seen C before:

 $\mathcal{C} \cong \mathcal{M}_{g,1}.$

We will construct cohomology classes from an intrinsic complex line bundle on C.

Let \mathcal{L} be the cotangent line over the universal curve,



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Since $\mathcal{L} \to \mathcal{C}$ is a line bundle, we can define

$$\psi = c_1(\mathcal{L}) \in H^2(\mathcal{C},\mathbb{Q})$$
.

Chern class: Poincaré dual to the cycle defined by the zeros and poles of a meromorphic section of \mathcal{L} .

Via integration along the fiber of $\pi: \mathcal{C} \to \mathcal{M}_g$, we define

$$\kappa_i = \pi_*(\psi^{i+1}) \in H^{2i}(\mathcal{M}_g, \mathbb{Q})$$
.

Let $R^*(\mathcal{M}_g) \subset H^*(\mathcal{M}_g, \mathbb{Q})$ denote the subring generated by the κ classes, also called the Miller-Morita-Mumford classes.

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Answer: No, but yes stably.

Mumford's conjecture / Madsen-Weiss Theorem:

$$\lim_{g\to\infty} H^*(\mathcal{M}_g,\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\kappa_3,\ldots] \; .$$



For fixed genus g, we take Mumford's conjecture as motivation to restrict our attention to the tautological subring

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Other motivation comes from classical constructions in algebraic geometry: many interesting classes lie in $R^*(\mathcal{M}_g)$.



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Question: What is the structure of the ring $R^*(\mathcal{M}_g)$?

Question: What is the ideal of relations

$$0 \rightarrow \mathcal{I}_{g} \rightarrow \mathbb{Q}[\kappa_{1}, \kappa_{2}, \kappa_{3}, \ldots] \rightarrow R^{*}(\mathcal{M}_{g}) \rightarrow 0$$
?

§VI. Faber-Zagier Conjecture

Results of Looijenga and Faber determine the *lower end* of the tautological ring

$$R^{g-2}(\mathcal{M}_g) = \mathbb{Q} \ , \quad R^{>g-2}(\mathcal{M}_g) = 0 \ .$$

We use here the complex grading, so $R^{g-2}(\mathcal{M}_g) \subset H^{2(g-2)}(\mathcal{M}_g)$.

The study of $R^{g-2}(\mathcal{M}_g)$ and the κ proportionalities is a rich subject, but we take a different direction here.

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We are interested in the full ideal of relations of $R^*(\mathcal{M}_g)$,

 $\mathcal{I}_g \subset \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots].$

Mumford started the study of \mathcal{I}_g , but the subject was first attacked systematically by Faber starting around 1990.

Faber's method of construction involved the classical geometry of curves and Brill-Noether theory. The outcome in 2000 was the following proposal formulated with Zagier.





To write the Faber-Zagier relations, let the variable set

 $\mathbf{p} = \{ p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \dots \}$

be indexed by positive integers not congruent to 2 modulo 3.





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be indexed by positive integers *not* congruent to 2 modulo 3. Define the series

$$\Psi(t,\mathbf{p}) = (1 + tp_3 + t^2p_6 + t^3p_9 + \dots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i + (p_1 + tp_4 + t^2p_7 + \dots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i$$

Since Ψ has constant term 1, we may take the logarithm.

Define the constants $C_r^{FZ}(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{FZ}(\sigma) t^r \mathbf{p}^{\sigma}$$

The sum is over all partitions σ of size $|\sigma|$ which avoid parts congruent to 2 modulo 3. To the partition

$$\sigma=1^{n_1}3^{n_3}4^{n_4}\cdots,$$

we associate the monomial $\mathbf{p}^{\sigma} = p_1^{n_1} p_3^{n_3} p_4^{n_4} \cdots$.

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$$\gamma^{\text{FZ}} = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\text{FZ}}(\sigma) \kappa_r t^r \mathbf{p}^{\sigma}$$

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The coefficient of $t^r \mathbf{p}^{\sigma}$ in the exponential

$$\exp(-\gamma^{FZ})$$

is a polynomial in the variables κ_i .

Theorem (P-Pixton 2010)

In $R^d(\mathcal{M}_g)$, the Faber-Zagier relation

$$\left[\exp(-\gamma^{FZ})
ight]_{t^d\mathbf{p}^{\sigma}}=0$$

holds when $3d > g - 1 + |\sigma|$ and $d \equiv g - 1 + |\sigma| \mod 2$.

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The g dependence in the Faber-Zagier relations of the Theorem occurs in the inequality and the modulo 2 restriction.

For a given genus g and codimension d, the Theorem provides finitely many relations.

Examples of Faber-Zagier relations in genus g=6:

$$d = 3, \ \sigma = \emptyset \quad : \quad -36000 \,\kappa_1^3 + 1555200 \,\kappa_1 \kappa_2 - 22913280 \,\kappa_3 \,,$$

 $d = 3, \ \sigma = (1^2): -5453280 \kappa_1^3 + 167650560 \kappa_1 \kappa_2 - 1745452800 \kappa_3,$

 $\begin{aligned} d &= 4 \,, \ \sigma = (1) : 10584000 \,\kappa_1^4 - 783820800 \,\kappa_1^2 \kappa_2 + 19734865920 \,\kappa_1 \kappa_3 \\ &+ 4702924800 \,\kappa_2^2 - 363065794560 \,\kappa_4 \,. \end{aligned}$

The coefficients are large – the relations can be manipulated by theory or by computer, but not really by hand.

§VII. Three questions immediately arise from the Theorem: (A) Do the Faber-Zagier relations span the ideal of all κ relations? (B) What is the path of the proof of the Faber-Zagier relations? (C) What about the cohomology of the compactification

 $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$?

§VII. Three questions immediately arise from the Theorem: (A) Do the Faber-Zagier relations span the ideal of all κ relations? (B) What is the path of the proof of the Faber-Zagier relations? (C) What about the cohomology of the compactification

 $\mathcal{M}_g\subset\overline{\mathcal{M}}_g$?

The Q-linear span of the Faber-Zagier relations determines an ideal

$$\mathcal{I}_{g}^{FZ} \subset \mathbb{Q}[\kappa_{1}, \kappa_{2}, \kappa_{3}, \ldots].$$

By the Theorem, $\mathcal{I}_g^{FZ} \subset \mathcal{I}_g$.

<u>Question A</u>: Is $\mathcal{I}_g^{FZ} = \mathcal{I}_g$?

Despite serious efforts using different methods (Clader, Faber, Janda, Q. Yin, Randal-Williams), no relation not in \mathcal{I}_g^{FZ} has been found.

Conjecture A: $\mathcal{I}_{g}^{FZ} = \mathcal{I}_{g}$.

As presented, the Faber-Zagier relations appear from nowhere, but the proof puts the set on conceptual footing related to the theory of semisimple CohFTs.

Question B: Path of proof?

We know three proofs (all via Gromov-Witten theory and properties of the virtual fundamental class).

• P.-Pixton-Zvonkine (2013) proved the Faber-Zagier relations using Witten's 3-spin class (mathematical development by Polishchuk-Vaintrob) together with the Givental-Teleman classification of semisimple CohFTs.

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A Cohomological Field Theory (CohFT) on the \mathbb{Q} -vector space V with inner product \langle,\rangle is a set of \mathbb{Q} -linear maps

$$\left\{\Omega_{g,n}: V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})\right\}_{g,n}$$

which satisfies several axioms of compatibility with the boundary structure of the moduli space.

The genus 0, 3-pointed map $\Omega_{0,3}$ determines a quantum product $\langle v_1 \star v_2, v_3 \rangle = \Omega_{0,3}(v_1, v_2, v_3).$

When (V, \star) is a semisimple algebra, the Givental-Teleman classification determines $\Omega_{g>0,n}$ from $\Omega_{0,n}$ and an R-matrix.

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For the 3-spin CohFT,

$$\mathsf{R} = \begin{pmatrix} \boldsymbol{B}_1^{\mathrm{even}} \left(\frac{z}{1728}\right) & -\boldsymbol{B}_1^{\mathrm{odd}} \left(\frac{z}{1728}\right) \\ \\ -\boldsymbol{B}_0^{\mathrm{odd}} \left(\frac{z}{1728}\right) & \boldsymbol{B}_0^{\mathrm{even}} \left(\frac{z}{1728}\right) \end{pmatrix},$$

where the hypergeometric series

$$\boldsymbol{B}_{0}(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!} (-T)^{i}, \quad \boldsymbol{B}_{1}(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!} \frac{1+6i}{1-6i} (-T)^{i}$$

are precisely those of the Faber-Zagier relations!

• For the 3-spin CohFT, the vector space is $V = \mathbb{Q}e_0 \oplus \mathbb{Q}e_1$, and the classes are of pure dimension,

$$\Omega_{g,n}(e_1,\ldots,e_1)\in H^{2(\frac{g-1+n}{3})}(\overline{\mathcal{M}}_{g,n},\mathbb{Q}).$$

The Givental-Teleman classification generates a CohFT of impure dimension. The two descriptions must agree

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 \implies Faber-Zagier relations.

• Janda views the same mechanism as a pole cancellation result. Pole cancellations are required by the structure of every (suitable) semisimple CohFT as a non-semisimple limit is taken

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Question C: Relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$?

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Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable pointed curves:



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For such a graph Γ , let $[\Gamma] \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the class associated to the closure of the stratum.

To each stable graph Γ , we associate the moduli space

$$\overline{\mathcal{M}}_{\Gamma} = \prod_{\nu \in \mathsf{Vert}(\Gamma)} \overline{\mathcal{M}}_{g(\nu), n(\nu)}.$$

There is a canonical morphism

$$\xi_{\Gamma}: \overline{\mathcal{M}}_{\Gamma} \to \overline{\mathcal{M}}_{g,n} , \quad \frac{1}{|\operatorname{Aut}(\Gamma)|} \cdot \xi_{\Gamma*}[\overline{\mathcal{M}}_{\Gamma}] = [\Gamma] .$$

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The first boundary relation is almost trivial:



an equivalence of two points in $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$ from the cross-ratio.

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$$\xi_{\Gamma}: \overline{\mathcal{M}}_{\Gamma} o \overline{\mathcal{M}}_{g,n} , \quad \frac{1}{|\mathsf{Aut}(\Gamma)|} \cdot \xi_{\Gamma*}[\overline{\mathcal{M}}_{\Gamma}] = [\Gamma] .$$

The first boundary relation is almost trivial:



an equivalence of two points in $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$ from the cross-ratio. Getzler (1996) found the first really interesting relation:



$$-2\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] + 2\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] + 3\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] - 3\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right]$$
$$+ \frac{2}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] + \frac{12}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] - \frac{18}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right]$$
$$+ \frac{9}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] + \frac{1}{60}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] - \frac{3}{20}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] + \frac{3}{20}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right]$$
$$- \frac{1}{60}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] + \frac{1}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}\right] + \frac{1}{5}\left[\overbrace{a}^{\bullet} \overbrace{a}\right] - \frac{1}{10}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] - \frac{1}{10}\left[\overbrace{a}^{\bullet} \overbrace{a}^{\bullet}\right] = 0$$

in $H^4(\overline{\mathcal{M}}_{2,3},\mathbb{Q})$.

$$\begin{split} -2\left[\overbrace{a}^{\bullet}\right] + 2\left[\overbrace{a}^{\bullet}\right] + 3\left[\overbrace{a}^{\bullet}\right] - 3\left[\overbrace{a}^{\bullet}\right] \\ + \frac{2}{5}\left[\overbrace{a}^{\bullet}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet}\right] + \frac{12}{5}\left[\overbrace{a}^{\bullet}\right] - \frac{18}{5}\left[\overbrace{a}^{\bullet}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet}\right] \\ + \frac{9}{5}\left[\overbrace{a}^{\bullet}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet}\right] + \frac{1}{60}\left[\bigtriangledown_{a}^{\bullet}\right] - \frac{3}{20}\left[\bigtriangledown_{a}^{\bullet}\right] + \frac{3}{20}\left[\bigtriangledown_{a}^{\bullet}\right] \\ - \frac{1}{60}\left[\bigtriangledown_{a}^{\bullet}\right] + \frac{1}{5}\left[\overbrace{a}^{\bullet}\right] - \frac{3}{5}\left[\overbrace{a}^{\bullet}\right] + \frac{1}{5}\left[\overbrace{a}^{\bullet}\right] - \frac{1}{10}\left[\bigtriangledown_{a}^{\bullet}\right] - \frac{1}{10}\left[\bigtriangledown_{a}^{\bullet}\right] = 0 \\ & \text{in } H^{4}(\overline{\mathcal{M}}_{2,3}, \mathbb{Q}) \;. \end{split}$$

Question C': Is there any structure to these formulas?

$$\begin{split} -2\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + 2\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + 3\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - 3\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] \\ + \frac{2}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{12}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{18}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] \\ + \frac{9}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{1}{60}\left[\bigtriangledown_{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{3}{20}\left[\bigtriangledown_{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{3}{20}\left[\bigtriangledown_{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] \\ - \frac{1}{60}\left[\bigtriangledown_{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{1}{5}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{3}{5}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{1}{10}\left[\bigtriangledown_{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{1}{10}\left[\curvearrowleft_{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] = 0 \\ & \text{in } H^{4}\left(\overline{\mathcal{M}}_{2,3}, \mathbb{Q}\right) \,. \end{split}$$

Question C': Is there any structure to these formulas? Question C'': Is there a connection to the Faber-Zagier relations?

$$\begin{split} -2\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + 2\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + 3\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - 3\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] \\ + \frac{2}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{12}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{18}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] \\ + \frac{9}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{6}{5}\left[\overbrace{a}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{1}{60}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{3}{20}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{3}{20}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] \\ - \frac{1}{60}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] + \frac{1}{5}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{3}{5}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{1}{10}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] - \frac{1}{10}\left[\overbrace{\mathbf{a}}^{\bullet} \stackrel{\mathbf{a}}{\mathbf{a}}\right] = 0 \\ & \text{in } H^{4}\left(\overline{\mathcal{M}}_{2,3}, \mathbb{Q}\right) \,. \end{split}$$

Question C': Is there any structure to these formulas? Question C'': Is there a connection to the Faber-Zagier relations? Answer: Yes! (Pixton) §VIII. Pixton's relations on $\overline{\mathcal{M}}_{g,n}$

\S VIII. Pixton's relations on $\overline{\mathcal{M}}_{g,n}$

We define tautological classes $\mathcal{R}^{d}_{g,A}$ associated to the data

- $g, n \in \mathbb{Z}_{\geq 0}$ in the stable range 2g 2 + n > 0,
- $A = (a_1, ..., a_n), a_i \in \{0, 1\},$
- $d \in \mathbb{Z}_{\geq 0}$ satisfying $d > \frac{g-1+\sum_{i=1}^{n} a_i}{3}$.

Pixton's relations then take the form

$$\mathcal{R}^{d}_{g,A} = 0 \in H^{2d}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$
.

The formula for $\mathcal{R}_{g,A}^d$ requires more detail than can be given here, but the shape can be easily shown.
We have already seen the following two series:

$$B_0(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!} (-T)^i = 1 - 60T + 27720T^2 \cdots,$$

$$B_1(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!} \frac{1+6i}{1-6i} (-T)^i = 1 + 84T - 32760T^2 \cdots.$$

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- These series control the original set of Faber-Zagier relations.
- These series control Pixton's relations.

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- These series control Pixton's relations.

Let $G_{g,n}$ be the finite set of stable graphs of genus g with n legs. For example, $G_{1,2}$ has 5 elements:

The formula for $\mathcal{R}^d_{g,A}$ is a sum over stable graphs,

$$\mathcal{R}_{g,A}^{d} = \sum_{\Gamma \in \mathsf{G}_{g,a}} \frac{1}{2^{h^{1}(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_{v} \prod \Psi_{\ell} \prod \Delta_{e} \right]_{a}$$

where $\overline{\mathcal{M}}_{\Gamma}$ is the moduli space associated to Γ ,

$$\mathcal{K}_{\mathsf{v}}, \ \Psi_{\ell}, \ \Delta_{e} \in H^{*}(\overline{\mathcal{M}}_{\Gamma}),$$

 $[\Gamma, \prod \mathcal{K}_{v} \prod \Psi_{\ell} \prod \Delta_{e}]$ is the push-forward to $\overline{\mathcal{M}}_{g,n}$ of

$$\frac{1}{|\mathsf{Aut}(\Gamma)|} \prod_{\nu \in \mathsf{Vertex}(\Gamma)} \mathcal{K}_{\nu} \prod_{\ell \in \mathsf{Leg}(\Gamma)} \Psi_{\ell} \prod_{e \in \mathsf{Edge}(\Gamma)} \Delta_{e} \cap [\overline{\mathcal{M}}_{\Gamma}]$$

and $[...]_d$ extracts the part in $H^{2d}(\overline{\mathcal{M}}_{g,n})$.

$$\mathcal{R}_{g,A}^{d} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{2^{h^{1}(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_{v} \prod \Psi_{\ell} \prod \Delta_{e} \right]_{d}$$

• Vertex \mathcal{K}_{ν} , leg Ψ_{ν} , and edge Δ_e factors have explicit formulas in terms of the κ and ψ classes and the series B_0 and B_1 .

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• Vertex \mathcal{K}_{ν} , leg Ψ_{ν} , and edge Δ_e factors have explicit formulas in terms of the κ and ψ classes and the series B_0 and B_1 .

• Edge factor is the most interesting:



For $e \in Edge(\Gamma)$, the formula for the edge factor is:

$$\Delta_e = \frac{2 - B_0(\psi')B_1(\psi'') - B_1(\psi')B_0(\psi'')}{\psi' + \psi''}$$

$$= -24 + 5040(\psi' + \psi'') + \dots$$

The numerator of Δ_e is divisible by the denominator by the identity

$$B_0(T)B_1(-T) + B_1(T)B_0(-T) = 2.$$

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Warning: A parity factor has been omitted for simplicity.

Theorem (P-Pixton-Zvonkine 2013)

For 2g - 2 + n > 0, $a_i \in \{0, 1\}$, and $d > \frac{g - 1 + \sum_{i=1}^{n} a_i}{3}$, Pixton's relations hold

$$\mathsf{R}^{d}_{g,\mathcal{A}} = 0 \in H^{2d}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$
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Theorem (P-Pixton-Zvonkine 2013)

For
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, $a_i \in \{0, 1\}$, and $d > \frac{g - 1 + \sum_{i=1}^{n} a_i}{3}$, Pixton's relations hold
$$R_{g,A}^d = 0 \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) .$$

- Proof is by the CohFT path used for the Faber-Zagier relations. The geometry there naturally concerns $\overline{\mathcal{M}}_{g,n}$.
- Theorem captures everything we have seen: the cross-ratio, Getzler's relation, the Faber-Zagier relations.
- By Janda's results, Pixton's relations hold in the Chow theory of algebraic cycles.







Mumford (1983), in his foundational paper

Towards an enumerative geometry of the moduli space of curves,

opened the study of the algebra of tautological classes.

Pixton's relations provide the first proposal for their calculus parallel to the Schubert calculus for Gr(r, n).

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Questions:

• Are Pixton's relations the complete set of relations among tautological classes?

• Is there an abstract algebraic structure which realizes Pixton's relations?

The End



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