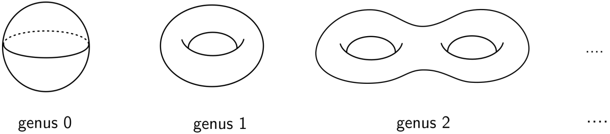


# Geometry of the moduli of curves

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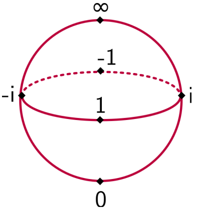
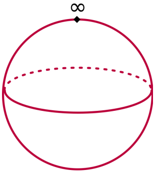
## §I. **Genus 0**

The Riemann sphere $\mathbb{CP}^1$ is a compact 1-dimensional complex manifold obtained by adding a point at infinity to $\mathbb{C}$,

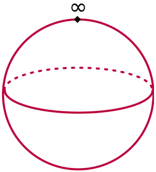



$$= \quad \mathbb{C} \cup \{\infty\}.$$

All the biholomorphisms of $\mathbb{CP}^1$ are given by linear fractional transformations




$$\ni \quad z \mapsto \frac{az+b}{cz+d} \quad \in$$



$\mathcal{M}_{0,n}$ is the moduli space of $n$-pointed Riemann spheres.

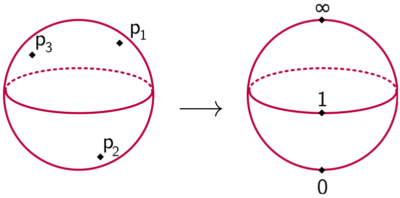
The moduli space $\mathcal{M}_{0,n}$ parameterizes $n$ distinct points on $\mathbb{CP}^1$ up to biholomorphism,

$$[\mathbb{CP}^1, p_1, \ldots, p_n] \in \mathcal{M}_{0,n}\,.$$

- Let $p_1, p_2, p_3 \in \mathbb{CP}^1$ be three distinct points.

There exists a unique linear fractional transformation

$f$ :




satisfying $f(p_1) = 0,\ f(p_2) = 1,\ f(p_3) = \infty$.

$$\Longrightarrow \qquad \mathcal{M}_{0,3} \text{ is a single point.}$$

• Given four distinct points $p_1, p_2, p_3, p_4 \in \mathbb{CP}^1$, the first three can be moved via linear fractional transformation to $0, 1, \infty \in \mathbb{CP}^1$.

$$\Longrightarrow \quad \mathcal{M}_{0,4} \stackrel{\sim}{=} \mathbb{CP}^1 \setminus \{0, 1, \infty\}.$$

The statement may also be approached via the classical cross-ratio (which goes back to Pappus of Alexandria 300 AD).

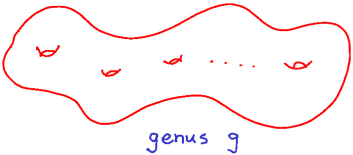
• We can always fix the first three points to be $0, 1, \infty \in \mathbb{CP}^1$.

$$\Longrightarrow \quad \mathcal{M}_{0,n} \stackrel{\sim}{=} \left(\mathbb{CP}^1 \setminus \{0, 1, \infty\}\right)^{n-3} \setminus \text{Diagonals}\,.$$

While there are open questions about $\mathcal{M}_{0,n}$, we will go immediately to higher genus.

## §II. **Higher genus**

A Riemann surface $C$ is a compact connected 1-dimensional complex manifold.




The genus $g$ is the number of holes as a topological surface.

- genus 0: there is a unique complex structure (up to biholomorphism) – the Riemann sphere.
- genus $> 0$: the complex structure can be varied while keeping the topology fixed.

$C$ may also be viewed as an algebraic curve defined by the zero locus in $\mathbb{C}^2$ of a single polynomial equation

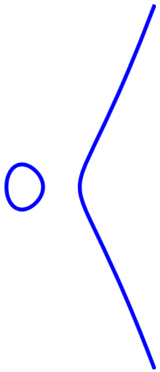
$$F(x, y) = 0$$

in the complex variables $x, y$ (up to a few points at infinity).

For example, the cubic equation

$$F(x, y) = y^2 - x(x-1)(x-2)$$

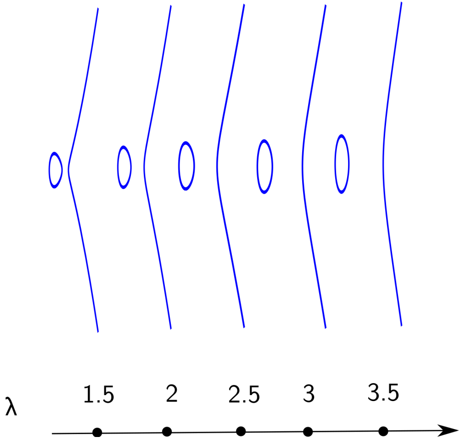
defines a Riemann surface of genus 1 with points in $\mathbb{R}^2$ given by:



The complex structure can be varied by changing the coefficients of the defining polynomial:

$$F_\lambda(x, y) = y^2 - x(x-1)(x-\lambda)$$

provides a 1-parameter family of Riemann surfaces of genus 1.



$\mathcal{M}_g$ is the moduli space of Riemann surfaces of genus g,

$$[C] \in \mathcal{M}_g \,.$$

There are several approaches to $\mathcal{M}_g$:

- we have seen complex analysis and algebraic geometry,
- hyperbolic geometry (Mirzakhani),
- geometry of the mapping class group $\Gamma_g$,
- more recently, topological string theory.

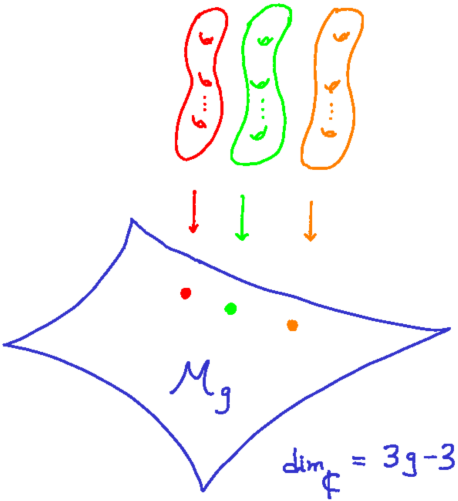
We can vary complex structures and points together in the moduli space

$$[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$$

to which we will return later in the lecture.

§III. **Riemann**

Riemann studied the moduli space $\mathcal{M}_g$:




Riemann knew $\mathcal{M}_g$ was (essentially) a complex manifold of dimension $3g - 3$.

**Theorie der *Abel*'schen Functionen.**

(Von Herrn *B. Riemann.*)

Riemann constructs the variations of complex structure, states the dimension, and coins the term moduli in a single sentence.

Die $3p-3$ übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter $\mu$werthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $\overline{2p+1}$fach zusammenhangender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Gröfsen ab, welche die Moduln dieser Klasse genannt werden sollen.

Os restantes $3p-3$ valores de ramificação nesses sistemas de funções com $\mu$-valores e igualmente ramificadas podem tomar valores arbitrários; e assim uma classe de sistemas de funções $\overline{2p+1}$-vezes conexas e a correspondente classe de equações algébricas dependem continuamente de $3p-3$ quantidades, as quais deverão ser chamadas moduli desta classe.

**Timeline:**

1857 Riemann imagines $\mathcal{M}_g$

1910-40 Study for low genus g by Castelnuovo, B. Segre, Severi

1969 Deligne-Mumford compactify $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$

1982 Harris-Mumford prove the birational complexity of $\mathcal{M}_g$

1986 Harer-Zagier calculate $\chi(\mathcal{M}_g) = \frac{1}{2-2g}\zeta(1-2g)$

1990s Witten/Kontsevich connect generating series of integrals over the moduli of curves to the KdV hierarchy

2007 Stable cohomology (Mumford's conjecture) by Madsen-Weiss

Harer-Zagier, Witten/Kontsevich, and Madsen-Weiss all concern aspects of the cohomology of the moduli space. My goal here is to present a new direction in the cohomological study which has developed in the past few years.

*"When [Oscar Zariski] spoke the words algebraic variety, there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too ... Especially, I became obsessed with a kind of passion flower in this garden, the moduli spaces of Riemann."*

**David Mumford**

§IV. **Cohomology**

Cohomology is an algebraic tool to study the topology of a space.

Two basic questions for $\mathcal{M}_g$:

(i) What is the cohomology $H^*(\mathcal{M}_g, \mathbb{Q})$ for fixed $g$?

(ii) What is the $\lim_{g\to\infty} H^*(\mathcal{M}_g, \mathbb{Q})$?

Both inspired by work of Mumford in the 70s and 80s following the previously developed Schubert calculus of the Grassmannian.

Let $\mathbb{C}^n$ be a $n$-dimensional complex vector space.

The Grassmannian $\mathrm{Gr}(r, n)$ parameterizes all $r$-dimensional linear subspaces of $\mathbb{C}^n$.

(i) What is the cohomology $H^*(\mathrm{Gr}(r, n), \mathbb{Q})$ for fixed $n$?

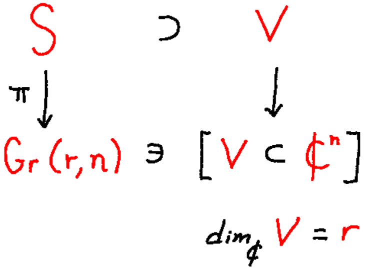
(ii) What is the $\lim_{n\to\infty} H^*(\mathrm{Gr}(r, n), \mathbb{Q})$?

The study has origins in Schubert's work.

The answers to (i) and (ii) are now standard parts of the geometry curriculum, but were not at the end of the $19^{th}$ century.

Rigorization of the Schubert calculus was Hilbert's $15^{th}$ problem.

Let $S \subset \mathbb{C}^n \times \mathrm{Gr}(r, n)$ be the universal subbundle.




Questions (i) and (ii) can be answered via the geometry of $S$.

$H^*(\mathrm{Gr}(r,n),\mathbb{Q})$ is generated by the Chern classes of S,

$$c_1, \ldots, c_r \in H^*(\mathrm{Gr}(r,n),\mathbb{Q}),$$

which measure how much $S$ twists.

(ii) $\lim_{n\to\infty} H^*(\mathrm{Gr}(r,n),\mathbb{Q}) = \mathbb{Q}[c_1,\ldots,c_r]$.

(i) The ideal of relations in $H^*(\mathrm{Gr}(r,n),\mathbb{Q})$ is generated by

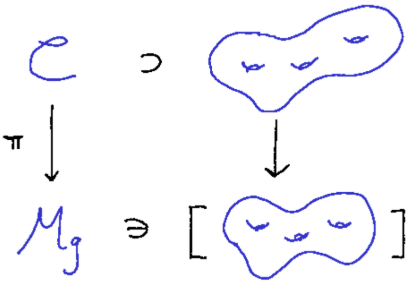
$$\left[\frac{1}{1+c_1 t + c_2 t^2 + \ldots + c_r t^r}\right]_{t^d} = 0$$

for $n-r+1 \leq d \leq n$.

## §V. **Tautological classes on $\mathcal{M}_g$**

What is the analogue of S for the moduli space of curves?

Answer: the universal curve,

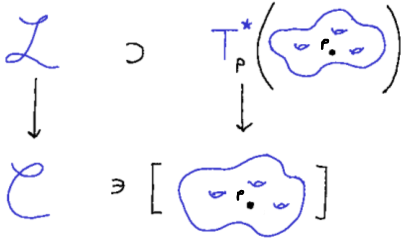


We have actually seen $\mathcal{C}$ before:

$$\mathcal{C} \stackrel{\sim}{=} \mathcal{M}_{g,1}\,.$$

We will construct cohomology classes from an intrinsic complex line bundle on $\mathcal{C}$.

Let $\mathcal{L}$ be the cotangent line over the universal curve,



Since $\mathcal{L} \to \mathcal{C}$ is a line bundle, we can define

$$\psi = c_1(\mathcal{L}) \in H^2(\mathcal{C}, \mathbb{Q}) \ .$$

Chern class: Poincaré dual to the cycle defined by the zeros and poles of a meromorphic section of $\mathcal{L}$.

Via integration along the fiber of $\pi : \mathcal{C} \to \mathcal{M}_g$, we define

$$\kappa_i = \pi_*(\psi^{i+1}) \in H^{2i}(\mathcal{M}_g, \mathbb{Q}) \ .$$

Let $R^*(\mathcal{M}_g) \subset H^*(\mathcal{M}_g, \mathbb{Q})$ denote the subring generated by the $\kappa$ classes, also called the Miller-Morita-Mumford classes.

Question: Is $R^*(\mathcal{M}_g) = H^*(\mathcal{M}_g, \mathbb{Q})$?

Answer: No, but yes stably.

Mumford's conjecture / Madsen-Weiss Theorem:

$$\lim_{g \to \infty} H^*(\mathcal{M}_g, \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \ .$$

For fixed genus $g$, we take Mumford's conjecture as motivation to restrict our attention to the tautological subring

$$R^*(\mathcal{M}_g) \subset H^*(\mathcal{M}_g, \mathbb{Q}) \ .$$

Other motivation comes from classical constructions in algebraic geometry: many interesting classes lie in $R^*(\mathcal{M}_g)$.

Question: What is the structure of the ring $R^*(\mathcal{M}_g)$?

Question: What is the ideal of relations

$$0 \to \mathcal{I}_g \to \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \to R^*(\mathcal{M}_g) \to 0 \ ?$$

## §VI. Faber-Zagier **Conjecture**

Results of Looijenga and Faber determine the *lower end* of the tautological ring

$$R^{g-2}(\mathcal{M}_g) = \mathbb{Q} \; , \quad R^{>g-2}(\mathcal{M}_g) = 0 \; .$$

We use here the complex grading, so $R^{g-2}(\mathcal{M}_g) \subset H^{2(g-2)}(\mathcal{M}_g)$.

The study of $R^{g-2}(\mathcal{M}_g)$ and the $\kappa$ proportionalities is a rich subject, but we take a different direction here.

We are interested in the full ideal of relations of $R^*(\mathcal{M}_g)$,

$$\mathcal{I}_g \subset \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \, .$$

Mumford started the study of $\mathcal{I}_g$, but the subject was first attacked systematically by Faber starting around 1990.

Faber's method of construction involved the classical geometry of curves and Brill-Noether theory. The outcome in 2000 was the following proposal formulated with Zagier.

To write the Faber-Zagier relations, let the variable set

$$\mathbf{p} = \{ \ p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \ldots \ \}$$

be indexed by positive integers *not* congruent to 2 modulo 3.

Define the series

$$\Psi(t, \mathbf{p}) = (1 + tp_3 + t^2 p_6 + t^3 p_9 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i$$
$$+ (p_1 + tp_4 + t^2 p_7 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i \ .$$

Since $\Psi$ has constant term 1, we may take the logarithm.

Define the constants $C_r^{\mathrm{FZ}}(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathrm{FZ}}(\sigma)\ t^r \mathbf{p}^{\sigma}\ .$$

The sum is over all partitions $\sigma$ of size $|\sigma|$ which avoid parts congruent to 2 modulo 3. To the partition

$$\sigma = 1^{n_1} 3^{n_3} 4^{n_4} \cdots,$$

we associate the monomial $\mathbf{p}^{\sigma} = p_1^{n_1} p_3^{n_3} p_4^{n_4} \cdots$. Let

$$\gamma^{\mathrm{FZ}} = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathrm{FZ}}(\sigma)\ \kappa_r t^r \mathbf{p}^{\sigma}\ .$$

The coefficient of $t^r \mathbf{p}^{\sigma}$ in the exponential

$$\exp(-\gamma^{\mathrm{FZ}})$$

is a polynomial in the variables $\kappa_i$.

**Theorem (P-Pixton 2010)**

*In $R^d(\mathcal{M}_g)$, the Faber-Zagier relation*

$$\left[\exp(-\gamma^{\mathsf{FZ}})\right]_{t^d \mathbf{p}^\sigma} = 0$$

*holds when $3d > g - 1 + |\sigma|$ and $d \equiv g - 1 + |\sigma| \mod 2$.*

The $g$ dependence in the Faber-Zagier relations of the Theorem occurs in the inequality and the modulo 2 restriction.

For a given genus $g$ and codimension $d$, the Theorem provides finitely many relations.

Examples of Faber-Zagier relations in genus g=6:

$$d = 3\,,\ \sigma = \emptyset \quad : \quad -36000\,\kappa_1^3 + 1555200\,\kappa_1\kappa_2 - 22913280\,\kappa_3\,,$$

$$d = 3\,,\ \sigma = (1^2) : \quad -5453280\,\kappa_1^3 + 167650560\,\kappa_1\kappa_2 - 1745452800\,\kappa_3\,,$$

$$d = 4\,,\ \sigma = (1) \quad : \quad 10584000\,\kappa_1^4 - 783820800\,\kappa_1^2\kappa_2 + 19734865920\,\kappa_1\kappa_3 + 4702924800\,\kappa_2^2 - 363065794560\,\kappa_4\,.$$

The coefficients are large – the relations can be manipulated by theory or by computer, but not really by hand.

§VII. Three questions immediately arise from the Theorem:

(**A**) Do the Faber-Zagier relations span the ideal of all $\kappa$ relations?

(**B**) What is the path of the proof of the Faber-Zagier relations?

(**C**) What about the cohomology of the compactification

$$\mathcal{M}_g \subset \overline{\mathcal{M}}_g\,?$$

The $\mathbb{Q}$-linear span of the Faber-Zagier relations determines an ideal

$$\mathcal{I}_g^{FZ} \subset \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]\,.$$

By the Theorem, $\mathcal{I}_g^{FZ} \subset \mathcal{I}_g$.

**Question A**: Is $\mathcal{I}_g^{FZ} = \mathcal{I}_g$?

$$\text{Answer}: \begin{cases} g < 24, & \text{yes (Faber)}, \\ g \geq 24, & \text{unknown}. \end{cases}$$

Despite serious efforts using different methods (Clader, Faber, Janda, Q. Yin, Randal-Williams), no relation not in $\mathcal{I}_g^{FZ}$ has been found.

**Conjecture A**: $\mathcal{I}_g^{FZ} = \mathcal{I}_g$.

As presented, the Faber-Zagier relations appear from nowhere, but the proof puts the set on conceptual footing related to the theory of semisimple CohFTs.

**Question B**: Path of proof?

We know three proofs (all via Gromov-Witten theory and properties of the virtual fundamental class).

• P.-Pixton-Zvonkine (2013) proved the Faber-Zagier relations using Witten's 3-spin class (mathematical development by Polishchuk-Vaintrob) together with the Givental-Teleman classification of semisimple CohFTs.

• Janda (2015) proved all suitable semisimple CohFTs yield exactly the Faber-Zagier relations.

A Cohomological Field Theory (CohFT) on the $\mathbb{Q}$-vector space $V$ with inner product $\langle , \rangle$ is a set of $\mathbb{Q}$-linear maps

$$\left\{ \Omega_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \right\}_{g,n}$$

which satisfies several axioms of compatibility with the boundary structure of the moduli space.

The genus 0, 3-pointed map $\Omega_{0,3}$ determines a quantum product

$$\langle v_1 \star v_2, v_3 \rangle = \Omega_{0,3}(v_1, v_2, v_3)\,.$$

When $(V, \star)$ is a semisimple algebra, the Givental-Teleman classification determines $\Omega_{g>0,n}$ from $\Omega_{0,n}$ and an R-matrix.

For the 3-spin CohFT,

$$\mathsf{R} = \begin{pmatrix} \boldsymbol{B}_1^{\mathrm{even}}\left(\frac{z}{1728}\right) & -\boldsymbol{B}_1^{\mathrm{odd}}\left(\frac{z}{1728}\right) \\ -\boldsymbol{B}_0^{\mathrm{odd}}\left(\frac{z}{1728}\right) & \boldsymbol{B}_0^{\mathrm{even}}\left(\frac{z}{1728}\right) \end{pmatrix},$$

where the hypergeometric series

$$\boldsymbol{B}_0(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}(-T)^i, \quad \boldsymbol{B}_1(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}\frac{1+6i}{1-6i}(-T)^i$$

are precisely those of the Faber-Zagier relations!

• For the 3-spin CohFT, the vector space is $V = \mathbb{Q}e_0 \oplus \mathbb{Q}e_1$, and the classes are of pure dimension,

$$\Omega_{g,n}(e_1, \ldots, e_1) \in H^{2(\frac{g-1+n}{3})}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) .$$

The Givental-Teleman classification generates a CohFT of impure dimension. The two descriptions must agree

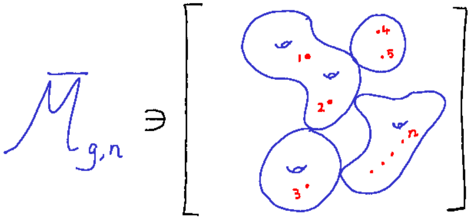
$$\Longrightarrow \qquad \text{Faber-Zagier relations.}$$

• Janda views the same mechanism as a pole cancellation result. Pole cancellations are required by the structure of every (suitable) semisimple CohFT as a non-semisimple limit is taken

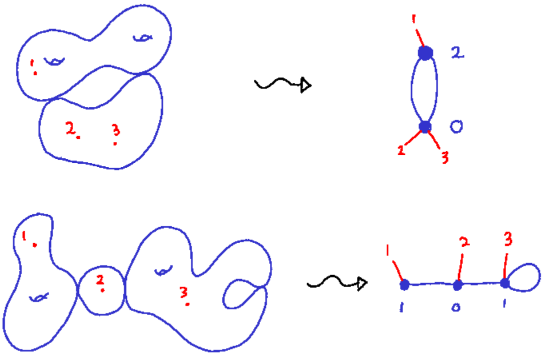
$$\Longrightarrow \qquad \text{Faber-Zagier relations.}$$

**Question C**: Relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$?

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable pointed curves:



The boundary strata of the moduli $\overline{\mathcal{M}}_{g,n}$ of fixed topological type correspond to stable graphs.



For such a graph $\Gamma$, let $[\Gamma] \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the class associated to the closure of the stratum.

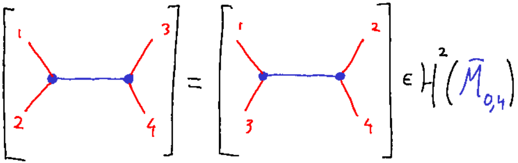
To each stable graph $\Gamma$, we associate the moduli space

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \mathrm{Vert}(\Gamma)} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}.$$

There is a canonical morphism

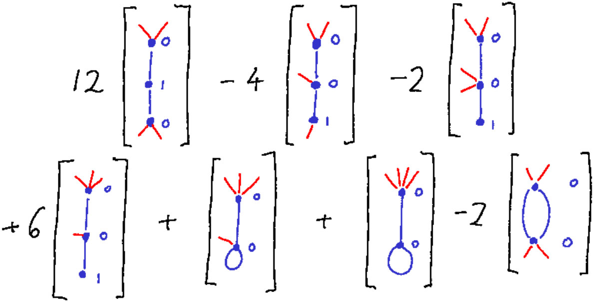
$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} , \qquad \frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot \xi_{\Gamma *}[\overline{\mathcal{M}}_\Gamma] = [\Gamma] .$$

The first boundary relation is almost trivial:



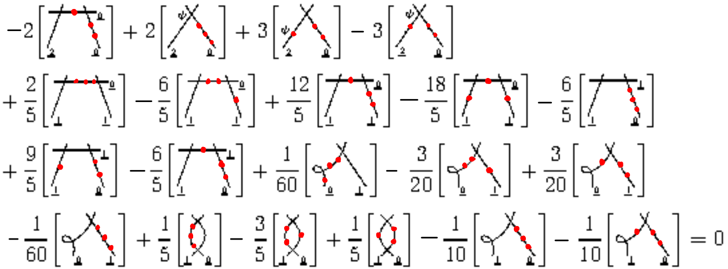

an equivalence of two points in $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$ from the cross-ratio.

Getzler (1996) found the first really interesting relation:



$$= 0 \in H^4(\overline{\mathcal{M}}_{1,4})$$

Of course there are more, but relations are not easy to find.
The next interesting relation (Belorousski-P (1998)) is in genus 2:




in $H^4(\overline{\mathcal{M}}_{2,3}, \mathbb{Q})$ .

Question C′: Is there any structure to these formulas?

Question C″: Is there a connection to the Faber-Zagier relations?

Answer: Yes! (Pixton)

## §VIII. Pixton's **relations on $\overline{\mathcal{M}}_{g,n}$**

We define tautological classes $\mathcal{R}^d_{g,A}$ associated to the data

- $g, n \in \mathbb{Z}_{\geq 0}$ in the stable range $2g - 2 + n > 0$,
- $A = (a_1, \ldots, a_n), \quad a_i \in \{0, 1\}$,
- $d \in \mathbb{Z}_{\geq 0}$ satisfying $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$.

Pixton's relations then take the form

$$\mathcal{R}^d_{g,A} = 0 \quad \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

The formula for $\mathcal{R}^d_{g,A}$ requires more detail than can be given here, but the shape can be easily shown.

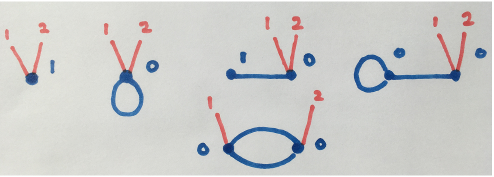
We have already seen the following two series:

$$B_0(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}(-T)^i = 1 - 60T + 27720T^2 \cdots ,$$

$$B_1(T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}\frac{1+6i}{1-6i}(-T)^i = 1 + 84T - 32760T^2 \cdots .$$

- These series control the original set of Faber-Zagier relations.
- These series control Pixton's relations.

Let $\mathsf{G}_{g,n}$ be the finite set of stable graphs of genus $g$ with $n$ legs. For example, $\mathsf{G}_{1,2}$ has 5 elements:



The formula for $\mathcal{R}^d_{g,A}$ is a sum over stable graphs,

$$\mathcal{R}^d_{g,A} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e\right]_d$$

where $\overline{\mathcal{M}}_\Gamma$ is the moduli space associated to $\Gamma$,

$$\mathcal{K}_v\,,\ \Psi_\ell\,,\ \Delta_e \in H^*(\overline{\mathcal{M}}_\Gamma)\,,$$

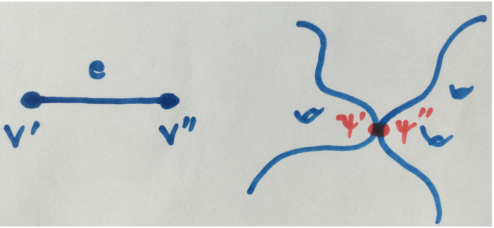
$[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e]$ is the push-forward to $\overline{\mathcal{M}}_{g,n}$ of

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{v \in \mathrm{Vertex}(\Gamma)} \mathcal{K}_v \prod_{\ell \in \mathrm{Leg}(\Gamma)} \Psi_\ell \prod_{e \in \mathrm{Edge}(\Gamma)} \Delta_e \cap [\overline{\mathcal{M}}_\Gamma]$$

and $[...]_d$ extracts the part in $H^{2d}(\overline{\mathcal{M}}_{g,n})$.

$$\mathcal{R}^d_{g,A} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{2^{h^1(\Gamma)}} \left[\Gamma, \prod \mathcal{K}_v \prod \Psi_\ell \prod \Delta_e\right]_d$$

- Vertex $\mathcal{K}_v$, leg $\Psi_v$, and edge $\Delta_e$ factors have explicit formulas in terms of the $\kappa$ and $\psi$ classes and the series $B_0$ and $B_1$.
- Edge factor is the most interesting:



For $e \in \text{Edge}(\Gamma)$, the formula for the edge factor is:

$$\Delta_e = \frac{2 - B_0(\psi')B_1(\psi'') - B_1(\psi')B_0(\psi'')}{\psi' + \psi''}$$

$$= -24 + 5040(\psi' + \psi'') + \dots .$$

The numerator of $\Delta_e$ is divisible by the denominator by the identity

$$B_0(T)B_1(-T) + B_1(T)B_0(-T) = 2 .$$

Warning: A parity factor has been omitted for simplicity.

## Theorem (P-Pixton-Zvonkine 2013)

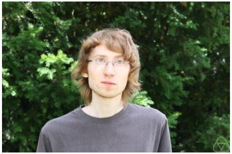
*For* $2g - 2 + n > 0$, $a_i \in \{0, 1\}$, *and* $d > \frac{g-1+\sum_{i=1}^{n} a_i}{3}$, Pixton's relations *hold*

$$R^d_{g,A} = 0 \ \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

• Proof is by the CohFT path used for the Faber-Zagier relations. The geometry there naturally concerns $\overline{\mathcal{M}}_{g,n}$.

• Theorem captures everything we have seen: the cross-ratio, Getzler's relation, the Faber-Zagier relations.

• By Janda's results, Pixton's relations hold in the Chow theory of algebraic cycles.



Mumford (1983), in his foundational paper

*Towards an enumerative geometry of the moduli space of curves*,

opened the study of the algebra of tautological classes.

Pixton's relations provide the first proposal for their calculus parallel to the Schubert calculus for $\mathrm{Gr}(r, n)$.

**Questions**:

- Are Pixton's relations the complete set of relations among tautological classes?

- Is there an abstract algebraic structure which realizes Pixton's relations?

The End

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