QUOT SCHEMES OF CURVES AND SURFACES: VIRTUAL CLASSES, INTEGRALS, EULER CHARACTERISTICS

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Abstract. We compute tautological integrals over Quot schemes on curves and surfaces. After obtaining several explicit formulas over Quot schemes of dimension 0 quotients on curves (and finding a new symmetry), we apply the results to tautological integrals against the virtual fundamental classes of Quot schemes of dimension 0 and 1 quotients on surfaces (using also universality, torus localization, and cosection localization). The virtual Euler characteristics of Quot schemes of surfaces, a new theory parallel to the Vafa-Witten Euler characteristics of the moduli of bundles, is defined and studied. Complete formulas for the virtual Euler characteristics are found in the case of dimension 0 quotients on surfaces. Dimension 1 quotients are studied on $K3$ surfaces and surfaces of general type with connections to the Kawai-Yoshioka formula and the Seiberg-Witten invariants respectively. The dimension 1 theory is completely solved for minimal surfaces of general type admitting a nonsingular canonical curve. Along the way, we find a new connection between weighted tree counting and multivariate Fuss-Catalan numbers which is of independent interest.

Contents

1. Introduction

1.1. **Overview.** The main goal of the paper is to study the virtual fundamental classes of Quot schemes of surfaces. The parallel study for 3-folds was undertaken in [37, 38] and led to the MacMahon function for Hilbert schemes of points and the GW/DT correspondence for Hilbert schemes of curves. For the surface case, we use several techniques: the universality results of [8], $\mathbb{C}^\star$-equivariant localization of the virtual class [17], and

Date: March 2019.

cosection localization [20]. However, the most important input to the surface theory concerns the parallel study of Quot schemes of curves of quotients with dimension 0 support, which we develop first. By applying the curve results to the surface theory, we prove several basic results about the virtual fundamental classes of Quot schemes of quotients with supports of dimension 0 and 1 on surfaces. The subject is full of open questions.

1.2. **Curves.** Let C be a nonsingular projective curve. Let $\mathsf{Quot}_C(\mathbb{C}^N, n)$ parameterize short exact sequences

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \to Q \to 0\,,$$

where Q is a rank 0 sheaf on C with

$$\chi(Q) = n\,.$$

The scheme $\mathsf{Quot}_C(\mathbb{C}^N, n)$ was viewed in [33] as the stable quotient compactification of *degree n maps to the point*, where the point is the degenerate Grassmannian $\mathbb{G}(N, N)$. By analyzing the Zariski tangent space, $\mathsf{Quot}_C(\mathbb{C}^N, n)$ is easily seen to be a nonsingular projective variety of dimension Nn, see [33, Section 4.7].

For a vector bundle $V \to C$ of rank r, the assignment

$$Q \mapsto H^0(C, V \otimes Q)$$

for $[\mathbb{C}^N \otimes O_C \to Q] \in \mathsf{Quot}_C(\mathbb{C}^N, n)$ defines a tautological vector bundle

$$V^{[n]} \to \mathsf{Quot}_C(\mathbb{C}^N, n)$$

of rank rn. The construction descends to K-theory via locally free resolutions. We define generating series of Segre[1] classes on Quot schemes of curves as follows.

Definition 1. *Let $\alpha_1, \ldots, \alpha_\ell$ be K-theory classes on C. Let*

$$\mathsf{Z}_{C,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \sum_{n=0}^{\infty} q^n \int_{\mathsf{Quot}_C(\mathbb{C}^N, n)} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_\ell}(\alpha_\ell^{[n]})\,.$$

Since the integrals in Definition 1 depend upon C only through the genus g of the curve, we will often write

$$\mathsf{Z}_{g,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \mathsf{Z}_{C,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell)\,.$$

By the arguments of [8], there exists a factorization

$$\mathsf{Z}_{g,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \mathsf{A}_1^{c_1(\alpha_1)} \cdots \mathsf{A}_\ell^{c_1(\alpha_\ell)} \cdot \mathsf{B}^{1-g}, \tag{1}$$

[1]For a vector bundle V on a scheme X, we write

$$s_t(V) = 1 + t s_1(V) + t^2 s_2(V) + \ldots$$

for the total Segre class.

for universal series

$$\mathsf{A}_1\,,\;\ldots\;,\;\mathsf{A}_\ell\,,\;\mathsf{B}\;\in\mathbb{Q}[[q,x_1,\ldots,x_\ell]] \tag{2}$$

which *do not* depend on the genus g or the degrees $c_1(\alpha_i)$. However, the series (2) *do* depend on the ranks

$$\mathsf{r}=(r_1,\ldots,r_\ell)\,,\quad r_i=\mathrm{rank}\;\alpha_i$$

and N. The complete notation for the series (2) is

$$\mathsf{A}_{1,\mathsf{r},N}\,,\;\ldots\;,\;\mathsf{A}_{\ell,\mathsf{r},N}\,,\;\mathsf{B}_{\mathsf{r},N}\;\in\mathbb{Q}[[q,x_1,\ldots,x_\ell]]\,, \tag{3}$$

but we will often use the abbreviated notation (2) with the ranks r_i and N suppressed.

Question 2. *Find closed-form expressions for the series* $\mathsf{A}_{i,\mathsf{r},N}$ *and* $\mathsf{B}_{\mathsf{r},N}$.

Integrals over Quot schemes of curves were also studied in [32] via equivariant localization. In particular, formulas of Vafa-Intriligator [2, 18, 49] were recovered and extended.

1.3. **Symmetric products ($N=1$).** For curves, the symmetric product $C^{[n]}$ is the Quot scheme in the $N=1$ case,

$$C^{[n]}=\mathsf{Quot}_C(\mathbb{C}^1,n)\,.$$

We give a complete answer to Question 2 for $N=1$. The result will later play an important role in our study of Quot schemes of surfaces.

Theorem 3. *Let $\alpha_1,\ldots,\alpha_\ell$ have ranks $r_1,\ldots,r_\ell$, and let $N=1$. Then*

$$\mathsf{Z}_{g,1}(q,x_1,\ldots,x_\ell\,|\,\alpha_1,\ldots,\alpha_\ell)=\mathsf{A}_1(q)^{\,c_1(\alpha_1)}\cdots\mathsf{A}_\ell(q)^{\,c_1(\alpha_\ell)}\cdot\mathsf{B}(q)^{1-g}\,,$$

where, for the change of variables

$$q=t(1-x_1t)^{r_1}\cdots(1-x_\ell t)^{r_\ell}\,, \tag{4}$$

we set

$$\mathsf{A}_i(q)=1-x_i\cdot t\,,\qquad \mathsf{B}(q)=\left(\frac{q}{t}\right)^2\cdot\frac{dt}{dq}\,.$$

To compute the series[2] $\mathsf{A}_i(q)$ and $\mathsf{B}(q)$, the change of variables (4) must be inverted to write t as a function of q with $x_1,\ldots,x_\ell$ viewed as parameters. By Theorem 3, the series $\mathsf{Z}_{g,1}(q,x_1,\ldots,x_\ell\,|\,\alpha_1,\ldots,\alpha_\ell)$ is a function in q which is *algebraic* over the field $\mathbb{Q}(x_1,\ldots,x_\ell)$.

[2]For Theorem 3, the complete notation is $\mathsf{A}_i=\mathsf{A}_{i,\mathsf{r},1}$ and $\mathsf{B}=\mathsf{B}_{\mathsf{r},1}$.

Remark 4. Specializing to the case $\ell = 1$, $x_1 = 1$, and $r_1 = r$, and letting $V \to C$ be a rank r vector bundle, we recover the result of [36]:

$$\sum_{n=0}^{\infty} q^n \int_{C^{[n]}} s_n(V^{[n]}) = \exp\left(c_1(V) \cdot \widehat{\mathsf{A}}(q) + (1-g) \cdot \widehat{\mathsf{B}}(q)\right) \tag{5}$$

for the series

$$\begin{aligned} \widehat{\mathsf{A}}(t(1-t)^r) &= \log(1-t)\,, \\ \widehat{\mathsf{B}}(t(1-t)^r) &= (r+1)\log(1-t) - \log(1-t(r+1))\,. \end{aligned} \tag{6}$$

These expressions confirmed and expanded predictions of [57]. The $r = 1$ case is related to the counts of secants to projectively embedded curves [6, 24].

Remark 5. To go beyond numerical invariants, we consider a flat family

$$\pi : C \to S$$

of nonsingular projective curves with line bundles $L_1, \ldots, L_\ell \to C$. We write

$$\pi^{[n]} : C^{[n]} \to S$$

for the relative symmetric product. A more difficult question concerns the calculation of the push-forwards

$$\sum_{n=0}^{\infty} q^n \pi_\star^{[n]} \left(s_{x_1}(L_1^{[n]}) \cdots s_{x_\ell}(L_\ell^{[n]})\right) \in A^\star(S)$$

in terms of the classes

$$\kappa[a_1, \ldots, a_\ell, b] = \pi_\star\left(c_1(L_1)^{a_1} \cdots c_1(L_\ell)^{a_\ell} \cdot c_1(\omega_\pi)^b\right) \in A^\star(S)\,.$$

When π is the universal family over the moduli space of curves, such constructions play a role in the study of tautological classes [41, 42].

1.4. Higher N (for $\ell = 1$).

Our second result concerns the case of arbitrary N, but we assume $\ell = 1$. The corresponding series is

$$\mathsf{Z}_{g,N}(q \,|\, V) = \sum_{n=0}^{\infty} q^n \int_{\mathsf{Quot}_C(\mathbb{C}^N, n)} s(V^{[n]})\,,$$

where $V \to C$ is a rank r vector bundle.

Theorem 6. *The universal Segre series is*

$$\mathsf{Z}_{g,N}(q \,|\, V) = \mathsf{A}(q)^{c_1(V)} \cdot \mathsf{B}(q)^{1-g}\,,$$

where

$$\log \mathsf{A}(q) = \sum_{n=1}^{\infty} (-1)^{(N+1)n+1} \binom{(r+N)n-1}{Nn-1} \cdot \frac{q^n}{n}\,.$$

Remark 7. In case $N = 1$, Theorem 6 is a special case of Theorem 3. The agreement of the formulas follows from the identity

$$-\log(1-t) = \sum_{n=1}^{\infty} \binom{(r+1)n-1}{n-1} \cdot \frac{q^n}{n} \qquad \text{for} \qquad q = t(1-t)^r$$

which will be proven in Lemma 33 below.

Theorem 6 identifies the $\ell = 1$ series $\mathsf{A} = \mathsf{A}_{1,r,N}$, but does not specify the series $\mathsf{B} = \mathsf{B}_{r,N}$. However, for rank $r = 1$, *closed-form* expressions for the A and B-series are determined by the following result.

Theorem 8. *For rank $V = 1$, after the change of variables*

$$q = (-1)^N t(1+t)^N$$

we have

$$\mathsf{A}_{1,1,N}(q) = (1+t)^N \quad \textit{and} \quad \mathsf{B}_{1,N}(q) = \frac{(1+t)^{N+1}}{1+t(N+1)}.$$

We also write an explicit power series expansion for the B-series parallel to Theorem 6.

Corollary 9. *For rank $V = 1$, we have*

$$\mathsf{B}_{1,N}(q) = \sum_{n=0}^{\infty} (-1)^{n(N+1)} \cdot \binom{(n-1)(N+1)}{n} \cdot q^n.$$

By comparing the expressions of Theorem 8 with those of equation (6), we obtain the following new symmetry exchanging N and the rank.

Corollary 10. *For any line bundle $L \to C$, we have*

$$\int_{\mathsf{Quot}_C(\mathbb{C}^N, n)} s(L^{[n]}) = (-1)^{n(N-1)} \int_{C^{[n]}} s(L^{[n]})^N.$$

In particular, for $C = \mathbb{P}^1$, we have

$$\int_{\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)} s(L^{[n]}) = (-1)^{Nn} \binom{N \deg L - N(n-1)}{n}.$$

1.5. **Catalan numbers.** By specializing Theorem 3 to the case of an elliptic curve C and using Wick expansion techniques, we are led to a combinatorial identity for Catalan numbers which appears to be new.[3]

The m^{th} Catalan number

$$\mathsf{C}_m = \frac{1}{m+1}\binom{2m}{m}$$

[3]There are many realizations of the Catalan numbers! But we have asked several experts and ours does not appear to be in the literature. If you know a reference, please tell us.

is well-known to count unlabelled ordered trees with $m+1$ vertices [50]. The *multivariate Fuss-Catalan* numbers were introduced and studied in [1]. A special case of the definition is used here. For non-negative integers $p_1, \ldots, p_k$, the multivariate Fuss-Catalan number of interest to us is

$$\mathsf{C}(p_1, \ldots, p_k) = \frac{1}{p_1 + \ldots + p_k + 1} \binom{2p_1 + p_2 + \ldots + p_k}{p_1} \cdots \binom{p_1 + p_2 + \ldots + 2p_k}{p_k}.$$

The case $k = 1$ corresponds to the usual Catalan number $\mathsf{C}(m) = \mathsf{C}_m$. The multivariate Fuss-Catalan numbers were shown to count certain k-Dyck paths or, alternatively, k-nary trees, and also arise in connection with algebras of B-quasisymmetric polynomials [1].

We interpret the Catalan and multivariate Fuss-Catalan numbers as a *weighted* count of trees. Let non-negative integers $p_1, \ldots, p_k$ be given. Let

$$n = p_1 + \ldots + p_k + 1.$$

A *labelled k-colored tree of type* $(p_1, \ldots, p_k)$ is a tree T with

- n vertices labelled $\{1, 2, \ldots, n\}$,
- $n - 1$ edges each painted with one of the k different colors such that exactly p_j edges are painted with the j^{th} color.

For each vertex v, we write

$$d_v^1, \ldots, d_v^k$$

for the *out-degrees*[4] of v corresponding to each of the k colors. More precisely, d_v^j counts edges e incident to v, of color j, such that e connects v to a vertex w satisfying

$$v > w.$$

We define the *weight* of T as the product

$$\mathsf{wt}(T) = \frac{1}{(n-1)!} \prod_{v \text{ vertex}} d_v^1! \cdots d_v^k!.$$

Theorem 11. *The Fuss-Catalan number is the weighted count of ordered k-colored trees of type $(p_1, \ldots, p_k)$:*

$$\mathsf{C}(p_1, \ldots, p_k) = \sum_T \mathsf{wt}(T).$$

Example 12. Let us now specialize to the single color ($k = 1$) case with $m = p_1$ and $n = m + 1$. The weights then take the form:

$$\mathsf{wt}(T) = \frac{1}{m!} \prod_{v \text{ vertex}} d_v! = \binom{m}{d_1, \ldots, d_n}^{-1},$$

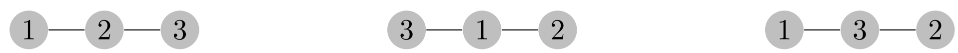
[4]The term *out-degree* comes from regarding T as an oriented graph with each edge oriented in the direction of *decreasing* vertex label.

where for each v, d_v denotes the out-degree of v. We then obtain the standard m^{th} Catalan number as a weighted count of labelled trees with $m+1$ vertices:

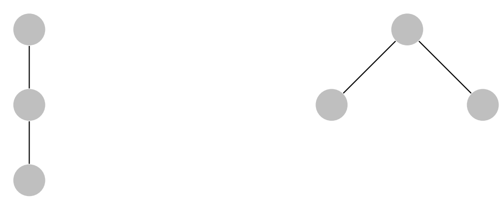
$$\mathsf{C}(m) = \sum_T \mathsf{wt}(T)\,. \tag{7}$$

The result (7) should perhaps be compared with the realization of $\mathsf{C}(m)$ as the unweighted count of unlabelled ordered trees with $m+1$ vertices (see [50] for instance). The following diagram shows the two counts for $\mathsf{C}(2)$:

– weighted count



– unweighted count



In the first count, the weights are $\frac{1}{2}, \frac{1}{2}$ and 1 respectively and

$$\mathsf{C}(2) = \frac{1}{2} + \frac{1}{2} + 1.$$

1.6. **Surfaces: dimension 0 quotients.** We can apply the above results for curves to the calculation of tautological integrals over Quot schemes of dimension 0 quotients of nonsingular projective surfaces X.

The Quot scheme $\mathsf{Quot}_X(\mathbb{C}^N, n)$ of short exact sequences

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0, \quad \chi(Q) = n\,,\ c_1(Q) = 0\,,\ \mathrm{rank}(Q) = 0$$

is known [9, 29] to be irreducible of dimension $n(N+1)$, but may be singular.[5] Since the higher obstructions for the standard deformation theory lie in

$$\mathrm{Ext}^2(S,Q) = \mathrm{Ext}^0(Q, S \otimes K_X)^\vee = 0, \tag{8}$$

the Quot scheme carries a 2-term perfect obstruction theory and a virtual fundamental cycle of dimension

$$\mathrm{Ext}^0(S,Q) - \mathrm{Ext}^1(S,Q) = \chi(S,Q) = Nn\,.$$

[5] An example is given in Section 4 below.

Question 13. *Evaluate the integrals*

$$\mathsf{Z}_{X,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \sum_{n=0}^{\infty} q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N, n)]^{\mathrm{vir}}} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_\ell}(\alpha_\ell^{[n]})$$

where $\alpha_1, \ldots, \alpha_\ell$ are K-theory classes on X.

By our next result, the surface series of Question 13 are obtained from the parallel curves series of Question 2. The relationship is not unlike the localization result for the Gromov-Witten theory of surfaces of general type with respect to a canonical divisor [20, 25, 39].

Theorem 14. *Let the ranks of the classes $\alpha_1, \ldots, \alpha_\ell$ be given by $\mathsf{r} = (r_1, \ldots, r_\ell)$. Let the series $\mathsf{A}_{1,\mathsf{r},N}, \ldots, \mathsf{A}_{\ell,\mathsf{r},N}, \mathsf{B}_{\mathsf{r},N}$ be defined by the curve integrals (1). Then, we have*

$$\mathsf{Z}_{X,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \\ \mathsf{A}_{1,\mathsf{r},N}(-q)^{c_1(\alpha_i)\cdot K_X} \cdots \mathsf{A}_{\ell,\mathsf{r},N}(-q)^{c_1(\alpha_\ell)\cdot K_X} \cdot B_{\mathsf{r},N}(-q)^{-K_X^2} .$$

In case X is a surface of general type with a nonsingular canonical divisor

$$C \subset X ,$$

then $c_1(\alpha_i) \cdot K_X$ is the degree of the restriction of α_i to C and

$$-K_X^2 = 1 - \mathrm{genus}(C)$$

by adjunction. We may therefore write Theorem 14 as

$$\mathsf{Z}_{X,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \mathsf{Z}_{g(C),N}\left(-q, x_1, \ldots, x_\ell \,\middle|\, \alpha_1|_C, \ldots, \alpha_\ell|_C\right) .$$

However, Theorem 14 holds for all X (even if X is not of general type).

For $N = 1$, Theorems 3 and 14 together yield a complete answer for the virtual Segre integrals over the Hilbert scheme of points,

$$X^{[n]} = \mathsf{Quot}_X(\mathbb{C}^1, n) .$$

Corollary 15. *Let X be a nonsingular projective surface. Then*

$$\sum_{n=0}^{\infty} q^n \int_{[X^{[n]}]^{\mathrm{vir}}} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_\ell}(\alpha_\ell^{[n]}) = \mathsf{A}_1(q)^{c_1(\alpha_1)\cdot K_X} \cdots \mathsf{A}_\ell(q)^{c_1(\alpha_\ell)\cdot K_X} \cdot \mathsf{B}(q)^{-K_X^2}$$

where, for the change of variable

$$q = -t(1 - x_1 t)^{r_1} \cdots (1 - x_r t)^{r_\ell} ,$$

we set

$$\mathsf{A}_i(q) = 1 - x_i \cdot t , \quad \mathsf{B}(q) = -\left(\frac{q}{t}\right)^2 \cdot \frac{dt}{dq} .$$

Similarly, for higher N, Theorems 8 and 14 yield the following evaluation.

Corollary 16. *Let $L \to X$ be a line bundle on a nonsingular projective surface. Then*

$$\sum_{n=0}^{\infty} q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N,n)]^{\mathrm{vir}}} s(L^{[n]}) = \mathsf{A}(q)^{c_1(L)\cdot K_X} \cdot \mathsf{B}(q)^{-K_X^2}$$

where, for the change of variables

$$q = (-1)^{N+1}\, t\,(1+t)^N\,,$$

we set

$$\mathsf{A}(q) = (1+t)^N, \qquad \mathsf{B}(q) = \frac{(1+t)^{N+1}}{1+(N+1)t}\,.$$

Remark 17. Question 13 is well-posed for integrals against the actual fundamental class of dimension $n(N+1)$ of $\mathsf{Quot}_X(\mathbb{C}^N, n)$ instead of the virtual fundamental class of dimension nN. The calculation for the actual fundamental class is more complicated. The $N = 1$ case is by far the most studied. Then, the series

$$\mathsf{Z}_X(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \sum_{n=0}^{\infty} q^n \int_{X^{[n]}} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_r}(\alpha_\ell^{[n]})$$

are generalizations of the Segre integrals considered by Lehn [26]. In fact, Lehn's case corresponds to $\ell = 1$ and rank $\alpha_1 = 1$, and was studied in [35, 36, 56]. The case

$$x_1 = \ldots = x_\ell = 1$$

was studied in [34], and a complete solution was given for K-trivial surfaces. The case $\ell = 2$ was analyzed in [58], and the answer was found for all surfaces if

$$\text{rank } \alpha_1 = \text{rank } \alpha_2 = -1$$

via connections to K-theory.

1.7. **Virtual Euler characteristics: dimension 0 quotients.** The topological Euler characteristics of the schemes $\mathsf{Quot}_C(\mathbb{C}^N, n)$ and $\mathsf{Quot}_X(\mathbb{C}^N, n)$ can be easily computed via equivariant localization:

$$\sum_{n=0}^{\infty} q^n \mathsf{e}(\mathsf{Quot}_C(\mathbb{C}^N, n)) = (1-q)^{N(2g-2)}\,,$$

$$\sum_{n=0}^{\infty} q^n \mathsf{e}(\mathsf{Quot}_X(\mathbb{C}^N, n)) = \prod_{n=1}^{\infty} (1-q^n)^{-N\chi(X)}\,.$$

More subtle is the virtual Euler characteristic of $\mathsf{Quot}_X(\mathbb{C}^N, n)$ defined via the 2-term obstruction theory. A basic result for dimension 0 quotients, proven using a reduction to the Quot schemes of curves, is the following rationality statement.

Theorem 18. *The generating series of virtual Euler characteristics of* $\mathsf{Quot}_X(\mathbb{C}^N, n)$ *is a rational function in* q *which depends only upon* K_X^2 *and* N*,*

$$\sum_{n=0}^{\infty} q^n \mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, n)) = \mathsf{U}_N^{K_X^2}, \quad \mathsf{U}_N \in \mathbb{Q}(q).$$

We can calculate U_1 directly using the evaluations given in Theorem 3:

$$\mathsf{U}_1 = \frac{(1-q)^2}{1-2q}.$$

For higher N, a more involved computation in Section 4.3 yields an an exact expression in a different form:

$$\mathsf{U}_N(q) = \frac{(1-q)^{2N}}{(1-2^N q)^N} \cdot \prod_{i<j} (1 - (r_i - r_j)^2), \tag{9}$$

where $r_1(q), \ldots, r_N(q)$ are the N distinct roots of the polynomial equation

$$z^N - q(z-1)^N = 0$$

in the variable z. The shape of the answer is reminiscent of the Vafa-Intriligator formulas for Quot schemes of curves [2, 18, 32, 49] which yield expressions depending on the roots of unity.

Using (9), we can easily calculate U_N as a rational function of q. The next few cases are:

$$\begin{aligned}
\mathsf{U}_2 &= \frac{(1-q)^2(1-6q+q^2)}{(1-4q)^2}, \\
\mathsf{U}_3 &= \frac{(1-q)^2(1-22q+150q^2-22q^3+q^4)}{(1-8q)^3}, \\
\mathsf{U}_4 &= \frac{(1-q)^2(1-62q+1407q^2-15492q^3+1407q^4-62q^5+q^6)}{(1-16q)^4}.
\end{aligned}$$

Formula (9) implies

$$\mathsf{U}_N(q) = \frac{(1-q)^2}{(1-2^N q)^N} \cdot \mathsf{P}_N(q), \tag{10}$$

where $\mathsf{P}_N(q) \in \mathbb{Z}[q]$ is a palindromic polynomial of degree $2N-2$. A simple functional equation holds for the transformation $q \leftrightarrow q^{-1}$.

1.8. **Surfaces: dimension 1 quotients.** Let X be a nonsingular, simply connected, projective surface, and let $D \in A^1(X)$ be a divisor class. As observed in [35], the Quot scheme $\mathsf{Quot}_X(\mathbb{C}^N, n, D)$ of short exact sequences

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0, \quad \chi(Q) = n, \ c_1(Q) = D, \ \mathrm{rank}(Q) = 0$$

carries a 2-term perfect obstruction theory and a virtual fundamental class of dimension

$$\chi(S, Q) = Nn + D^2.$$

Indeed, the higher obstructions vanish

$$\mathrm{Ext}^2(S,Q) = \mathrm{Ext}^0(Q, S \otimes K_X)^\vee = 0\,,$$

since Q is a torsion sheaf. Using the above obstruction theory, we define generating series of virtual Euler characteristics.

Definition 19. *Let X be a nonsingular, simply connected* [6]*, projective surface. For a divisor class $D \in A^1(X)$ and an integer $N \geq 1$, let*

$$\mathsf{Z}^{\mathcal{E}}_{X,N,D}(q) = \sum_{n\in\mathbb{Z}} q^n \mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, n, D))\,.$$

For fixed N and D, the Quot schemes $\mathsf{Quot}_X(\mathbb{C}^N, n, D)$ are empty for all n sufficiently negative, so

$$\mathsf{Z}^{\mathcal{E}}_{X,N,D}(q) \,\in\, \mathbb{Z}((q))\,.$$

The virtual Euler characteristic results described in Section 1.7 concern the generating series $\mathsf{Z}^{\mathcal{E}}_{X,N,0}(q)$ with vanishing divisor class D. In case $D \neq 0$, exact calculations are more difficult to obtain.

(i) Rational surfaces

A very rich theory arises for rational surfaces. In Section 5.2, we write general tautological integrals over Hilbert schemes of points which compute the virtual Euler characteristics. The following result provides an example of an exact solution.

Proposition 20. *Let X be the blowup of a rational surface with exceptional divisor E. We have*

$$\mathsf{Z}^{\mathcal{E}}_{X,1,E}(q) = q\left(\frac{(1-q)^2}{1-2q}\right)^{K_X^2+1}.$$

The formula of Proposition 20 concerns only the case $N = 1$. The proof makes use again of Theorem 3 for curves. Further exact calculations for rational surfaces will require new techniques. However, we can calculate much more for $K3$ surfaces and surfaces of general type.

(ii) $K3$ surfaces

For $K3$ surfaces, the standard obstruction theory contains a trivial factor which forces the virtual invariants to vanish. The natural generating series therefore concerns the

[6] There is no difficulty to define the generating series in the non-simply connected case, but then D should be taken in $H^2(X,\mathbb{Z})$ instead of $A^1(X)$.

virtual Euler characteristics of the *reduced* obstruction theory:

$$\mathsf{Z}^{\mathrm{r}\mathcal{E}}_{X,N,D}(q) = \sum_{n\in\mathbb{Z}} q^n \mathsf{e}^{\mathrm{red}}(\mathsf{Quot}_X(\mathbb{C}^N, n, D))\,.$$

In the $N = 1$ case, the reduced obstruction theory leads to expressions matching the curve counts on $K3$ surfaces. Specifically, let $N_{g,n}$ be defined by the Kawai-Yoshioka [19] formula:

$$\sum_{g=0}^{\infty}\sum_{n=1-g}^{\infty} N_{g,n}\, y^n\, q^g = \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}\,. \tag{11}$$

The Kawai-Yoshioka formula has played a central role in the Gromov-Witten and the stable pairs theory of $K3$ surfaces [40, 45, 46]. For primitive classes, we have complete results.

Theorem 21. *Let X be a $K3$ surface, and let D be a primitive divisor class of genus $2g-2 = D^2$ which is big and nef. We have*

$$\mathsf{e}^{\mathrm{red}}(\mathsf{Quot}_X(\mathbb{C}^1, n, D)) = N_{g,n}\,.$$

The argument matches the reduced virtual Euler characteristic integral of the Quot scheme to the topological Euler characteristic integral of the moduli space of stable pairs (the integrands however are *not* the same).

(iii) Surfaces of general type

Let X be a simply connected surface of general type with $p_g > 0$. In the class of the canonical divisor K_X, we show the vanishing of the virtual Euler characteristics for $N = 1$ in almost all cases. The single exception is significant: the Poincaré-Seiberg-Witten invariants of [5, 7] are recovered,

$$\mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1, n = -K_X^2, K_X)) = (-1)^{\chi(\mathcal{O}_X)}\,.$$

For arbitrary N, a vanishing holds for minimal surfaces.[7]

Proposition 22. *Let X be a simply connected minimal surface of general type with $p_g > 0$. If D is a curve class with*

$$\left[\mathsf{Quot}_X(\mathbb{C}^N, n, D)\right]^{\mathrm{vir}} \neq 0\,,$$

then $D = \ell K_X$ for $0 \leq \ell \leq N$.

---

[7]We thank M. Kool for very helpful discussions about Seiberg-Witten classes.

If we further assume the canonical class of X is represented by a nonsingular curve, we can calculate $\mathsf{Z}^{\mathcal{E}}_{X,\,N,\,\ell K_X}(q)$ completely in all cases. By Proposition 22, we need only consider ℓ in the range

$$0 \leq \ell \leq N\,.$$

Theorem 23. *Let X be a simply connected minimal surface of general type with a nonsingular canonical curve of genus $g = K_X^2 + 1$. Then,*

$$\mathsf{Z}^{\mathcal{E}}_{X,\,N,\,\ell K_X}(q) = (-1)^{\ell\cdot\chi(\mathcal{O}_X)}\; q^{\ell(1-g)} \cdot \sum_{1\leq i_1<\ldots<i_{N-\ell}\leq N} \mathsf{A}(r_{i_1},\ldots,r_{i_{N-\ell}})^{1-g}\,,$$

where the sum is taken over all $\binom{N}{N-\ell}$ choices of $N-\ell$ distinct roots $r_i(q)$ of the polynomial equation

$$z^N - q(z-1)^N = 0$$

in the variable z. The function A is defined by

$$\mathsf{A}(x_1,\ldots,x_{N-\ell}) = \frac{(-1)^{\binom{N-\ell}{2}}}{N^{N-\ell}} \cdot \prod_{i=1}^{N-\ell} \frac{(1+x_i)^N(1-x_i)}{x_i^{N-1}} \cdot \prod_{i<j} \frac{(x_i-x_j)^2}{1-(x_i-x_j)^2}.$$

Since the answer of Theorem 23 is a symmetric function of the roots $r_1(q),\ldots,r_N(q)$, we have

$$\mathsf{Z}^{\mathcal{E}}_{X,N,\ell K_X}(q) \,\in\, \mathbb{Q}(q)\,.$$

Theorem 23 is the most advanced calculation of paper. The proof uses essentially all of the ideas and methods that we have developed.

Example 24. Theorem 23 for $N = 2$ and $\ell = 1$ specializes to the following formula:

$$\mathsf{Z}^{\mathcal{E}}_{X,2,K_X}(q) = (-1)^{\chi(\mathcal{O}_X)} \left(\frac{q}{2}\right)^{1-g} \left(\left(\frac{(1+r_1)^2(1-r_1)}{r_1}\right)^{1-g} + \left(\frac{(1+r_2)^2(1-r_2)}{r_2}\right)^{1-g} \right), \tag{12}$$

where $r_1(q)$ and $r_2(q)$ are the two roots of the quadratic equation

$$z^2 - q(z-1)^2 = 0$$

in the variable z. For a minimal surface of general type X with a canonical curve of genus 2, formula (12) yields:

$$\mathsf{Z}^{\mathcal{E}}_{X,2,K_X}(q) = (-1)^{\chi(\mathcal{O}_X)}\,\frac{(16q-8)}{(1-4q)^2}\,.$$

For X with a canonical curve of genus 3, the answer is

$$\mathsf{Z}^{\mathcal{E}}_{X,2,K_X}(q) = (-1)^{\chi(\mathcal{O}_X)}\,\frac{(128q^4-64q^3+8q^2-16q+8)}{q(1-4q)^4}\,.$$

1.9. **Rationality.** By Theorem 18, the series $\mathsf{Z}^{\mathcal{E}}_{X,N,0}(q)$ is the expansion of a rational function in q. Rationality also holds for all the examples discussed in Section 1.8 for Quot schemes of quotients with dimension 1 support on surfaces.

Conjecture 25. *For a nonsingular, simply connected, projective surface X,*

$$\mathsf{Z}^{\mathcal{E}}_{X,N,D}(q) \in \mathbb{Q}(q)\,.$$

A natural further direction is to study the associated series in algebraic cobordism:

$$\mathsf{Z}^{\mathsf{Cobord}}_{X,N,D} \;=\; \sum_{n\in\mathbb{Z}} [\mathsf{Quot}_X(\mathbb{C}^N, n, D)]^{\mathrm{vir}} q^n \;\in\; \Omega_*(\mathsf{point})((q))\,.$$

The algebraic cobordism[8] class

$$[\mathsf{Quot}_X(\mathbb{C}^N, n, D)]^{\mathrm{vir}} \in \Omega_*(\mathsf{point})$$

is well-defined by [48]. Are there formulas for $\mathsf{Z}^{\mathsf{Cobord}}_{X,N,D}(q)$?

The parallel question for the virtual classes in algebraic cobordism of the moduli spaces of stable pairs on 3-folds is conjectured to have an affirmative answer, see [48, Conjecture 0.3]. In the case of toric geometries, Shen is able to prove the rationality of the cobordism series via the rationality results for the descendent theory of stable pairs [43, 44].

1.10. **Vafa-Witten theory.** There has been a series of recent papers studying the virtual Euler characteristics of the moduli spaces of stable bundles (and stable Higgs pairs) on surfaces [12, 13, 14, 15, 23, 51, 52]. The outcome has been a clear mathematical proposal for the theory studied earlier by Vafa and Witten [55].

Definition 19 here is motivated by the Vafa-Witten developments. The Quot scheme geometry, with the associated obstruction theory, provides a straightforward approach to sheaf counting on surfaces. The idea is that given a stable bundle B of rank N on an algebraic surface X, we can pick N sections (assuming B is sufficiently positive) which will generically generate B:

$$0 \to \mathbb{C}^N \otimes \mathcal{O}_X \to B \to F \to 0\,, \tag{13}$$

where F is supported in dimension 1. By dualizing (13), we obtain a quotient sequence

$$\left[0 \to B^\vee \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0\right] \;\in\; \mathsf{Quot}_X(\mathbb{C}^N, \chi(Q), c_1(B))\,.$$

Of course, $\chi(Q)$ can be computed from the Chern classes of B and X.

The calculations that we have presented, which may be viewed as the beginning of the study of the virtual Euler characteristics of Quot schemes of surfaces, already show some features of Vafa-Witten theory: the appearance of the Kawai-Yoshioka formula (in

[8]See [27] for a foundational treatment of algebraic cobordism and [28] for applications to enumerative geometry.

the $K3$ case) and the appearance of the Seiberg-Witten invariants (in the general type case). A difference is the rationality in the variable q for the Quot scheme theory versus modularity in the variable

$$q = \exp(2\pi i\tau)$$

for Vafa-Witten theory. A basic open question is the following.

Question 26. *Formulate the precise relationship of the Quot scheme theory of surfaces for all N to Vafa-Witten theory and Seiberg-Witten theory.*

Moduli spaces of bundles on *curves* with sections have been considered by many authors, see [3, 53]. Moreover, the relationship between the intersection theory of Quot schemes and the moduli space of stable bundles on curves has been successfully studied in [30, 31].

1.11. **Higher rank quotients.** Let X be a nonsingular projective surface, and consider the Quot scheme $\mathsf{Quot}_X(\mathbb{C}^N, n, D, r)$ of quotients with dimension 2 support,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0, \quad \chi(Q) = n\,,\ c_1(Q) = D\,,\ \mathrm{rank}(Q) = r > 0\,.$$

The existence of a virtual fundamental class of $\mathsf{Quot}_X(\mathbb{C}^N, n, D, r)$ for del Pezzo surfaces was first noted in [47], but the study can be pursued more generally.

As in the cases of support of dimension 0 and 1, the higher obstructions of the standard deformation theory of $\mathsf{Quot}_X(\mathbb{C}^N, n, D, r)$ lie in $\mathrm{Ext}^2(S, Q)$. We have

$$\mathrm{Ext}^2(S, Q) = \mathrm{Ext}^0(Q, S \otimes K_X)^\vee \quad \textit{and} \quad \mathrm{Ext}^0(Q, S \otimes K_X) \hookrightarrow \mathrm{Ext}^0(Q, \mathbb{C}^N \otimes K_X)\,.$$

Hence, if $\mathrm{Ext}^2(S, Q) \neq 0$, then $\mathrm{Ext}^0(Q, \mathbb{C}^N \otimes K_X) \neq 0$. Since Q is generated by global sections, we conclude that $H^0(X, K_X) \neq 0$.

By the above logic, we obtain the following condition: *if X satisfies*

$$H^0(X, K_X) = 0\,, \tag{14}$$

then the standard deformation theory of $\mathsf{Quot}_X(\mathbb{C}^N, n, D, r)$ is 2-term and yields a virtual fundamental class of dimension $\chi(S, Q)$.

There are many surfaces which satisfy $H^0(X, K_X) = 0$ including rational surfaces, ruled surfaces, Enriques surfaces, and even some surfaces of general type. The Quot scheme virtual Euler characteristic theory for such surfaces is well defined for all r, D, and n. We leave the investigation for higher r to a future paper.

1.12. **Plan of the paper.** We start by computing Segre integrals over the symmetric product

$$C^{[n]} = \mathsf{Quot}_C(\mathbb{C}^1, n)$$

in Section 2. In particular, Theorem 3 is proven in Section 2.3. Theorem 11 about the Fuss-Catalan numbers is obtained via Wick expansion in Section 2.5. Segre integrals over Quot schemes of curves for higher N are studied in Section 3 where the proofs of Theorem 6 and the first part of Theorem 8 are presented.

We then consider Quot schemes of surfaces. Section 4 concerns the case of quotients with dimension 0 support. The second part of Theorem 8 as well as Theorems 14 and 18 are proven there by reducing surface integrals to curve integrals. Section 5 concerns the case of dimension 1 support. The proofs of Theorems 21 and 23 are presented in Sections 5.3.2 and 5.4 respectively.

1.13. **Acknowledgements.** Many of the ideas developed here are related to collaborations with Alina Marian in [31, 32, 33, 34, 35, 36]. We are grateful to her for numerous discussions over the years related to Quot schemes of curves and surfaces.

Our study of the virtual Euler characteristics of the Quot scheme of surfaces was motivated in part by the Euler characteristic calculations of L. Göttsche and M. Kool [14, 15] for the moduli spaces of rank 2 and 3 stable sheaves on surfaces. We thank I. Gessel, B. Rhoades, and J. Verstraete for helpful conversations about the Catalan numbers. Discussions about several related topics with A. Okounkov and R. Thomas have been valuable.

D. O. was supported by the NSF through grant DMS 1802228. R.P. was supported by the Swiss National Science Foundation and the European Research Council through grants SNF-200020-182181, ERC-2017-AdG-786580-MACI, SwissMAP, and the Einstein Stiftung. We thank the Shanghai Center for Mathematical Science at Fudan University for a very productive visit in September 2018 at the start of the project. R.P. also thanks the Hebrew University of Jerusalem where part of the research was pursued during a visit in October 2018.

The project has received funding from the European Research Council (ERC) under the European Union Horizon 2020 Research and Innovation Program (grant No. 786580).

2. Symmetric products of curves

2.1. **Overview.** We first present the proof of Theorem 3. Theorem 11 will be obtained in Section 2.5 by specializing Theorem 3 to genus 1. In fact, *all* other main results of the paper (Theorems 6, 8, 14, 18 and 23) proven in later Sections, rely either directly upon Theorem 3 or upon the analysis of the integrals over $C^{[n]}$ developed here.

2.2. **Projective line.** To begin the proof of Theorem 3, we observe that the factorization

$$\sum_{n=0}^{\infty} q^n \int_{C^{[n]}} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_\ell}(\alpha_\ell^{[n]}) = \mathsf{A}_1^{c_1(\alpha_1)} \cdots \mathsf{A}_\ell^{c_1(\alpha_\ell)} \cdot \mathsf{B}^{1-g}\,, \tag{15}$$

allows us to specialize the calculation to genus 0 where

$$C \simeq \mathbb{P}^1 \quad \text{and} \quad C^{[n]} \simeq \mathbb{P}^n .$$

We write h for the hyperplane class on $\mathbb{P}^n$.

Lemma 27. *For a K-theory class α on $\mathbb{P}^1$ of rank r and degree $d = c_1(\alpha)$ we have*

$$s_x(\alpha^{[n]}) = (1 - xh)^{d-nr+r} .$$

Proof. Both expressions are multiplicative in short exact sequences of vector bundles

$$0 \to V_1 \to V \to V_2 \to 0 .$$

The claim is clear for the right hand side. For the left hand side, claim is a consequence of the induced sequence

$$0 \to V_1^{[n]} \to V^{[n]} \to V_2^{[n]} \to 0 \implies s_x(V^{[n]}) = s_x(V_1^{[n]}) \cdot s_x(V_2^{[n]}) .$$

Since the K-theory of $\mathbb{P}^1$ is generated by line bundles, we can restrict to $\alpha = \mathcal{O}_{\mathbb{P}^1}(d)$. By the proof of Theorem 2 in [36], we have

$$\mathrm{ch}\,((\mathcal{O}_{\mathbb{P}^1}(d))^{[n]}) = (d+1) - (d-n+1)\exp(-h) ,$$

which then gives

$$s_x((\mathcal{O}_{\mathbb{P}^1}(d))^{[n]}) = (1 - xh)^{d-n+1} ,$$

completing the argument. ◻

2.3. **Proof of Theorem 3 (using $\mathbb{P}^1$).** Let $\alpha_1, \ldots, \alpha_\ell$ be K-theory classes of ranks r_i and degree d_i. Using the Lemma 27, we obtain

$$\begin{aligned}
\mathsf{Z}_{\mathbb{P}^1,1}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) &= \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_\ell}(\alpha_\ell^{[n]}) \\
&= \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} (1 - x_1 h)^{d_1 - nr_1 + r_1} \cdots (1 - x_\ell h)^{d_\ell - nr_\ell + r_\ell} \\
&= \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} f(h)^n \cdot g(h) \\
&= \sum_{n=0}^{\infty} q^n \cdot ([t^n]\, f(t)^n \cdot g(t)) .
\end{aligned}$$

In the third equality,

$$\begin{aligned}
f(t) &= (1 - x_1 t)^{-r_1} \cdots (1 - x_\ell t)^{-r_\ell} , \\
g(t) &= (1 - x_1 t)^{d_1 + r_1} \cdots (1 - x_\ell t)^{d_\ell + r_\ell} .
\end{aligned}$$

The brackets denote the coefficient of the suitable power of t.

We can evaluate such expressions using the Lagrange-Bürmann formula [59]. Assuming $f(0) \neq 0$, for the change of variables $q = \frac{t}{f(t)}$, the following general identity holds

$$\sum_{n=0}^{\infty} \left([t^n]\, f(t)^n \cdot g(t)\right) \cdot q^n = \frac{g(t)}{f(t)} \cdot \frac{dt}{dq}\,. \tag{16}$$

We will use the above identity repeatedly.

In our case, the change of variables takes the form

$$q = t(1-x_1 t)^{r_1} \cdots (1-x_r t)^{r_\ell}\,,$$

and the Segre series becomes

$$\mathsf{Z}_{\mathbb{P}^1,1}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) = \prod_{i=1}^{\ell}(1-x_i t)^{d_i} \cdot \prod_{i=1}^{\ell}(1-x_i t)^{2r_i} \cdot \frac{dt}{dq}\,.$$

Combined with the factorization (15),

$$\mathsf{Z}_{\mathbb{P}^1,1}(q, x_1, \ldots, x_\ell | \alpha_1, \ldots, \alpha_\ell) = \mathsf{A}_1^{d_1} \cdots \mathsf{A}_\ell^{d_\ell} \cdot \mathsf{B}\,,$$

the above calculation yields

$$\mathsf{A}_i(q) = 1 - x_i t\,, \qquad \mathsf{B}(q) = \prod_{i=1}^{\ell}(1-x_i t)^{2r_i} \cdot \frac{dt}{dq} = \left(\frac{q}{t}\right)^2 \cdot \frac{dt}{dq}.$$

This completes the proof of Theorem 3. $\square$

For future use, we also record a formula for the *logarithms* of the functions A_i. Of course, we may take $i = 1$ without loss of generality.

Lemma 28. *We have*

$$\log \mathsf{A}_1 = \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{n} \cdot \mathsf{a}_n$$

where

$$\mathsf{a}_n(x_1, \ldots, x_\ell) = x_1 \cdot \sum_{p_1 + \ldots + p_\ell = n-1} \binom{-nr_1 - 1}{p_1} \cdot \binom{-nr_2}{p_2} \cdots \binom{-nr_\ell}{p_\ell} \cdot x_1^{p_1} \cdots x_\ell^{p_\ell}\,. \tag{17}$$

Proof. The argument consists in another application of the Lagrange-Bürmann formula (16). Indeed, write $\widetilde{\mathsf{a}}_n$ for the right hand side of (17) and let

$$\mathsf{L}(q) = \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{n} \cdot \widetilde{\mathsf{a}}_n\,.$$

We must prove $\log \mathsf{A}_1 = \mathsf{L}$. Clearly,

$$\begin{aligned} \widetilde{\mathsf{a}}_n(x_1, \ldots, x_\ell) &= (-1)^{n-1} x_1 \cdot \left(\left[t^{n-1}\right] (1-x_1 t)^{-nr_1 - 1} \cdot (1-x_2 t)^{-nr_2} \cdots (1-x_\ell t)^{-nr_\ell}\right) \\ &= (-1)^{n-1} x_1 \cdot \left(\left[t^{n-1}\right] f(t)^{n-1} \cdot h(t)\right). \end{aligned}$$

where we write as before

$$f(t) = (1 - x_1 t)^{-r_1} \cdots (1 - x_\ell t)^{-r_\ell}$$

$$h(t) = (1 - x_1 t)^{-r_1 - 1} \cdot (1 - x_2 t)^{-r_2} \cdots (1 - x_\ell t)^{-r_\ell} .$$

We further compute

$$\begin{aligned} \frac{d\mathsf{L}}{dq} &= \sum_{n=1}^{\infty} (-1)^n q^{n-1} \cdot \widetilde{\mathsf{a}}_n \\ &= -x_1 \cdot \sum_{n=1}^{\infty} q^{n-1} \cdot \left(\left[t^{n-1} \right] f(t)^{n-1} \cdot h(t) \right) \\ &= -x_1 \cdot \frac{h(t)}{f(t)} \cdot \frac{dt}{dq} \\ &= -\frac{x_1}{1 - x_1 t} \cdot \frac{dt}{dq} , \end{aligned}$$

where the Lagrange-Bürmann formula (16) was applied in the third equality, for the same change of variables $q = \frac{t}{f(t)}$ which we used previously. Therefore

$$d\mathsf{L} = -\frac{x_1}{1 - x_1 t} dt \quad \Longrightarrow \quad \mathsf{L} = \log(1 - x_1 t).$$

Combined with Theorem 3, we obtain $\mathsf{L} = \log \mathsf{A}_1$. $\square$

2.4. **Wick's formalism for an elliptic curve.** Let C be a nonsingular genus 1 curve. Let $L_1, \ldots, L_\ell$ be line bundles on C of degrees $d_1, \ldots, d_\ell$. We lift the integrals over the symmetric product to the n-fold ordinary product via the morphism

$$p_n : C^{\times n} = C \times \cdots \times C \to C^{[n]} .$$

We write D_{ij} for the diagonals

$$D_{ij} = \{ x_i = x_j \} \subset C^{\times n}$$

and further set

$$\Delta_i = D_{1,i} + D_{2,i} + \ldots + D_{i-1,i} .$$

We also write $\pi_i : C^{\times n} \to C$ for the canonical projections, $1 \leq i \leq n$. From the exact sequence

$$0 \to \pi_n^\star L(-\Delta_n) \to p_n^\star L^{[n]} \to (\pi_1 \times \cdots \times \pi_{n-1})^\star p_{n-1}^\star L^{[n-1]} \to 0 ,$$

we inductively obtain

$$p_n^\star s_x(L^{[n]}) = \prod_{i=1}^{n} \frac{1}{1 + x(\pi_i^\star c_1(L) - \Delta_i)} .$$

Consequently,

$$\int_{C^{[n]}} s_{x_1}(L_1^{[n]}) \cdots s_{x_\ell}(L_\ell^{[n]}) = \frac{1}{n!} \int_{C^{\times n}} \prod_{j=1}^{\ell} \prod_{i=1}^{n} \frac{1}{1 + x_j(\pi_i^\star c_1(L_j) - \Delta_i)}\,.$$

By Lemma 28, we know

$$\log \left(\sum_{n=0}^{\infty} \frac{q^n}{n!} \int_{C^{\times n}} \prod_{j=1}^{\ell} \prod_{i=1}^{n} \frac{1}{1 + x_j(\pi_i^\star c_1(L_j) - \Delta_i)} \right) = \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{n} \cdot \left(\sum_{j=1}^{\ell} \mathsf{a}_n^{(j)} \cdot \deg L_j \right), \tag{18}$$

where

$$\mathsf{a}_n^{(1)}(x_1, \ldots, x_\ell) = x_1 \cdot \sum_{p_1 + \ldots + p_\ell = n-1} \binom{-n-1}{p_1} \cdot \binom{-n}{p_2} \cdots \binom{-n}{p_\ell} \cdot x_1^{p_1} \cdots x_\ell^{p_\ell}$$

and $\mathsf{a}_n^{(j)}(x_1, \ldots, x_\ell)$ is given by the correspondingly permuted formula.

We will expand the left hand side of (18) using Wick's formalism. To connect with Theorem 11, write

$$w_n(p_1, \ldots, p_\ell) = \sum_T \mathsf{wt}(T)$$

weighted count of ordered ℓ-colored trees of type $(p_1, \ldots, p_\ell)$ with

$$n = p_1 + \ldots + p_\ell + 1\,.$$

Theorem 11 is equivalent to the following claim:

$$w_n(p_1, \ldots, p_\ell) = \frac{(-1)^{n-1}}{n} \binom{-n}{p_1} \cdots \binom{-n}{p_\ell}. \tag{19}$$

To establish (19), we set

$$\mathsf{W}_n = \sum_{p_1 + \ldots + p_{n-1} = n-1} w_n(p_1, \ldots, p_\ell) \cdot x_1^{p_1} \cdots x_\ell^{p_\ell}\,.$$

Define the differential operator

$$\mathsf{D}_1 = 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \ldots + x_\ell \frac{\partial}{\partial x_\ell} + \mathbf{1}$$

and define $\mathsf{D}_2, \ldots, \mathsf{D}_\ell$ by the correspondingly permuted formulas.

Lemma 29. *The following identity holds*

$$\log \left(\sum_{n=0}^{\infty} \frac{q^n}{n!} \int_{C^{\times n}} \prod_{j=1}^{\ell} \prod_{i=1}^{n} \frac{1}{1 + x_j(\pi_i^\star c_1(L_j) - \Delta_i)} \right) = -\sum_{n=1}^{\infty} \frac{q^n}{n} \cdot \left(\sum_{j=1}^{\ell} x_j \cdot \deg L_j \cdot \mathsf{D}_j \mathsf{W}_n \right).$$

Proof. We refer the reader to Section 1.3 of [42] for a gentle introduction to the Wick formalism in precisely the context which we require here. By Wick, the logarithm on the left hand side is given by

$$\sum_{n=1}^{\infty} \frac{q^n}{n!} S[n]\,, \tag{20}$$

where $S[n]$ is the *connected contribution* on n vertices. We will match the connected contributions $S[n]$ with the right hand side of Lemma 29.

Consider an arbitrary monomial in the diagonal classes. Such a monomial determines a graph with n vertices, whose edges are given by the diagonal associations. Since C is an elliptic curve, the squares of diagonals vanish

$$D_{i,j}^2 = 0 \ \in H^*(C^{\times n}, \mathbb{Z})\,.$$

Hence, a connected graph on n vertices cannot have any cycles, thus it corresponds exactly to a tree with $n-1$ edges determined by the diagonals. The diagonals come from the expansions of the terms

$$\prod_{j=1}^{\ell}\prod_{i=1}^{n} \frac{1}{1 + x_j(\pi_i^\star c_1(L_j) - \Delta_i)}\,,$$

and therefore may be considered as carrying colors between $1, \ldots, \ell$ depending on the j index.

Let us first analyze the (simpler) connected contribution for the terms

$$\prod_{j=1}^{\ell}\prod_{i=1}^{n} \frac{1}{1 + x_j(-\Delta_i)}\,. \tag{21}$$

We see that the coefficient of $x_1^{p_1} \cdots x_\ell^{p_\ell}$ in the connected contribution with

$$n = p_1 + \ldots + p_\ell + 1$$

vertices is exactly a sum over *labelled ℓ-colored trees of type* $(p_1, \ldots, p_\ell)$. The vertices of the trees T are labelled by the n ordered factors of $C^{\times n}$. To calculate the weight, we must expand (21) as

$$\prod_{j=1}^{\ell}\prod_{i=1}^{n} (1 + x_j\Delta_i + x_j^2\Delta_i^2 + x_j^3\Delta_i^3 + \ldots)\,. \tag{22}$$

If the i^{th} vertex v of T has d_v^j downward edges colored j, the weight receives a factor of $d_v^j!$ since the coefficient of the monomial in the corresponding diagonal in

$$x_j^{d_v^j} \Delta_i^{d_v^j}$$

is exactly $d_v^j!$. Hence, the full weight is

$$\prod_{v \text{ vertex}} d_v^1! \cdots d_v^k! = \mathsf{wt}(T) \cdot (n-1)! \ . \tag{23}$$

The actual connected contribution $S[n]$ of (20), which we must calculate, also includes the insertions of $\pi_i^\star(c_1(L_j))$. Since the diagonal edges already cut $C^{\times n}$ to just a single elliptic curve C, exactly one insertion from the set

$$\left\{ \pi_i^\star(c_1(L_j)) \right\}_{1\le i\le n,\ 1\le j\ \le \ell}$$

must be chosen. We separate the contribution

$$S[n] = \sum_{j=1}^{\ell} S[n,j]$$

by which L_j is chosen as an insertion. The connected contribution $S[n,j]$ will be matched with

$$-x_j \cdot \deg L_j \cdot \mathsf{D}_j \mathsf{W}_n \cdot (n-1)!$$

to complete the proof.

To this end, we calculate the effect of the insertion L_j on the weight of a labelled ℓ-colored tree of type $(p_1, \dots, p_\ell)$ generated by the diagonals. The insertion L_j can occur at any vertex $1 \le i \le n$. When $\pi_i^\star(c_1(L_j))$ is selected, the weight receives the factor

$$-\deg L_j \cdot (d_v^j + 1)!$$

since the coefficient of the corresponding monomial in

$$x_j^{d_v^j+1}\big(-\pi_i^\star(c_1(L_j)) + \Delta_i\big)^{d_v^j+1}$$

is exactly $(d_v^j+1)!$. Since the insertion L_j can be placed at any vertex $1 \le i \le n$, we must modify the weight (23) of T by the prefactor

$$\sum_{i=1}^{n} (d_v^j + 1) = p_j + n = 2p_j + \sum_{j' \ne j} p_{j'} + 1\,.$$

This prefactor is achieved precisely by the action of the differential operator D_j on W_n.

Collecting all terms, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{q^n}{n!} S[n] &= \sum_{n=1}^{\infty}\sum_{j=1}^{\ell} \frac{q^n}{n!} S[n,j] \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{\ell} \left(-\frac{q^n}{n} x_j \cdot \deg L_j \cdot \mathsf{D}_j \mathsf{W}_n \right)\end{aligned}$$

which completes the calculation. □

2.5. **Proof of Theorem 11 (using an elliptic curve).** We prove Theorem 11 here geometrically by specializing Theorem 3 and Lemma 28 to genus 1 and using the Wick result of Lemma 29. Alternatively, a direct combinatorial proof of Theorem 11 is provided in the Appendix.

By setting the right hand side of (18) equal to the right hand side of the formula of Lemma 29, we obtain

$$x_1 \mathsf{D}_1 \mathsf{W}_n = (-1)^{n-1} \mathsf{a}_n^{(1)} . \tag{24}$$

The operator D_1 acts on the monomial $x_1^{p_1} \cdots x_\ell^{p_\ell}$ as multiplication by $(n + p_1)$. By matching coefficients of $x_1^{p_1} \cdots x_\ell^{p_\ell}$ on both sides of (24), we solve

$$w_n(p_1, \ldots, p_\ell) = \frac{(-1)^{n-1}}{n} \binom{-n}{p_1} \cdots \binom{-n}{p_\ell} ,$$

which completes the argument. □

3. Quot schemes of curves for higher N

3.1. **Overview.** We prove here Theorem 6, part of Theorem 8, and the associated Corollaries 9 and 10.

We begin with Theorem 6. We specialize directly to the case of an elliptic curve C, seeking to show that

$$\sum_{n=0}^{\infty} q^n \int_{\mathsf{Quot}_C(\mathbb{C}^N, n)} s(V^{[n]}) = \mathsf{A}(q)^{\deg V} ,$$

with the specified formula for $\mathsf{A}(q) = \mathsf{A}_{1,r,N}(q)$.

3.2. **Equivariant localization.** The nonsingular projective variety $\mathsf{Quot}_C(\mathbb{C}^N, n)$ carries a natural action of the algebraic torus $\mathbb{C}^\star$ defined as follows. Let $\mathbb{C}^\star$ act diagonally on $\mathbb{C}^N$ with weights

$$w_1 < w_2 < \ldots < w_N .$$

The $\mathbb{C}^\star$-action on $\mathsf{Quot}_C(\mathbb{C}^N, n)$ is then induced via the associated $\mathbb{C}^\star$-action on the middle term of the exact sequence

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \to Q \to 0 .$$

We will prove Theorem 6 by applying the Atiyah-Bott $\mathbb{C}^\star$-equivariant localization formula to compute integrals over $\mathsf{Quot}_C(\mathbb{C}^N, n)$. The fixed loci are indexed by partitions $n_1 + \ldots + n_N = n$ where

$$\mathsf{F}[n_1, \ldots, n_N] = C^{[n_1]} \times \ldots \times C^{[n_N]}$$

parameterizes tuples $(Z_1, \ldots, Z_N)$ of divisors on C with

$$\mathrm{length}(Z_i) = n_i .$$

The inclusion

$$j : \mathsf{F}[n_1, \ldots, n_N] \hookrightarrow \mathsf{Quot}_C(\mathbb{C}^N, n)$$

corresponds to the invariant sequences

$$0 \to S = \bigoplus_{i=1}^{N} \mathcal{O}_C(-Z_i) \hookrightarrow \bigoplus_{i=1}^{N} \mathcal{O}_C \to Q = \bigoplus_{i=1}^{N} \mathcal{O}_{Z_i} \to 0\,.$$

The normal bundle to the fixed locus is found from the moving part of the tangent bundle:

$$\begin{aligned}
\mathsf{N}[n_1, \ldots, n_N] &= \mathrm{Hom}(S, Q)^{\mathrm{mov}} = \bigoplus_{i \neq j} \mathrm{Hom}(\mathcal{O}(-Z_i), \mathcal{O}_{Z_j})[w_j - w_i] \\
&= \bigoplus_{i \neq j} H^\bullet(\mathcal{O}(Z_i)|_{Z_j})[w_j - w_i] \\
&= \bigoplus_{i \neq j} \left(H^\bullet(\mathcal{O}(Z_i)) - H^\bullet(\mathcal{O}(Z_i - Z_j))\right)[w_j - w_i]
\end{aligned}$$

with the brackets denoting the equivariant weights. We combine the mixed (i, j) and (j, i) terms by setting

$$\mathbb{V}_{ij} = H^\bullet(\mathcal{O}(Z_i - Z_j))[w_j - w_i] \oplus H^\bullet(\mathcal{O}(Z_j - Z_i))[w_i - w_j]\,.$$

Since C is an elliptic curve, Serre duality yields the $\mathbb{C}^\star$-equivariant isomorphism

$$\mathbb{V}_{ij} \simeq \mathbb{V}_{ij}^\vee[-1]\,.$$

Therefore,

$$\mathsf{e}_{\mathbb{C}^\star}(\mathbb{V}_{ij}) = (-1)^{\chi(\mathcal{O}(Z_i - Z_j))} = (-1)^{n_i + n_j}\,.$$

For the remaining terms, we use the K-theoretic relation

$$H^\bullet(\mathcal{O}(Z_i)) = H^{1-\bullet}(\mathcal{O}(-Z_i))^\vee = -H^\bullet(\mathcal{O})^\vee + H^0(\mathcal{O}_{Z_i})^\vee$$

obtained from the exact sequence

$$0 \to \mathcal{O}(-Z_i) \to \mathcal{O} \to \mathcal{O}_{Z_i} \to 0\,.$$

While the first summand is trivial, the second summand corresponds to the bundle $\left(\mathcal{O}^{[n_i]}\right)^\vee$. We conclude

$$\begin{aligned}
\mathsf{e}_{\mathbb{C}^\star}(\mathsf{N}[n_1, \ldots, n_N]) &= \prod_{i<j} (-1)^{n_i + n_j} \prod_{i \neq j} \mathsf{e}\left((\mathcal{O}^{[n_i]})^\vee[w_j - w_i]\right) \\
&= \prod_{i \neq j} \mathsf{e}_{\mathbb{C}^\star}(\mathcal{O}^{[n_i]}[w_i - w_j]).
\end{aligned}$$

Furthermore, the restriction of $V^{[n]}$ to $\mathsf{F}[n_1, \ldots, n_N]$ splits equivariantly as

$$\iota^\star V^{[n]} = V^{[n_1]}[w_1] \oplus \ldots \oplus V^{[n_N]}[w_N]\,.$$

Atiyah-Bott localization then yields

$$\begin{aligned} \mathsf{Z}_{C,N}(q \,|\, V) &= \sum_{n=0}^{\infty} q^n \int_{\mathsf{Quot}_C(\mathbb{C}^N,n)} s(V^{[n]}) \\ &= \sum_{n_1+\ldots+n_N=n} q^n \int_{C^{[n_1]}\times\ldots C^{[n_N]}} \frac{\prod_i s(V^{[n_i]}[w_i])}{\prod_i \prod_{j\neq i} e_{\mathbb{C}^\star}(\mathcal{O}^{[n_i]}[w_i-w_j])} \,. \end{aligned}$$

An important aspect of the above formula is that, in the genus 1 case, the integral on the right hand side splits over the individual factors. For any tuple of equivariant weights $(a, b_1, \ldots, b_{N-1})$, we write

$$\mathsf{P}_C(q \,|\, a \,|\, b_1, \ldots, b_{N-1}) = \sum_{n=0}^{\infty} q^n \int_{C^{[n]}} \frac{s(V^{[n]}[a])}{e_{\mathbb{C}^\star}(\mathcal{O}^{[n]}[b_1]) \cdots e_{\mathbb{C}^\star}(\mathcal{O}^{[n]}[b_{N-1}])} \,.$$

We can write the splitting explicitly as

$$\begin{aligned} (25)\quad & \mathsf{Z}_{C,N}(q \,|\, V) = \\ & \mathsf{P}_C(q \,|\, w_1 \,|\, w_1 - w_2, \ldots, w_1 - w_N) \cdots \mathsf{P}_C(q \,|\, w_N \,|\, w_N - w_1, \ldots, w_N - w_{N-1}) \,. \end{aligned}$$

In fact, equation (25) holds equivariantly. To prove Theorem 6, we must take the non-equivariant limit: we must extract the free term with respect to the variables $w_1, \ldots, w_N$ on the right hand side.

3.3. **Symmetric products.** Our next step is to evaluate the expressions

$$\mathsf{P}_C(q \,|\, a \,|\, b_1, \ldots, b_{N-1})$$

by relating them to the integrals of Theorem 3. For convenience of notation, we write

$$\widehat{s}_t(V) = t^{-\operatorname{rank} V} \cdot s_{1/t}(V) = \prod_i \frac{1}{t+v_i} \,,$$

where the v_i are the roots of a vector bundle V on a scheme S.

Write $\mathsf{R} = H^\star_{\mathbb{C}^\star}(\mathrm{pt})$ for the equivariant coefficient ring. For $\alpha, \beta_1, \ldots, \beta_{N-1} \in \mathsf{R}$, we introduce the function

$$\mathsf{Q}_C(q \,|\, \alpha \,|\, \beta_1, \ldots, \beta_{N-1}) = \sum_{n=0}^{\infty} q^n \int_{C^{[n]}} \widehat{s}_\alpha(V^{[n]}) \cdot \widehat{s}_{\beta_1}(\mathcal{O}^{[n]}) \cdots \widehat{s}_{\beta_{N-1}}(\mathcal{O}^{[n]}) \,.$$

Note that

$$\mathsf{Q}_C(q \,|\, \alpha \,|\, \beta_1, \ldots, \beta_{N-1}) \in \mathsf{K}[[q]]$$

where K denotes the fraction field of R. The calculations below will take place in the power series ring $\mathsf{K}[[q]]$.

For a scheme S endowed with a trivial torus action, and a vector bundle $V \to S$ with nontrivial equivariant weight t, we have

$$\widehat{s}_t(V) = \mathsf{e}_{\mathbb{C}^\star}(V[t])^{-1} \in H^\star(S) \otimes \mathsf{K} \,.$$

Applied to our setting, we obtain

$$\mathsf{P}_C(q\,|\,a\,|\,b_1,\ldots,b_{N-1}) = \mathsf{Q}_C(q\,|\,1+a\,|\,b_1,\ldots,b_{N-1})\,. \tag{26}$$

The next result computes the logarithm of Q_C.

Lemma 30. *For an elliptic curve C, we have*

$$\mathsf{Q}_C(q\,|\,\alpha\,|\,\beta_1,\ldots,\beta_{N-1}) = \mathsf{F}(q\,|\,\alpha\,|\,\beta_1,\ldots,\beta_{N-1})^{\deg V}\,,$$

where we define

$$\log \mathsf{F}(q\,|\,\alpha\,|\,\beta_1,\ldots,\beta_{N-1}) = \sum_{n=1}^{\infty}(-1)^n\frac{q^n}{n}\cdot \mathsf{f}_n(\alpha\,|\,\beta_1,\ldots,\beta_{N-1})\,,$$

with

$$\mathsf{f}_n(\alpha\,|\,\beta_1,\ldots,\beta_{N-1}) = \sum_{p+q_1+\ldots+q_{N-1}=n-1}\binom{-nr-1}{p}\binom{-n}{q_1}\cdots\binom{-n}{q_{N-1}} \cdot \alpha^{-nr-p-1}\beta_1^{-n-q_1}\cdots\beta_{N-1}^{-n-q_{N-1}}\,. \tag{27}$$

Proof. Using the definitions, we compute

$$\begin{aligned}
\mathsf{Q}_C\quad & (q\,|\,\alpha\,|\,\beta_1,\ldots,\beta_{N-1}) = \sum_{n=0}^{\infty} q^n \int_{C^{[n]}} \widehat{s}_\alpha(V^{[n]})\cdot \widehat{s}_{\beta_1}(\mathcal{O}^{[n]})\cdots \widehat{s}_{\beta_{N-1}}(\mathcal{O}^{[n]})\\
= \quad & \sum_{n=0}^{\infty}(q\alpha^{-r}\beta_1^{-1}\cdots\beta_{N-1}^{-1})^n\int_{C^{[n]}} s_{\frac{1}{\alpha}}(V^{[n]})s_{\frac{1}{\beta_1}}(\mathcal{O}^{[n]})\cdots s_{\frac{1}{\beta_{N-1}}}(\mathcal{O}^{[n]})\\
= \quad & \mathsf{Z}_C(\widehat{q},\alpha^{-1},\beta_1^{-1},\ldots,\beta_{N-1}^{-1}\,|\,V,\mathcal{O},\ldots,\mathcal{O})\,.
\end{aligned}$$

Here we set

$$\widehat{q} = q\alpha^{-r}\beta_1^{-1}\cdots\beta_{N-1}^{-1}\,,$$

and we remind the reader that the Segre series Z_C was introduced in Definition 1. Since most of the bundles appearing are trivial, only one universal function appears in the answer:

$$\mathsf{Z}_C(\widehat{q},\alpha^{-r},\beta_1^{-1},\ldots,\beta_{N-1}^{-1}\,|\,V,\mathcal{O},\ldots,\mathcal{O}) = \mathsf{F}^{\deg V}\,.$$

The proof is completed by invoking Lemma 28 which gives an expression for $\log \mathsf{F}$ matching the one claimed here. $\square$

3.4. **Proof of Theorem 6.** By equation (25), equation (26), and Lemma 30, we obtain

$$\mathsf{Z}_{C,N}(q\,|\,V) = \mathsf{A}(q)^{\deg V}\,,$$

where $\log \mathsf{A}(q)$ equals

$$\mathsf{F}(q\,|\,1+w_1\,|\,w_1-w_2,\ldots,w_1-w_N)+\ldots+\mathsf{F}(q\,|\,1+w_N\,|\,w_N-w_1,\ldots,w_N-w_{N-1})=$$
$$\sum_{n=1}^{\infty}\frac{(-q)^n}{n}\left[\mathsf{f}_n(1+w_1\,|\,w_1-w_2,\ldots,w_1-w_N)+\ldots+\mathsf{f}_n(1+w_N\,|\,w_N-w_1,\ldots,w_N-w_{N-1})\right].$$

Our goal is to prove

$$\log \mathsf{A}(q)=\sum_{n=1}^{\infty}(-1)^{(N+1)n+1}\binom{(r+N)n-1}{Nn-1}\cdot\frac{q^n}{n}.$$

Equivalently, we will show that the free term, with respect to the variables $w_1,\ldots,w_N$, in the expression[9]

$$\mathsf{f}_n(1+w_1\,|\,w_1-w_2,\ldots,w_1-w_N)+\ldots+\mathsf{f}_n(1+w_N\,|\,w_N-w_1,\ldots,w_N-w_{N-1})$$

equals

$$(-1)^{Nn+1}\binom{(r+N)n-1}{Nn-1}.$$

To establish the last claim, we will use the expression for f_n provided by equation (27). Each monomial in the formula contributes the following sum to the final answer

$$(1+w_1)^{-nr-p-1}(w_1-w_2)^{-n-q_1}\cdots(w_1-w_N)^{-n-q_{N-1}}+\ldots$$
$$+(1+w_N)^{-nr-p-1}(w_N-w_1)^{-n-q_1}\cdots(w_N-w_{N-1})^{-n-q_{N-1}}.$$

By Lemma 31 below, the free term of the sum equals

$$(-1)^{(n+q_1)+\ldots+(n+q_{N-1})}\cdot\binom{(nr+p)+(n+q_1)+\ldots+(n+q_{N-1})}{nr+p}$$
$$=(-1)^{n(N-1)+q}\cdot\binom{(N+r)n-1}{nr+p},$$

where

$$q=q_1+\ldots+q_{N-1}\implies p+q=n-1.$$

Therefore, the free term we seek is

$$\sum_{p+q_1+\ldots+q_{N-1}=n-1}(-1)^{n(N-1)+q}\binom{-nr-1}{p}\binom{-n}{q_1}\cdots\binom{-n}{q_{N-1}}\cdot\binom{(N+r)n-1}{nr+p}.$$

By the Vandermonde identity the middle binomials can be summed:

$$\sum_{p+q=n-1}(-1)^{n(N-1)+q}\binom{-nr-1}{p}\binom{-n(N-1)}{q}\binom{(N+r)n-1}{nr+p}.$$

After substituting $p=n-1-q$ and rearranging the factorials, we obtain

$$(-1)^{nN+1}\cdot\frac{(nN-1)!}{(n-1)!(n(N-1)-1)!}\binom{(r+N)n-1}{Nn-1}\cdot\sum_{q=0}^{n-1}\binom{n-1}{q}\cdot\frac{(-1)^q}{n(N-1)+q}.$$

[9]We use here that taking the free term can be done before or after taking the logarithm.

Lemma 32 in case $x = n(N-1)$ evaluates the final sum as

$$(-1)^{nN+1}\binom{(r+N)n-1}{Nn-1},$$

which completes the proof of Theorem 6 . $\square$

Lemma 31. *Let $x_1, \ldots, x_N$ be fixed positive integers. Set*

$\mathsf{S}(w_1, \ldots, w_N) = (1+w_1)^{-x_1} \cdot (w_1 - w_2)^{-x_2} \cdots (w_1 - w_N)^{-x_N} +$ *all symmetric combinations.*

Expand $\mathsf{S}(w_1, \ldots, w_N)$ in the region

$$w_1 << w_2 << \ldots << w_N .$$

The free term of this expansion equals

$$(-1)^{x_2+\ldots+x_N}\binom{x_1+\ldots+x_N-1}{x_1-1}.$$

Proof. We have

$$(1+w_N)^{-x_1} \cdot (w_N - w_1)^{-x_2} \cdots (w_N - w_{N-1})^{-x_N} = \\ w_N^{-x_2-\ldots-x_N} \cdot (1+w_N)^{-x_1} \cdot \left(1 - \frac{w_1}{w_N}\right)^{-x_2} \cdots \left(1 - \frac{w_{N-1}}{w_N}\right)^{-x_N} .$$

To extract the free term, we need the coefficient of $w_1^0 \cdots w_{N-1}^0 \cdot w_N^{x_2+\ldots+x_N}$ in

$$(1+w_N)^{-x_1} \cdot \left(1 - \frac{w_1}{w_N}\right)^{-x_2} \cdots \left(1 - \frac{w_{N-1}}{w_N}\right)^{-x_N} .$$

This coefficient equals

$$\binom{-x_1}{x_2+\ldots+x_N} = (-1)^{x_2+\ldots+x_N} \cdot \binom{x_1+\ldots+x_N-1}{x_1-1}.$$

An entirely parallel computation shows that the remaining terms

$$(1+w_j)^{-x_1} \cdot (w_j - w_1)^{-x_2} \cdots (w_j - w_N)^{-x_N} ,$$

for $j \neq N$, do *not* contribute. $\square$

Lemma 32. *For positive integers x and n, we have*

$$\sum_{q=0}^{n-1} \frac{(-1)^q}{x+q}\binom{n-1}{q} = \frac{(x-1)!(n-1)!}{(x+n-1)!} .$$

Proof. We induct on n. For the inductive step, we compute

$$\begin{aligned}\sum_{q=0}^{n}\frac{(-1)^q}{x+q}\binom{n}{q} &= \sum_{q=0}^{n}\frac{(-1)^q}{x+q}\binom{n-1}{q-1}+\sum_{q=0}^{n}\frac{(-1)^q}{x+q}\binom{n-1}{q}\\ &= -\frac{x!(n-1)!}{(x+n)!}+\frac{(x-1)!(n-1)!}{(x+n-1)!}\\ &= \frac{(x-1)!n!}{(x+n)!}.\end{aligned}$$

The first line is Pascal's identity, while the second line uses the induction hypothesis. ☐

3.5. **Binomial identities.** We prove here part of Theorem 8 stated in Section 1.4 together with Corollaries 9 and 10.

Proof of first half of Theorem 8. The first statement in Theorem 8 is purely combinatorial. In case rank $V=1$, the expression of Theorem 6 simplifies:

$$\begin{aligned}\log \mathsf{A}_{1,1,N}(q) &= \sum_{n=1}^{\infty}(-1)^{(N+1)n+1}\binom{(N+1)n-1}{Nn-1}\cdot\frac{q^n}{n}\\ &= N\cdot\sum_{n=1}^{\infty}(-1)^{Nn}\binom{-Nn-1}{n-1}\frac{q^n}{n}.\end{aligned}$$

The result can be rewritten in the form

$$\mathsf{A}_{1,1,N}(q)=(1+t)^N \quad \text{for} \quad q=(-1)^N t(1+t)^N$$

using Lemma 33 (ii) below.

The main point of Theorem 8, however, is the formula

$$\mathsf{B}_{1,N}(q)=\frac{(1+t)^{N+1}}{1+(N+1)t} \qquad \text{for} \ \ q=(-1)^N t(1+t)^N \tag{28}$$

proven in Section 4.4 below. The calculation uses a specialization to genus 0 and is similar in spirit to the computations carried out in Section 4. ☐

Proof of Corollary 9. Similarly,

$$\begin{aligned}\mathsf{B}_{1,N}(q) &= \frac{(1+t)^{N+1}}{1+t(N+1)}\\ &= \sum_{n=0}^{\infty}(-1)^{nN}\cdot\binom{-Nn+N}{n}\cdot q^n\\ &= \sum_{n=0}^{\infty}(-1)^{n(N+1)}\cdot\binom{(n-1)(N+1)}{n}\cdot q^n\end{aligned}$$

where we have used Lemma 33(i) with $d=0$ on the second line. ☐

Lemma 33. *For the change of variables $q=t(1+t)^r$, we have*

(i)

$$\sum_{n=0}^{\infty} \binom{d-rn+r}{n} \cdot q^n = \frac{(1+t)^{d+r+1}}{1+t(r+1)}\,,$$

(ii)

$$\log(1+t) = \sum_{n=1}^{\infty} \binom{-rn-1}{n-1} \cdot \frac{q^n}{n}\,.$$

Proof. Part (i) is the content of [36, Lemma 3]. For part (ii), the identity to be established is

$$\sum_{n=1}^{\infty} \frac{1}{n} \binom{-rn-1}{n-1} \cdot t^n (1+t)^{rn} = \log(1+t)\,. \tag{29}$$

For the proof, we set $d = -2r-1$ in equation (i)

$$\sum_{n=0}^{\infty} \binom{-r(n+1)-1}{n} \cdot t^n (1+t)^{rn} = \frac{(1+t)^{-r}}{1+t(r+1)}\,,$$

which we rewrite as

$$\sum_{n=1}^{\infty} \binom{-rn-1}{n-1} \cdot t^{n-1}(1+t)^{rn-1} \cdot (1+t(r+1)) = \frac{1}{1+t}\,.$$

The identity (29) is obtained by integration. □

Proof of Corollary 10. The first statement in Corollary 10 follows by directly comparing Theorem 8 and equation (6). Indeed, up to signs, the two universal functions A and B agree for both sides. For the second statement, we observe

$$\int_{(\mathbb{P}^1)^{[n]}} s(L^{[n]})^N = \int_{\mathbb{P}^n} (1-h)^{N(\deg L - n+1)} = (-1)^n \binom{N(\deg L - n + 1)}{n},$$

where Lemma 27 has been used in the first identity. □

4. Virtual invariants of surfaces: dimension 0 quotients

4.1. **Overview.** We prove here Theorems 8, 14, and 18. In particular, we study the virtual intersection theory of the Quot scheme $\mathsf{Quot}_X(\mathbb{C}^N, n)$ of short exact sequences

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0, \quad \chi(Q) = n\,,\ c_1(Q) = 0\,,\ \mathrm{rank}(Q) = 0$$

on nonsingular projective surfaces X. As noted in Section 1.6, $\mathsf{Quot}_X(\mathbb{C}^N, n)$ carries a virtual fundamental class

$$\left[\mathsf{Quot}_X(\mathbb{C}^N, n)\right]^{\mathrm{vir}}$$

of dimension Nn. Our basic technique is to relate integrals against the virtual class of Quot schemes of surfaces to integrals over Quot schemes of curves which we have already studied. Theorem 14 is the first outcome.

The idea of dimensional reduction plays a central role in the proof of Theorem 18. In the $N = 1$ case, the integrals over the Quot schemes of curves which arise are covered by Theorem 3. For higher N, a more delicate analysis of the curve integrals is required. A similar analysis is used to complete the proof of Theorem 8 in Section 4.4 (and appears also in the proof of Theorem 23 for surfaces of general type in Section 5.4).

4.2. **Virtual integrals.**

4.2.1. *Strategy.* We first prove Theorem 14. The argument requires the following two steps:

(i) We show a universality statement allowing us to reduce to the case of a surface with nonsingular canonical curve $C \subset X$.
(ii) The claim will then be obtained by direct comparison of the obstruction theories of the Quot schemes of X and of C.

4.2.2. *Universality.* We will use equivariant localization to compute the series

$$\mathsf{Z}_{X,N}(q, x_1, \ldots, x_\ell | \alpha_1, \ldots, \alpha_\ell) = \sum_{n=0}^{\infty} q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N, n)]^{\mathrm{vir}}} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_\ell}(\alpha_\ell^{[n]}) \,.$$

The Quot scheme $\mathsf{Quot}_X(\mathbb{C}^N, n)$ carries torus action via the diagonal $\mathbb{C}^\star$-action on the middle term of the sequence

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0 \,.$$

We write $w_1, \ldots, w_N$ for the equivariant weights. Just as in Section 3.2, the fixed loci are products of Hilbert schemes

$$\mathsf{F}[n_1, \ldots, n_N] = X^{[n_1]} \times \cdots \times X^{[n_N]}$$

indexed by partitions $n_1 + \ldots + n_N = n$. We write

$$S = \oplus_{i=1}^N I_{Z_i} \,, \quad Q = \oplus_{i=1}^N \mathcal{O}_{Z_i} \,, \quad \text{length } (Z_i) = n_i$$

for the fixed kernel and quotient. Furthermore, the induced obstruction theory of $\mathsf{F}[n_1, \ldots, n_N]$ splits:

$$\mathrm{Ext}^\bullet(S, Q)^{\mathrm{fix}} = \oplus_{i=1}^N \mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_i}) \,.$$

In fact, the $\mathbb{C}^\star$-fixed obstruction sheaf is locally free with obstruction bundle

$$\left(K_X^{[n_1]} \oplus \ldots \oplus K_X^{[n_N]}\right)^\vee \,. \tag{30}$$

This is a consequence of equation (31) below. The equivariant virtual normal bundle is the moving part of the tangent-obstruction theory

$$\mathsf{N}[n_1, \ldots, n_N]^{\mathrm{vir}} = \mathrm{Ext}^\bullet(S, Q)^{\mathrm{mov}} = \bigoplus_{i \neq j} \mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_j})[w_j - w_i] \,.$$

Using the virtual localization theorem of [17], the integral

$$\int_{[\mathsf{Quot}_X(\mathbb{C}^N,n)]^{\mathrm{vir}}} s_{x_1}(\alpha_1^{[n]})\cdots s_{x_\ell}(\alpha_\ell^{[n]})$$

can be rewritten as

$$\sum_{n_1+\ldots+n_N=n}\int_{X^{[n_1]}\times\cdots\times X^{[n_N]}}\prod_{i=1}^{N}\mathsf{e}\left((K_X^{[n_i]})^\vee\right)\cdot\prod_{i=1}^{\ell}s_{x_i}(\alpha_i^{[n_i]}[w_i])\cdot\prod_{i\neq j}\mathsf{e}(\mathrm{Ext}^\bullet(I_{Z_i},\mathcal{O}_{Z_j})[w_j-w_i])^{-1}.$$

As in [16, Theorem 5.1], we regard the above expression as a tautological integral over the Hilbert scheme of the disconnected surface $Y = X\sqcup X\sqcup\ldots\sqcup X$, so that

$$Y^{[n]} = \bigsqcup_{n_1+\ldots+n_N=n} X^{[n_1]}\times\cdots\times X^{[n_N]}.$$

The answer depends solely on the Chern numbers of the data involved: monomials in the Chern classes of α_i and Chern classes of the surface X. In the absence of better notation, we write m_k for these monomials enumerated in some order. Thus

$$\mathsf{Z}_{X,N}(q,x_1,\ldots,x_\ell|\alpha_1,\ldots,\alpha_\ell) = \text{universal function of } \mathsf{m}_k.$$

Splitting the surface $X = X'\sqcup X''$ and the classes $\alpha_i = \alpha_i'\sqcup\alpha_i''$ one sees that

$$\mathrm{Quot}_X(\mathbb{C}^N,n) = \bigsqcup_{n'+n''=n}\mathrm{Quot}_{X'}(\mathbb{C}^N,n')\times\mathrm{Quot}_{X''}(\mathbb{C}^N,n''),$$

and the tangent-obstruction theory and the tautological elements $\alpha_i^{[n]}$ split as well. We then conclude the multiplicative form of the generating series

$$\mathsf{Z}_{X,N}(q,x_1,\ldots,x_\ell|\alpha_1,\ldots,\alpha_\ell) = \prod \mathsf{A}_k^{\mathsf{m}_k}.$$

As usual, A_k are universal functions in the variables $q, x_1,\ldots,x_\ell$ that may depend on the ranks of the α's and N.

To complete the proof of Theorem 14, we may assume X admits a nonsingular canonical curve

$$C\subset X\,,$$

since such surfaces X separate all the monomials m_k.

4.2.3. *Quot schemes of curves and surfaces.* For all nonsingular curves $C\subset X$, there is a natural embedding

$$\iota : \mathsf{Quot}_C(\mathbb{C}^N,n)\hookrightarrow\mathsf{Quot}_X(\mathbb{C}^N,n)\,,\quad \left[\mathbb{C}^N\otimes\mathcal{O}_C\to Q\right]\mapsto\left[\mathbb{C}^N\otimes\mathcal{O}_X\to Q\right]\,.$$

In the case of canonical curves, the following result relates the obstruction theories of the Quot schemes above and plays a crucial role in the proof of Theorem 14.

Lemma 34. *If C is a nonsingular canonical curve, we have*

$$\iota_\star \left[\mathsf{Quot}_C(\mathbb{C}^N, n)\right] = (-1)^n \left[\mathsf{Quot}_X(\mathbb{C}^N, n)\right]^{\mathrm{vir}}$$

in the localized $\mathbb{C}^\star$-equivariant Chow theory of $\mathsf{Quot}_X(\mathbb{C}^N, n)$.

Proof. We first consider the case $N = 1$. The Hilbert scheme $X^{[n]}$ has locally free obstruction sheaf $\left(K_X^{[n]}\right)^\vee$. The obstruction sheaf is obtained from the following sequence of canonical isomorphisms:

$$\begin{aligned} \mathrm{Ext}^1(I_Z, \mathcal{O}_Z) &= \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \\ &= \mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X)^\vee \\ &= \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_Z \otimes K_X)^\vee \\ &= \left(K_X^{[n]}\right)^\vee \Big|_Z . \end{aligned} \tag{31}$$

The defining equation s of the canonical curve $C \subset X$ yields a section $s^{[n]}$ of $K_X^{[n]}$ via the assignment

$$Z \mapsto s|_Z \in H^0(K_X \otimes \mathcal{O}_Z)\,.$$

The section $s^{[n]}$ vanishes precisely along

$$\iota : C^{[n]} \hookrightarrow X^{[n]}\,. \tag{32}$$

Therefore,

$$\left[X^{[n]}\right]^{\mathrm{vir}} = \mathsf{e}\left((K_X^{[n]})^\vee\right) \cap X^{[n]} = (-1)^n\, \iota_\star \left[C^{[n]}\right]\,, \tag{33}$$

which completes the proof of Lemma 34 in case $N = 1$.

Now let N be arbitrary. We apply $\mathbb{C}^\star$-equivariant localization to both Quot schemes over X and C using the same weights for the two torus actions. The fixed loci are

$$\mathsf{F}_C[n_1, \ldots, n_N] = C^{[n_1]} \times \cdots \times C^{[n_N]}\,, \quad \mathsf{F}_X[n_1, \ldots, n_N] = X^{[n_1]} \times \cdots \times X^{[n_N]}$$

respectively. Parallel to (32), there is a natural embedding

$$\iota : \mathsf{F}_C[n_1, \ldots, n_N] \hookrightarrow \mathsf{F}_X[n_1, \ldots, n_N].$$

We noted in (30) that the obstruction bundle of $\mathsf{F}_X[n_1, \ldots, n_N]$ splits as

$$\left((K_X)^{[n_1]} \oplus \ldots \oplus (K_X)^{[n_N]}\right)^\vee.$$

Using (33), we find that

$$\begin{aligned} \iota_\star \left[\mathsf{F}_C[n_1, \ldots, n_N]\right] &= (-1)^n \mathsf{e}\left(\left((K_X)^{[n_1]} \oplus \ldots \oplus (K_X)^{[n_N]}\right)^\vee\right) \cap \left[\mathsf{F}_X[n_1, \ldots, n_N]\right] \\ &= (-1)^n \left[\mathsf{F}_X[n_1, \ldots, n_N]\right]^{\mathrm{vir}}. \end{aligned} \tag{34}$$

We furthermore claim

$$\iota^\star \mathsf{e}(\mathsf{N}_X[n_1,\ldots,n_N]^{\mathrm{vir}}) = \mathsf{e}(\mathsf{N}_C[n_1,\ldots,n_N]) \tag{35}$$

where $\mathsf{N}_X^{\mathrm{vir}}$ and N_C are two normal bundles of the fixed loci.

The proof of (35) requires several steps. First, the difference

$$\iota^\star \mathsf{N}_X[n_1,\ldots,n_N]^{\mathrm{vir}} - \mathsf{N}_C[n_1,\ldots,n_N]$$

equals

$$\bigoplus_{i\neq j} \mathrm{Ext}^\bullet_X(I_{Z_i/X},\mathcal{O}_{Z_j})[w_j-w_i] - \bigoplus_{i\neq j} \mathrm{Ext}^\bullet_C(I_{Z_i/C},\mathcal{O}_{Z_j})[w_j-w_i]\,.$$

The latter expression can be further simplified using

$$\begin{aligned}\mathrm{Ext}^\bullet_X(I_{Z_i/X},\mathcal{O}_{Z_j}) - \mathrm{Ext}^\bullet_C(I_{Z_i/C},\mathcal{O}_{Z_j}) &= -\mathrm{Ext}^\bullet_X(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}) + \mathrm{Ext}^\bullet_C(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}) \\ &= -\mathrm{Ext}^\bullet_C(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}\otimes\Theta)[-1]\,,\end{aligned}$$

where $\Theta = \mathcal{O}_C(C)$ is the theta characteristic of C. For the first equality, we have expressed the ideal sheaves in terms of structure sheaves in K-theory. The second equality follows from the exact sequence

$$\ldots \to \mathrm{Ext}^i_C(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}) \to \mathrm{Ext}^i_X(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}) \to \mathrm{Ext}^{i-1}_C(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}\otimes\Theta) \to \ldots$$

proven, for instance, in [54, Lemma 3.42]. Next, in the difference of the normal bundles, we group the terms corresponding to the pairs (i,j) and (j,i). We define

$$\mathbb{V}_{ij} = \mathrm{Ext}^\bullet_C(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}\otimes\Theta)[w_j-w_i] \oplus \mathrm{Ext}^\bullet_C(\mathcal{O}_{Z_j},\mathcal{O}_{Z_i}\otimes\Theta)[w_i-w_j]\,,$$

and write

$$\iota^\star \mathsf{N}_X[n_1,\ldots,n_N]^{\mathrm{vir}} - \mathsf{N}_C[n_1,\ldots,n_N] = \bigoplus_{i<j} \mathbb{V}_{ij}\,.$$

By Serre duality, making use of the fact that Θ is a theta characteristic, we obtain

$$\mathbb{V}_{ij}^\vee = \mathbb{V}_{ij}[1]\,.$$

Therefore,

$$\mathsf{e}_{\mathbb{C}^\star}(\mathbb{V}_{ij}) = (-1)^{\chi(\mathcal{O}_{Z_i},\mathcal{O}_{Z_j}\otimes\Theta)} = 1\,,$$

which proves (35).

Finally, by the virtual localization formula [17], we have

$$\left[\mathsf{Quot}_X(\mathbb{C}^N,n)\right]^{\mathrm{vir}} = \sum_{n_1+\ldots+n_N=n} (j_X)_\star\left(\frac{1}{\mathsf{e}(\mathsf{N}_X[n_1,\ldots,n_N]^{\mathrm{vir}})} \cap [\mathsf{F}_X[n_1,\ldots,n_N]]^{\mathrm{vir}}\right)$$

$$\left[\mathsf{Quot}_C(\mathbb{C}^N,n)\right] = \sum_{n_1+\ldots+n_N=n} (j_C)_\star\left(\frac{1}{\mathsf{e}(\mathsf{N}_C[n_1,\ldots,n_N])} \cap [\mathsf{F}_C[n_1,\ldots,n_N]]\right).$$

Using equations (34) and (35) we obtain

$$\iota_\star\left[\mathsf{Quot}_C(\mathbb{C}^N,n)\right] = (-1)^n \left[\mathsf{Quot}_X(\mathbb{C}^N,n)\right]^{\mathrm{vir}}\,,$$

which proves the Lemma. □

Remark 35. The result of Lemma 34 should be expected. In fact, the canonical curve C gives a cosection

$$\mathrm{Ob} \to \mathcal{O}_{\mathsf{Quot}}$$

of the obstruction sheaf of $\mathsf{Quot}_X(\mathbb{C}^N, n)$ via the composition

$$\mathrm{Ext}^1(S, Q) \to \mathrm{Ext}^2(Q, Q) \stackrel{\text{Trace}}{\to} H^2(\mathcal{O}_X) = H^0(K_X)^\vee \to \mathbb{C}\,.$$

A careful analysis shows that the cosection vanishes along the quotients supported on C. By [20], the virtual fundamental cycle is localized along such quotients. However, the precise determination of the cycle still requires a calculation. The known techniques require stronger smoothness assumptions than what we can prove in our case, so we have given a different argument for the proof of Lemma 34.

For example, $\mathsf{Quot}_X(\mathbb{C}^N, n)$ is singular for every $N \geq 2$ and $n \geq 2$ even at quotients of the form

$$Q = \mathcal{O}_Z \oplus \mathcal{O}_Z\,, \quad \mathrm{length}(Z) = \frac{n}{2}\,.$$

Indeed, the Zariski tangent space

$$\mathrm{Hom}(S, Q) = \mathrm{Hom}(I_Z \,\oplus\, I_Z \,\oplus\, \mathbb{C}^{N-2} \,\otimes\, \mathcal{O}_X, \mathcal{O}_Z \,\oplus\, \mathcal{O}_Z)$$

has dimension $(N+2)n$ which is higher than the actual dimension $(N+1)n$.

4.2.4. *Proof of Theorem 14.* We argued in Section 4.2.2 that it suffices to consider the case when X admits a nonsingular canonical curve C. Let α_i be classes on X and set $\beta_i = \alpha_i|_C$. By Lemma 34, we have

$$\int_{[\mathsf{Quot}_X(\mathbb{C}^N,n)]^{\mathrm{vir}}} s_{x_1}(\alpha_1^{[n]}) \cdots s_{x_\ell}(\alpha_\ell^{[n]}) = (-1)^n \int_{\mathsf{Quot}_C(\mathbb{C}^N,n)} s_{x_1}(\beta_1^{[n]}) \cdots s_{x_\ell}(\beta_\ell^{[n]})\,.$$

Theorem 14 follows immediately

$$\begin{aligned} \mathsf{Z}_{X,N}(q, x_1, \ldots, x_\ell \,|\, \alpha_1, \ldots, \alpha_\ell) &= \mathsf{Z}_{g,N}(-q, x_1, \ldots, x_\ell, \beta_1, \ldots, \beta_\ell) \\ &= \mathsf{A}_1(-q)^{c_1(\alpha_i)\cdot K_X} \cdots \mathsf{A}_\ell(-q)^{c_1(\alpha_\ell)\cdot K_X} \cdot \mathsf{B}(-q)^{1-g}\,. \end{aligned}$$

□

4.3. **Virtual Euler characteristics.** Theorem 18 will be proven next. Before presenting the argument, we review general statements regarding virtual Euler characteristics.

4.3.1. *Generalities.* Let Z be a scheme admitting a 2-term perfect obstruction theory

$$\mathbb{E}^\bullet = [E_{-1} \to E_0] \to \tau^{[-1,0]}\mathbb{L}_Z,$$

and a virtual fundamental class $[Z]^{\text{vir}}$ of dimension

$$d = \text{rank } E_0 - \text{rank } E_{-1}\,.$$

The virtual tangent bundle $T^{\text{vir}}Z$ is defined in the K-theory of Z as the difference

$$(E_0)^\vee - (E_{-1})^\vee\,.$$

We define the virtual Euler characteristic

$$\mathsf{e}^{\text{vir}}(Z) = \int_{[Z]^{\text{vir}}} c_d(T^{\text{vir}}Z)\,, \tag{36}$$

see also [10]. Virtual Euler characteristics are deformation invariants.

In particular, if Z is nonsingular with a locally free obstruction bundle B, then

$$[Z]^{\text{vir}} = \mathsf{e}(B) \cap [Z]$$

and the virtual tangent bundle is the difference $TZ - B$. By definition, we obtain

$$\mathsf{e}^{\text{vir}}(Z) = \int_Z \mathsf{e}(B) \cdot \frac{c(TZ)}{c(B)}\,. \tag{37}$$

4.3.2. *Proof of Theorem 18 for $N = 1$.* We must prove

$$\sum_{n=0}^{\infty} q^n \cdot \mathsf{e}^{\text{vir}}\left(X^{[n]}\right) = \left(\frac{(1-q)^2}{1-2q}\right)^{K_X^2}\,. \tag{38}$$

Proof. We observed in Lemma 34 that the Hilbert schemes $X^{[n]}$ have locally free obstruction sheaves $\left(K_X^{[n]}\right)^\vee$. By (37), the virtual Euler characteristics are

$$\mathsf{e}^{\text{vir}}(X^{[n]}) = \int_{X^{[n]}} \mathsf{e}\left((K_X^{[n]})^\vee\right) \cdot \frac{c(TX^{[n]})}{c\left((K_X^{[n]})^\vee\right)}\,.$$

The above rewriting of the virtual Euler characteristic shows, via [8, Theorem 4.5], that expression (38) takes the universal form

$$\mathsf{U}(q)^{K_X^2} \cdot \mathsf{V}(q)^{c_2(X)}\,.$$

To prove

$$\mathsf{U}(q) = (1-q)^2 \cdot (1-2q)^{-1}\,, \quad \mathsf{V}(q) = 1\,,$$

we may specialize to surfaces X which admit a nonsingular canonical curve

$$C \subset X\,.$$

By (32), we have the embedding

$$\iota : C^{[n]} \hookrightarrow X^{[n]}$$

and furthermore, by (33), we have

$$\left[X^{[n]}\right]^{\mathrm{vir}} = \mathsf{e}\left((K_X^{[n]})^\vee\right) \cap X^{[n]} = (-1)^n \,\iota_\star \left[C^{[n]}\right].$$

We conclude

$$\mathsf{e}^{\mathrm{vir}}(X^{[n]}) = (-1)^n \int_{C^{[n]}} \iota^\star \frac{c(TX^{[n]})}{c\left((K_X^{[n]})^\vee\right)}.$$

Going further, let $\Theta = \mathcal{O}_C(C)$ be the theta characteristic of C. If $Z \subset C$, consider the exact sequence

$$0 \to \mathcal{O}_X(-C) \to I_{Z/X} \to \iota_\star I_{Z/C} \to 0.$$

Taking $\mathrm{Hom}(, \mathcal{O}_Z)$ we find

$$0 \to TC^{[n]} \to \iota^\star TX^{[n]} \to \Theta^{[n]} \to 0 \implies \iota^\star c(TX^{[n]}) = c(\Theta^{[n]}) \cdot c(TC^{[n]}).$$

Moreover, we have

$$\iota^\star K_X^{[n]} = \Theta^{[n]}.$$

We conclude

$$\mathsf{e}^{\mathrm{vir}}(X^{[n]}) = (-1)^n \int_{C^{[n]}} \frac{c(\Theta^{[n]}) \cdot c(TC^{[n]})}{c\left((\Theta^{[n]})^\vee\right)}. \tag{39}$$

There are now several ways to evaluate the integral (39), but the most direct path is to use Theorem 3. We observe

$$TC^{[n]} = \left(K_C^{[n]}\right)^\vee.$$

Then, we have

$$\begin{aligned}\mathsf{e}^{\mathrm{vir}}(X^{[n]}) &= (-1)^n \int_{C^{[n]}} \frac{c(\Theta^{[n]}) \cdot c\left((K_C^{[n]})^\vee\right)}{c\left((\Theta^{[n]})^\vee\right)} \\ &= (-1)^n \int_{C^{[n]}} s_1((-\Theta)^{[n]}) \cdot s_{-1}((-K_C)^{[n]}) \cdot s_{-1}(\Theta^{[n]}).\end{aligned}$$

Invoking Theorem 3, we find

$$\begin{aligned}\sum_{n=0}^\infty q^n \cdot \mathsf{e}^{\mathrm{vir}}\left(X^{[n]}\right) &= \mathsf{Z}_{C,1}(-q, x_1 = 1, x_2 = -1, x_3 = -1 \,|\, \alpha_1 = -\Theta, \alpha_2 = -K_C, \alpha_3 = \Theta) \\ &= \mathsf{A}_1^{\deg \alpha_1} \cdot \mathsf{A}_2^{\deg \alpha_2} \cdot \mathsf{A}_3^{\deg \alpha_3} \cdot \mathsf{B}^{1-g}.\end{aligned}$$

The change of variables specified by Theorem 3 takes the simple form

$$-q = \frac{t}{1-t},$$

and the universal functions are

$$\mathsf{A}_1 = 1 - t = (1-q)^{-1}, \quad \mathsf{A}_2 = \mathsf{A}_3 = 1 + t = (1-2q)(1-q)^{-1}, \quad \mathsf{B} = 1.$$

We conclude

$$\sum_{n=0}^{\infty} q^n \cdot \mathsf{e}^{\mathrm{vir}}\left(X^{[n]}\right) = \left((1-q)^2 \cdot (1-2q)^{-1}\right)^{K_X^2},$$

which completes the proof of the $N=1$ case of Theorem 18. $\square$

Remark 36. Using the same techniques, we can also compute the virtual χ_{-y} genera:

$$\sum_{n=0}^{\infty} q^n \cdot \chi_{-y}^{\mathrm{vir}}(X^{[n]}) = \left(\frac{(1-q)\cdot(1-yq)}{1-q-qy}\right)^{K_X^2}.$$

Theorem 18 is then recovered in the limit $y \to 1$.

Remark 37. For future reference, we record the following slight generalization of the above calculations. For any nonsingular projective surface X and $M \to X$ a line bundle, set

$$\mathsf{Z}_{X,M} = \sum_{n=0}^{\infty} q^n \int_{X^{[n]}} \mathsf{e}\left(\left(M^{[n]}\right)^{\vee}\right) \frac{c(TX^{[n]})}{c\left(\left(M^{[n]}\right)^{\vee}\right)}.$$

Without the duals placed on tautological bundles, such integrals also appear in the work [21] on stable pair invariants of local surfaces. The above calculations yield the following result.

Corollary 38. *We have*

$$\mathsf{Z}_{X,M} = \mathsf{U}(q)^{c_1(M)^2} \cdot \mathsf{V}(q)^{c_1(M)\cdot K_X} \tag{40}$$

where

$$\mathsf{U}(q) = 1-q\,, \quad \mathsf{V}(q) = (1-2q)^{-1}\cdot(1-q)\,.$$

4.3.3. *Proof of Theorem 18 for higher N.* Theorem 18 concerns the generating series

$$\mathsf{Z}^{\mathcal{E}}_{X,N,0} = \sum_{n=0}^{\infty} q^n \mathsf{e}^{\mathrm{vir}}(\mathrm{Quot}_X(\mathbb{C}^N, n))\,. \tag{41}$$

For notational convenience, we will denote the series (41) by $\mathsf{E}_X(q)$. We will follow a strategy similar to that of the proof of Theorem 14:

(i) We will first show the factorization

$$\mathsf{E}_X(q) = \mathsf{A}(q)^{K_X^2} \cdot \mathsf{B}(q)^{\chi(\mathcal{O}_X)}$$

holds for universal power series $\mathsf{A}, \mathsf{B} \in \mathbb{Q}[[q]]$.

(ii) To identify the series A, B, we will use Theorem 14 to localize the calculation to a nonsingular canonical curve

$$C \subset X\,.$$

(iii) The evaluation $\mathsf{B} = 1$ will follow for formal reasons.

(iv) To determine A, we will use equivariant localization on $\mathsf{Quot}_C(\mathbb{C}^N, n)$ for $C = \mathbb{P}^1$. We will find closed form expressions for the localization sums which will furthermore prove the rationality of Theorem 18.

Remark 39. We warn the reader that both the statement and the proof of the torus equivariant localization formula for virtual Euler characteristics stated in [10, Corollary 6.6 (3)] are wrong. In particular, application of [10, Corollary 6.6 (3)] to the diagonal $\mathbb{C}^\star$-action on $\mathbb{C}^N$ to calculate $\mathsf{Z}^{\mathcal{E}}_{X,N,0}$ in terms of $\mathsf{Z}^{\mathcal{E}}_{X,1,0}$ will give incorrect results.[10]

Step (i). We first apply the virtual localization formula to prove that the series $\mathsf{E}_X(q)$ depends only upon K_X^2 and $\chi(\mathcal{O}_X)$. By definition,

$$\mathsf{e}^{\mathrm{vir}}(\mathrm{Quot}_X(\mathbb{C}^N, n)) = \int_{[\mathrm{Quot}_X(\mathbb{C}^N,n)]^{\mathrm{vir}}} c(T^{\mathrm{vir}}\mathrm{Quot}_X(\mathbb{C}^N, n))$$

where

$$T^{\mathrm{vir}}\mathrm{Quot}_X(\mathbb{C}^N, n) = \mathrm{Ext}^0(S, Q) - \mathrm{Ext}^1(S, Q)$$

is the virtual tangent bundle. By the virtual localization formula of [17], we obtain

$$\mathsf{e}^{\mathrm{vir}}(\mathrm{Quot}_X(\mathbb{C}^N, n)) = \sum_{n_1+\ldots+n_N=n} \int_{\left[X^{[n_1]}\times\cdots\times X^{[n_N]}\right]^{\mathrm{vir}}} \frac{\iota^\star c(T^{\mathrm{vir}}\mathrm{Quot}_X(\mathbb{C}^N, n))}{\mathsf{e}_{\mathbb{C}^\star}(\mathsf{N}[n_1, \ldots, n_N]^{\mathrm{vir}})}\,.$$

Using

$$\iota^\star T^{\mathrm{vir}}\mathrm{Quot}_X(\mathbb{C}^N, n) = \bigoplus_{i,j} \mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_j})[w_j - w_i]$$

and

$$\mathsf{N}[n_1, \ldots, n_N]^{\mathrm{vir}} = \bigoplus_{i\neq j} \mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_j})[w_j - w_i]\,,$$

we rewrite the right hand side of the virtual localization as

$$\sum_{n_1+\ldots+n_N=n} \int_{X^{[n_1]}\times\cdots\times X^{[n_N]}} \prod_{i=1}^{N} \mathsf{e}\left((K_X^{[n_i]})^\vee\right) \cdot c(\mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_i})) \cdot \prod_{i\neq j} \frac{c(\mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_j})[w_j - w_i])}{\mathsf{e}(\mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_j})[w_j - w_i])}\,.$$

As in [16, Theorem 5.1], each Hilbert scheme integral depends solely on the Chern numbers of the surface X, so $\mathsf{E}_X(q)$ is a function of

$$K_X^2 \text{ and } \chi(\mathcal{O}_X)\,.$$

By splitting the surface $X = X' \sqcup X''$, we see

$$\mathrm{Quot}_X(\mathbb{C}^N, n) = \bigsqcup_{n'+n''=n} \mathrm{Quot}_{X'}(\mathbb{C}^N, n') \times \mathrm{Quot}_{X''}(\mathbb{C}^N, n'')$$

[10] B. Fantechi and L. Göttsche agree with Remark 39 about the error in part (3), but confirm that parts (1) and (2) of [10, Corollary 6.6] are correct.

with a splitting also of the obstruction theory. We therefore conclude

$$\mathsf{E}_X(q) = \mathsf{E}_{X'}(q) \cdot \mathsf{E}_{X''}(q)\,,$$

which implies the factorization

$$\mathsf{E}_X(q) = \mathsf{A}(q)^{K_X^2} \cdot \mathsf{B}(q)^{\chi(\mathcal{O}_X)}\,.$$

Step (ii). When $C \subset X$ is a nonsingular canonical curve, we can apply the result of Lemma 34 to write

$$\iota_\star \left[\mathsf{Quot}_C(\mathbb{C}^N, n)\right] = (-1)^n \left[\mathsf{Quot}_X(\mathbb{C}^N, n)\right]^{\mathrm{vir}}.$$

Here

$$\iota : \mathsf{Quot}_C(\mathbb{C}^N, n) \to \mathsf{Quot}_X(\mathbb{C}^N, n)$$

is the natural inclusion

$$\left[\mathbb{C}^N \otimes \mathcal{O}_C \to Q\right] \mapsto \left[\mathbb{C}^N \otimes \mathcal{O}_X \to Q\right].$$

As a consequence, we obtain

$$\begin{aligned} \mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, n)) &= \int_{[\mathrm{Quot}_X(\mathbb{C}^N,n)]^{\mathrm{vir}}} c(T^{\mathrm{vir}}\mathrm{Quot}_X(\mathbb{C}^N, n)) \\ &= (-1)^n \int_{\mathrm{Quot}_C(\mathbb{C}^N,n)} c(\iota^\star T^{\mathrm{vir}}\mathrm{Quot}_X(\mathbb{C}^N, n)) \\ &= (-1)^n \int_{\mathrm{Quot}_C(\mathbb{C}^N,n)} c(T\mathrm{Quot}_C(\mathbb{C}^N, n)) \cdot c(\mathcal{T}_n). \end{aligned}$$

Here, $\mathcal{T}_n \to \mathrm{Quot}_C(\mathbb{C}^N, n)$ is the virtual bundle given pointwise by

$$\mathcal{T}_n = \mathrm{Ext}^\bullet_C(Q, Q \otimes \Theta)\,,$$

where $\Theta = N_{C/X}$ is the associated theta characteristic. The last line follows from the K-theoretic decomposition

$$\iota^\star T^{\mathrm{vir}}\mathrm{Quot}_X(\mathbb{C}^N, n) = T\mathrm{Quot}_C(\mathbb{C}^N, n) + \mathcal{T}_n\,. \tag{42}$$

To prove (42), let S_C denote the kernel of the surjection

$$\mathbb{C}^N \otimes \mathcal{O}_C \to Q \to 0$$

on the curve C, and let S denote the kernel of the similar surjection

$$\mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0$$

on the surface X. The splitting (42) is a consequence of the following computation:

$$\begin{aligned} \mathrm{Ext}^\bullet_X(S, Q) - \mathrm{Ext}^\bullet_C(S_C, Q) &= -\mathrm{Ext}^\bullet_X(Q, Q) + \mathrm{Ext}^\bullet_C(Q, Q) \\ &= -\mathrm{Ext}^\bullet_C(Q, Q \otimes \Theta)[-1]\,. \end{aligned}$$

For the first equality, we have expressed S, S_C in terms of Q in the K-theory of X and C. The second equality follows from the exact sequence

$$\ldots \to \mathrm{Ext}^i_C(Q,Q) \to \mathrm{Ext}^i_X(Q,Q) \to \mathrm{Ext}^{i-1}_C(Q, Q\otimes\Theta) \to \ldots$$

provided by [54, Lemma 3.42].

Step (iii). By (ii), we are now left to evaluating the generating series

$$\mathsf{E}_C(q) = \sum q^n (-1)^n \cdot \int_{\mathrm{Quot}_C(\mathbb{C}^N, n)} c(T\mathrm{Quot}_C(\mathbb{C}^N, n)) \cdot c(\mathcal{T}_n).$$

By the argument in part (ii), the answer takes the form

$$\mathsf{E}_C(q) = \mathsf{A}(q)^{1-g}$$

with g the genus of C. The second series $\mathsf{B}(q) = 1$ since there is no $\chi(\mathcal{O}_X)$-dependence in the curve integral above.

Step (iv). To determine the series A, we specialize first to the $N = 2$ case. We prove

$$\mathsf{A}(q) = \frac{(1-4q)^2}{(1-q)^2 \cdot (1-6q+q^2)}\,.$$

The problem at hand is now purely a curve calculation. We can therefore discard the surface X and concentrate on the curve C. To find A, we take

$$C = \mathbb{P}^1\,.$$

Our goal is then to prove the second equality in the equation

$$\begin{aligned}\mathsf{A}(q) &= \sum_{n=0}^{\infty} q^n (-1)^n \cdot \int_{\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2, n)} c(T\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2, n)) \cdot c(\mathcal{T}_n) \\ &= \frac{(1-4q)^2}{(1-q)^2 \cdot (1-6q+q^2)}\,.\end{aligned} \tag{43}$$

We will apply $\mathbb{C}^\star$-equivariant localization on $\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2, n)$. We write

$$\mathbb{C}^2 = \mathbb{C}[w_1] \oplus \mathbb{C}[w_2]$$

for the weights of the diagonal $\mathbb{C}^\star$-action on $\mathbb{C}^2$. The fixed loci are

$$\mathsf{F}[n_1, n_2] = C^{[n_1]} \times C^{[n_2]} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \overset{\iota}{\hookrightarrow} \mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2, n)\,.$$

The fixed points correspond to the exact sequences

$$0 \to I_{Z_1} \oplus I_{Z_2} \to \mathbb{C}^2 \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2} \to 0\,.$$

Thus, by Atiyah-Bott localization, we find

$$\int_{\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2, n)} c(T\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2, n)) \cdot c(\mathcal{T}_n) = \sum_{n_1+n_2=n} \int_{\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}} \mathsf{Contr}(n_1, n_2)\,. \tag{44}$$

Here, we set

$$\mathsf{Contr}(n_1, n_2) = \frac{c(\iota^\star T\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2, n)) \cdot c(\iota^\star \mathcal{T}_n)}{\mathsf{e}_{\mathbb{C}^\star}(\mathsf{N}[n_1, n_2])}$$

for the contribution of the (n_1, n_2)-fixed locus, where $\mathsf{N}[n_1, n_2]$ denotes the normal bundle. We will evaluate (44) explicitly.

For the analysis of $\mathsf{Contr}(n_1, n_2)$, the notation $w = w_2 - w_1$ will be convenient. We compute

$$\iota^\star T\mathrm{Quot}_C(\mathbb{C}^2, n) = T\mathbb{P}^{n_1} + T\mathbb{P}^{n_2} + \mathrm{Ext}^\bullet(I_{Z_1}, \mathcal{O}_{Z_2})[w] + \mathrm{Ext}^\bullet(I_{Z_2}, \mathcal{O}_{Z_1})[-w]\,.$$

The last two terms come from the normal bundle

$$\mathsf{N}[n_1, n_2] = \mathrm{Ext}^\bullet(I_{Z_1}, \mathcal{O}_{Z_2})[w] + \mathrm{Ext}^\bullet(I_{Z_2}, \mathcal{O}_{Z_1})[-w]\,.$$

Similarly, $\iota^\star \mathcal{T}_n$ can be written as

$$\mathrm{Ext}^\bullet(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_1}(-1)) + \mathrm{Ext}^\bullet(\mathcal{O}_{Z_2}, \mathcal{O}_{Z_2}(-1))$$
$$+\mathrm{Ext}^\bullet(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-1))[w] + \mathrm{Ext}^\bullet(\mathcal{O}_{Z_2}, \mathcal{O}_{Z_1}(-1))[-w].$$

We now explicitly compute the various tautological structures appearing above. The arguments follow the proof of [36, Theorem 2]. We observe that the universal subschemes

$$\mathcal{Z}_1 \subset \mathbb{P}^1 \times \mathbb{P}^{n_1}\,, \quad \mathcal{Z}_2 \subset \mathbb{P}^1 \times \mathbb{P}^{n_2}$$

take the form

$$\mathcal{O}(-\mathcal{Z}_1) = \mathcal{O}_{\mathbb{P}^1}(-n_1) \boxtimes \mathcal{O}_{\mathbb{P}^{n_1}}(-1)\,, \quad \mathcal{O}(-\mathcal{Z}_2) = \mathcal{O}_{\mathbb{P}^1}(-n_2) \boxtimes \mathcal{O}_{\mathbb{P}^{n_2}}(-1)\,.$$

We require the following three calculations:

$$\begin{aligned}
\mathrm{Ext}^\bullet(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{O}_{\mathcal{Z}_1}(-1)) &= \mathrm{Ext}^\bullet(\mathcal{O} - \mathcal{O}(-\mathcal{Z}_1), \mathcal{O}_{\mathbb{P}^1}(-1) - \mathcal{O}(-\mathcal{Z}_1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \\
&= \mathrm{Ext}^\bullet(\mathcal{O} - \mathcal{O}_{\mathbb{P}^1}(-n_1) \boxtimes \mathcal{O}_{\mathbb{P}^{n_1}}(-1), \mathcal{O}_{\mathbb{P}^1}(-1) - \mathcal{O}_{\mathbb{P}^1}(-n_1-1) \boxtimes \mathcal{O}_{\mathbb{P}^{n_1}}(-1)) \\
&= \mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(-1) - \mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(1)\,,
\end{aligned}$$

$$\mathrm{Ext}^\bullet(I_{Z_1}, \mathcal{O}_{Z_2}) = \mathbb{C}^{n_1+1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(1) - \mathbb{C}^{n_1-n_2+1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(1) \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(-1)\,,$$

$$\mathrm{Ext}^\bullet(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-1)) = -\,\mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(1) + \mathbb{C}^{n_2} \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(-1) + \mathbb{C}^{n_1-n_2} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(1) \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(-1)\,.$$

As a consequence, we find

$$\iota^\star T\mathrm{Quot}_C(\mathbb{C}^2, n) + \iota^\star \mathcal{T}_n$$

can be calculated as

$$\begin{aligned}
T\mathbb{P}^{n_1} + T\mathbb{P}^{n_2} &+ (\mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(-1) - \mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(1)) + (\mathbb{C}^{n_2} \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(-1) - \mathbb{C}^{n_2} \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(1)) \\
&+ (\mathcal{O}_{\mathbb{P}^{n_1}}(1) + \mathbb{C}^{n_2} \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(-1) - \mathcal{O}_{\mathbb{P}^{n_1}}(1) \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(-1))\,[w] \\
&+ (\mathcal{O}_{\mathbb{P}^{n_2}}(1) + \mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(-1) - \mathcal{O}_{\mathbb{P}^{n_1}}(-1) \otimes \mathcal{O}_{\mathbb{P}^{n_2}}(1))\,[-w]\,.
\end{aligned}$$

We also have

$$\mathsf{N}[n_1,n_2] = \left(\mathbb{C}^{n_1+1}\otimes\mathcal{O}_{\mathbb{P}^{n_1}}(1) - \mathbb{C}^{n_1-n_2+1}\otimes\mathcal{O}_{\mathbb{P}^{n_1}}(1)\otimes\mathcal{O}_{\mathbb{P}^{n_2}}(-1)\right)[w]$$

$$+\left(\mathbb{C}^{n_2+1}\otimes\mathcal{O}_{\mathbb{P}^{n_2}}(1) - \mathbb{C}^{n_2-n_1+1}\otimes\mathcal{O}_{\mathbb{P}^{n_1}}(-1)\otimes\mathcal{O}_{\mathbb{P}^{n_2}}(1)\right)[-w]. \tag{45}$$

We write h_1 and h_2 for the hyperplane classes on $\mathbb{P}^{n_1}$ and $\mathbb{P}^{n_2}$ respectively. After substituting the last equation into (44), we find

$$\int_{\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2,n)} c(T\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^2,n))\cdot c(\mathcal{T}_n) = \sum_{n_1+n_2=n}\int_{\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}} \mathsf{Contr}(n_1,n_2)\,,$$

where $\mathsf{Contr}(n_1,n_2)$ is given by

$$\frac{(1-h_1)^{n_1}(1+h_1)(1-h_2)^{n_2}(1+h_2)(1+h_1+w)(1-h_2+w)^{n_2}(1+h_2-w)(1-h_1-w)^{n_1}}{(1+h_1-h_2+w)(1-h_1+h_2-w)}$$

$$\cdot\frac{(w+h_1-h_2)^{n_1-n_2+1}(-w-h_1+h_2)^{n_2-n_1+1}}{(h_1+w)^{n_1+1}(h_2-w)^{n_2+1}}\,.$$

While the expression may seem unwieldy, nonetheless, we will be able to sum the localization contributions explicitly via the Lagrange-Bürmann formula. We write

$$\Phi_1(h_1) = (1-h_1)\cdot(1-h_1-w)\cdot(h_1+w)^{-1} \tag{46}$$

$$\Phi_2(h_2) = (1-h_2)\cdot(1-h_2+w)\cdot(h_2-w)^{-1}$$

$$\Psi(h_1,h_2) = (1+h_1)\cdot(1+h_2)\cdot(1+h_1+w)\cdot(1+h_2-w)\cdot(1+h_1-h_2+w)^{-1}\cdot(1-h_1+h_2-w)^{-1}$$
$$(h_1+w)^{-1}\cdot(h_2-w)^{-1}\cdot(w+h_1-h_2)^2.$$

We obtain

$$\mathsf{Contr}(n_1,n_2) = (-1)^{n+1}\cdot\Phi_1(h_1)^{n_1}\cdot\Phi_2(h_2)^{n_2}\cdot\Psi(h_1,h_2)\,.$$

The sign in the last equality is a consequence of rewriting the numerator of the normal bundle:

$$(w-h_1+h_2)^{n_1-n_2+1}(-w-h_1+h_2)^{n_2-n_1+1} = (-1)^{n+1}(w+h_1-h_2)^2\,.$$

Therefore, we have

$$\begin{aligned}
\mathsf{A}(q) &= \sum_{n=0}^{\infty} q^n(-1)^n\cdot\int_{\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n)} c(T\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n))\cdot c(\mathcal{T}_n)\\
&= -\sum_{n=0}^{\infty} q^n\sum_{n_1+n_2=n}\int_{\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}}\Phi_1(h_1)^{n_1}\cdot\Phi_2(h_2)^{n_2}\cdot\Psi(h_1,h_2)\\
&= -\sum_{n=0}^{\infty}\sum_{n_1+n_2=n} q^n\cdot\left[h_1^{n_1}\cdot h_2^{n_2}\right]\left(\Phi_1(h_1)^{n_1}\cdot\Phi_2(h_2)^{n_2}\cdot\Psi(h_1,h_2)\right)\,.
\end{aligned}$$

As before, the brackets indicate taking the suitable coefficient of the expression following it. Omitted from the notation is the fact that we also need to take the w-free term at the end.

The multivariable Lagrange-Bürmann formula of [11, Theorem 2 (4.4)] is:

$$\sum_{n_1,n_2\geq 0} t_1^{n_1}t_2^{n_2}\cdot [h_1^{n_1}\cdot h_2^{n_2}]\left(\Phi_1(h_1)^{n_1}\cdot \Phi_2(h_2)^{n_2}\cdot \Psi(h_1,h_2)\right) = \frac{\Psi}{K}(h_1,h_2) \tag{47}$$

for the change of variables

$$t_1 = \frac{h_1}{\Phi_1(h_1)}\,, \qquad t_2 = \frac{h_2}{\Phi_2(h_2)}$$

and for

$$K(t_1,t_2) = \left(1-\frac{t_1}{\Phi_1(t_1)}\cdot \Phi_1'(t_1)\right)\cdot \left(1-\frac{t_2}{\Phi_2(t_2)}\cdot \Phi_2'(t_2)\right).$$

In our case, by (46), we have

$$t_1 = \frac{h_1(h_1+w)}{(1-h_1)(1-h_1-w)}\,, \qquad t_2 = \frac{h_2(h_2-w)}{(1-h_2)(1-h_2+w)}\,. \tag{48}$$

Using (46) again, by direct calculation, we find $\frac{\Psi}{K}(h_1,h_2)$ equals

$$\frac{(1-h_1^2)\cdot(1-(w+h_1)^2)\cdot(1-h_2^2)\cdot(1-(w-h_2)^2)\cdot(w+h_1-h_2)^2}{(2h_1^2+2h_1(w-1)+w(w-1))\cdot(2h_2^2-2h_2(w+1)+w(w+1))\cdot(1-(w+h_1-h_2)^2)}\,.$$

We set $t_1=t_2=q$ and use the above equations (48) to solve

$$h_1 = -\frac{q}{1-q}-\frac{w}{2}+\sqrt{\frac{q}{(1-q)^2}+\frac{w^2}{4}}, \quad h_2 = -\frac{q}{1-q}+\frac{w}{2}-\sqrt{\frac{q}{(1-q)^2}+\frac{w^2}{4}}.$$

A direct computation then shows that

$$\frac{\Psi}{K}\left(h_1(q),h_2(q)\right) = -\frac{(1-w^2)-4q(2-w^2)+4q^2(4-w^2)}{(1-q)^2(1-w^2-2q(3-w^2)+q^2(1-w^2))}$$

so that

$$\frac{\Psi}{K}\left(h_1(q),h_2(q)\right)\big|_{w=0} = -\frac{(1-4q)^2}{(1-q)^2(1-6q+q^2)}\,.$$

Therefore,

$$\mathsf{A}(q) = -\sum_{n=0}^{\infty}\sum_{n_1+n_2=n} q^n\cdot [h_1^{n_1}\cdot h_2^{n_2}]\left(\Phi_1^{n_1}\cdot\Phi_2^{n_2}\cdot\Psi\right) = \frac{(1-4q)^2}{(1-q)^2(1-6q+q^2)}\,.$$

We have completed the proof of the $N=2$ case of Theorem 18. ☐

4.3.4. *The case $N>2$.* . The calculation of $\mathsf{Z}^{\mathcal{E}}_{X,N=2,0}$ presented above can be exactly followed for all higher N. The universal series U_N of Theorem 18 is determined by the equation

$$\mathsf{U}_N^{-1} = \sum_{n=0}^{\infty} q^n(-1)^n\cdot \int_{\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n)} c(T\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n))\cdot c(\mathcal{T}_n)\,, \tag{49}$$

where $\mathcal{T}_n$ is the bundle

$$\mathcal{T}_n = \mathrm{Ext}^{\bullet}_{\mathbb{P}^1}(Q, Q\otimes\mathcal{O}(-1))\,.$$

Localization with respect to the diagonal $\mathbb{C}^\star$-action on $\mathbb{C}^N$ yields

$$\int_{\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n)} c(T\mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n))\cdot c(\mathcal{T}_n) = \sum_{n_1+\ldots+n_N=n}\int_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_N}} \mathsf{Contr}(n_1,\ldots,n_N)\,.$$

By an explicit analysis of $\mathsf{Contr}(n_1,\ldots,n_N)$, we can write

$$\mathsf{Contr}(n_1,\ldots,n_N) = (-1)^{n(N-1)+\binom{N}{2}}\cdot\Phi_1(h_1)^{n_1}\cdots\Phi_N(h_N)^{n_N}\cdot\Psi(h_1,\ldots,h_n) \tag{50}$$

for rational functions

$$\Phi_i(h_i) = \prod_{j=1}^N(1-h_i+w_i-w_j)\cdot\prod_{j\neq i}(h_i+w_j-w_i)^{-1}\,,$$

$$\Psi = \prod_i(1+h_i)\cdot\prod_{i<j}(h_i-h_j+w_j-w_i)^2$$
$$\cdot\prod_{j\neq i}(1+h_i+w_j-w_i)\cdot(1+h_i-h_j+w_j-w_i)^{-1}\cdot(h_i+w_j-w_i)^{-1}\,,$$

which depend upon N. After applying the Lagrange-Bürmann formula with

$$t_i = \frac{h_i}{\Phi_i(h_i)} = h_i\cdot\prod_{j=1}^N(1-h_i+w_i-w_j)^{-1}\cdot\prod_{j\neq i}(h_i+w_j-w_i)\,,$$

we find

$$\sum_{n_1,\ldots,n_N} t_1^{n_1}\cdots t_N^{n_N}\cdot\left([h_1^{n_1}\cdots h_N^{n_N}]\,\Phi_1(h_1)^{n_1}\cdots\Phi_N(h_N)^{n_N}\cdot\Psi(h_1,\ldots,h_n)\right) = \frac{\Psi}{K}(h_1,\ldots,h_N)\,.$$

After setting

$$t_1=\ldots=t_N=q(-1)^N$$

the series (49) becomes

$$\mathsf{U}_N^{-1} = (-1)^{\binom{N}{2}}\cdot\frac{\Psi}{K}(h_1,\ldots,h_N) \tag{51}$$

where h_i solves the equation

$$q(-1)^N = \prod_{j=1}^N\frac{h_i+w_j-w_i}{1-h_i+w_i-w_j}\,.$$

We must select the analytic solution $h_i(q)$ with

$$h_i|_{q=0}=0.$$

We prefer however to work with a single equation. Let $H_1,\ldots,H_N$ be all solutions to the $i=1$ equation

$$q(-1)^N = \prod_{j=1}^N\frac{h+w_j-w_1}{1-h+w_j-w_1}$$

with initial values $H_j(q=0) = w_1 - w_j$. Then, by direct computation, we see that

$$h_i = H_i + w_i - w_1$$

solves the i^{th} equation. By (51), we obtain

$$U_N^{-1} = (-1)^{\binom{N}{2}} \cdot \frac{\Psi}{K}\left(H_1, H_2 + w_1 - w_2, \ldots, H_N + w_N - w_1\right) . \tag{52}$$

Using the explicit expressions of Ψ and K, we see that the right hand side of (52) is symmetric in $H_1, \ldots, H_N$. Since symmetric functions in $H_1, \ldots, H_N$ are rational functions in w and q (with possible poles at $q = 1$), the same is true of U_N^{-1}.

In fact, there are no poles of U_N^{-1} at $w = 0$. Indeed, after setting the equivariant weights to 0, the series (52) is expressed as a symmetric rational function in the N roots $h_i = r_i$ of the polynomial equation

$$q(-1)^N = h^N(1-h)^{-N}. \tag{53}$$

A direct computation shows that the expression (52) becomes

$$\mathsf{U}_N^{-1} = (-1)^{\binom{N}{2}} \cdot \frac{\prod_{i=1}^N \left((1-r_i)\cdot(1+r_i)^N\right) \cdot \prod_{i<j}(r_i - r_j)^2}{N^N (r_1 \cdots r_N)^{N-1}} \cdot \prod_{i<j}(1 - (r_i - r_j)^2)^{-1}.$$

We write

$$f(h) = \frac{h^N - (h-1)^N q}{1-q} = \prod_{i=1}^N (h - r_i)$$

for the normalized equation (53). Then,

$$\begin{aligned} \prod_{i=1}^N (1+r_i) &= (-1)^N f(-1) = \frac{1 - 2^N q}{1-q}, \\ \prod_{i=1}^N (1-r_i) \cdot \frac{\prod_{i<j}(r_i - r_j)^2}{N^N (r_1 \cdots r_N)^{N-1}} &= (-1)^{\binom{N}{2}} \prod_{i=1}^N \frac{(1-r_i)\cdot f'(r_i)}{N r_i^{N-1}} = (-1)^{\binom{N}{2}} \cdot \prod_{i=1}^N \frac{1}{1-q}. \end{aligned}$$

Therefore, we find

$$\mathsf{U}_N = \frac{(1-q)^{2N}}{(1-2^N q)^N} \cdot \prod_{i \neq j} (1 - (r_i - r_j)^2). \tag{54}$$

We can easily calculate U_N for each N from formula (54) by elementary algebra. For instance

$$\begin{aligned} \mathsf{U}_3 &= \frac{(1-q)^2(1 - 22q + 150q^2 - 22q^3 + q^4)}{(1-8q)^3}, \\ \mathsf{U}_4 &= \frac{(1-q)^2(1 - 62q + 1407q^2 - 15492q^3 + 1407q^4 - 62q^5 + q^6)}{(1-16q)^4}. \end{aligned}$$

Moreover, since (54) is clearly a symmetric rational function of the roots $r_1, \ldots, r_N$, the series U_N is a rational function in the elementary symmetric functions of the roots and hence a rational function of q. $\square$

4.4. **Proof of Theorem 8.** The methods of Section 4.3 can also be used to give a proof of the second part of Theorem 8:

$$\mathsf{B}_{1,N}(q) = \frac{(1+t)^{N+1}}{1+(N+1)t} \qquad \text{for} \quad q = (-1)^N t(1+t)^N.$$

The first part of Theorem 8 was proven in Section 3.5.

Recall the A and B-series defined by

$$\sum_{n=0}^{\infty} q^n \int_{\mathsf{Quot}_C(\mathbb{C}^N, n)} s(L^{[n]}) = \mathsf{A}_{1,1,N}^{\deg L} \cdot \mathsf{B}_{1,N}(q)^{1-g},$$

for a line bundle $L \to C$. After specializing to $C = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}$, we obtain

$$\mathsf{B}_{1,N}(q) = \sum_{n=0}^{\infty} q^n \int_{\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)} s(\mathcal{O}^{[n]}).$$

As usual, we set $\mathsf{B}(q) = \mathsf{B}_{1,N}(q)$ for notational convenience.

Consider the standard $\mathbb{C}^\star$-action on $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)$ with weights $w_1, \ldots, w_N$. In order to keep the notation manageable, we specialize to $N = 2$ (the argument for arbitrary N is exactly parallel). By localizing, we obtain

$$\mathsf{B}(q) = \sum_{n=0}^{N} q^n \sum_{n_1+n_2=n} \mathsf{Contr}(n_1, n_2),$$

where each fixed locus $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ contributes

$$\mathsf{Contr}(n_1, n_2) = \int_{\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}} \frac{s(\mathcal{O}^{[n_1]}[w_1]) \cdot s(\mathcal{O}^{[n_2]}[w_2])}{\mathsf{e}_{\mathbb{C}^\star}(\mathsf{N}[n_1, n_2])}.$$

Using Lemma 27, we find

$$s(\mathcal{O}^{[n_i]}[w_i]) = (1-w_i)^{-1} \cdot (1-h_i+w_i)^{-n_i+1}.$$

For the normal bundle, we use equation (45):

$$\mathsf{e}_{\mathbb{C}^\star}(\mathsf{N}[n_1, n_2]) = (-1)^{n+1} \cdot (h_1+w_2-w_1)^{n_1+1} \cdot (h_2+w_1-w_2)^{n_2+1} \cdot (h_1-h_2+w_2-w_1)^{-2}.$$

We define

$$\Phi_1(h_1) = (1-h_1+w_1)^{-1} \cdot (h_1+w_2-w_1)^{-1},$$
$$\Phi_2(h_2) = (1-h_2+w_2)^{-1} \cdot (h_2+w_1-w_2)^{-1},$$

$$\begin{aligned}\Psi = (1-w_1)^{-1} \cdot (1-w_2)^{-1} \cdot (1-h_1+w_1) \cdot (1-h_2+w_2) \\ \cdot (h_1+w_2-w_1)^{-1} \cdot (h_2+w_1-w_2)^{-1} \cdot (h_1-h_2+w_2-w_1)^2.\end{aligned}$$

Therefore,

$$\mathsf{Contr}(n_1, n_2) = (-1)^{n+1} \cdot \int_{\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}} \Phi_1(h_1)^{n_1} \cdot \Phi_2(h_2)^{n_2} \cdot \, \Psi(h_1, h_2)$$

which gives

$$\mathsf{B}(q) = -\sum_{n=0}^{\infty} \sum_{n_1+n_2=n} (-q)^n \cdot \left([h_1^{n_1} \cdot h_2^{n_2}] \ \Phi_1(h_1)^{n_1} \cdot \Phi_2(h_2)^{n_2} \cdot \, \Psi(h_1, h_2)\right).$$

Using Lagrange-Bürmann inversion, we find

$$\mathsf{B}(q) = -\frac{\Psi}{K}(h_1, h_2)$$

for the change of variables

$$-q = \frac{h_1}{\Phi_1(h_1)} = h_1 \cdot (1 - h_1 + w_1) \cdot (h_1 + w_2 - w_1)\,,$$

$$-q = \frac{h_2}{\Phi_2(h_2)} = h_2 \cdot (1 - h_2 + w_2) \cdot (h_2 + w_1 - w_2)\,.$$

The first of the two equations

$$-q = h \cdot (1 - h + w_1) \cdot (h + w_2 - w_1)$$

has two solutions $H_1(q)$ and $H_2(q)$ with

$$H_1(0) = 0\,, \quad H_2(0) = w_1 - w_2\,.$$

The root of the second equation

$$-q = \frac{h_2}{\Phi_2(h_2)} = h_2 \cdot (1 - h_2 + w_2) \cdot (h_2 + w_1 - w_2)$$

with initial value 0 at $q = 0$ is then

$$\widetilde{H}_2(q) = H_2(q) + w_2 - w_1\,. \tag{55}$$

Equation (55) is easily seen by direct substitution. We conclude

$$\mathsf{B}(q) = -\frac{\Psi}{K}\left(H_1(q), \widetilde{H}_2(q)\right) = -\frac{\Psi}{K}\left(H_1(q), H_2(q) + w_2 - w_1\right).$$

Further direct calculation shows

$$\mathsf{B}(q) = -\frac{\Psi}{K}\left(H_1, H_2 + w_2 - w_1\right)$$

equals

$$-\frac{(1 - H_1 + w_1)^2 \cdot (1 - H_2 + w_1)^2 \cdot (H_1 - H_2)^2}{\prod_{i=1}^{2} (1 - w_i) \cdot (3H_i^2 - 2H_i \cdot (1 + 2w_1 - w_2) + (1 + w_1) \cdot (w_1 - w_2))}\,.$$

We finally take the limit $w_1, w_2 \to 0$. Write $\mathsf{h}_1, \mathsf{h}_2$ for the two roots of the equation

$$-q = h^2(1 - h)\,, \quad \mathsf{h}_1(0) = \mathsf{h}_2(0) = 0\,.$$

These are power series in $q^{1/2}$. In the limit $w_i \to 0$, we obtain

$$\mathsf{B}(q) = -\frac{(1-\mathsf{h}_1)^2 \cdot (1-\mathsf{h}_2)^2 \cdot (\mathsf{h}_1-\mathsf{h}_2)^2}{(3\mathsf{h}_1-2)\cdot(3\mathsf{h}_2-2)\cdot(\mathsf{h}_1\mathsf{h}_2)}\,.$$

For general N, a similar analysis yields

$$\mathsf{B}(q) = (-1)^{\binom{N+1}{2}} \cdot \frac{\prod_{i<j}(\mathsf{h}_i-\mathsf{h}_j)^2 \cdot (\mathsf{h}_1\cdots\mathsf{h}_N)^{-(N-1)} \cdot \prod_i (1-\mathsf{h}_i)^2}{\prod_i((N+1)\mathsf{h}_i - N)}$$

where $\mathsf{h}_1, \ldots, \mathsf{h}_N$ solve the equation

$$(-1)^{N-1} q = h^N(1-h)\,, \quad \mathsf{h}_i(0) = 0\,. \tag{56}$$

Equation (56) has an additional solution $\mathsf{h}(q)$ with $\mathsf{h}(0) = 1$, which we can express in simple form. Indeed, if

$$q = (-1)^N t(1+t)^N\,,$$

then by direct verification

$$\mathsf{h}(q) = 1 + t\,.$$

To complete the proof of Theorem 8, we must show

$$\mathsf{B}(q) = \frac{\mathsf{h}^{N+1}}{(N+1)\mathsf{h} - N} = \frac{(1+t)^{N+1}}{1+(N+1)t}\,.$$

Equivalently, we prove the identity

$$(-1)^{\binom{N+1}{2}} \cdot \frac{\prod_{i<j}(\mathsf{h}_i-\mathsf{h}_j)^2 \cdot (\mathsf{h}_1\cdots\mathsf{h}_N)^{-(N-1)} \cdot \prod_i (1-\mathsf{h}_i)^2}{\prod_i((N+1)\mathsf{h}_i - N)} = \frac{\mathsf{h}^{N+1}}{(N+1)\mathsf{h}-N}\,. \tag{57}$$

The identity (57) is straightforward to check. Let

$$f(h) = h^N(h-1) - q(-1)^N = (h - \mathsf{h})\prod_{i=1}^N (h - \mathsf{h}_i)\,.$$

After setting $h = 1$, we obtain

$$\prod_{i=1}^N (1-\mathsf{h}_i) = -\frac{q(-1)^N}{1-\mathsf{h}}\,.$$

We compute

$$f'(h) = h^{N-1}((N+1)h - N)\,.$$

In particular, we find

$$f'(\mathsf{h}_i) = (\mathsf{h}_i - \mathsf{h}) \cdot \prod_{j\neq i}(\mathsf{h}_i - \mathsf{h}_j) = \mathsf{h}_i^{N-1}((N+1)\mathsf{h}_i - N)$$

$$f'(\mathsf{h}) = \prod_{i=1}^N (\mathsf{h} - \mathsf{h}_i) = \mathsf{h}^{N-1}((N+1)\mathsf{h} - N).$$

Therefore,

$$(-1)^{\binom{N+1}{2}} \prod_{i<j} (\mathsf{h}_i - \mathsf{h}_j)^2 = \frac{\prod_{i=1}^N f'(\mathsf{h}_i)}{f'(\mathsf{h})} = \frac{\prod_{i=1}^N \mathsf{h}_i^{N-1}((N+1)\mathsf{h}_i - N)}{\mathsf{h}^{N-1}((N+1)\mathsf{h} - N)}\,.$$

After substitution, the left hand side of equation (57) becomes

$$\left(-\frac{q(-1)^N}{1-\mathsf{h}}\right)^2 \cdot \frac{1}{\mathsf{h}^{N-1}((N+1)\mathsf{h}-N)} = \frac{\mathsf{h}^{N+1}}{(N+1)\mathsf{h}-N}\,,$$

where equation (56) was used in the last step. □

The same method can be used to determine the series $\mathsf{B}_{r,N}$ for arbitrary values of $r = \operatorname{rank}(V)$. While in general the formulas are less explicit, for $\operatorname{rank}(V) = 2$ and $N = 2$, we obtain

$$\mathsf{B}_{2,2}(-t^2) = \frac{\left(1+\sqrt{1-4t}\right)^4 \cdot \left(1+\sqrt{1+4t}\right)^4 \cdot \left(1-\sqrt{1-16t^2}\right)}{2048t^2 \cdot \sqrt{1-16t^2}}\,.$$

5. Virtual invariants of surfaces: dimension 1 quotients

5.1. **Overview.** Let X be a nonsingular, simply connected, projective surface, and let D an effective divisor on X. We compute here invariants associated to the scheme $\mathsf{Quot}_X(\mathbb{C}^N, n, D)$ of short exact sequences

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0, \quad \chi(Q) = n\,,\ c_1(Q) = D\,,\ \operatorname{rank}(Q) = 0\,.$$

In particular, we will prove Proposition 20, Theorem 21, and Theorem 23.

5.2. **Tangent-obstruction theory.** Since X is simply connected, the Hilbert scheme of curves is isomorphic to

$$\mathsf{Quot}_X(\mathbb{C}^1, n, D) \simeq X^{[m]} \times \mathbb{P}$$

where $\mathbb{P} = |D|$. Here

$$m = n + \frac{D(D+K_X)}{2} = n + (g-1)\,,$$

where g is the genus of a nonsingular curve in the linear series $|D|$. Indeed, to each pair (Z, C) with $C \in |D|$, we can associate the sequence

$$0 \to I_Z(-C) \to \mathcal{O}_X \to Q \to 0\,.$$

While the actual dimension is $2m + h^0(D) - 1$, the expected dimension of the Hilbert scheme equals

$$m + \frac{D(D-K_X)}{2}\,.$$

The first term m comes from the Hilbert scheme of points, while the second is the virtual dimension of $|D|$ endowed with its natural obstruction theory as a Hilbert scheme.

We calculate the tangent-obstruction theory, following [35], in case $m > 0$. Let

$$\mathcal{L} \to |D| \text{ and } \mathcal{Z} \subset X^{[m]} \times X$$

denote the tautological bundle $\mathcal{O}_{|D|}(1)$ and the universal subscheme of the Hilbert scheme respectively. Over $\mathsf{Quot}_X(\mathbb{C}^1, n, D)$, we compute

$$\begin{aligned}
\mathrm{Tan} - \mathrm{Obs} &= \mathrm{Ext}^\bullet(\mathcal{S}, \mathcal{Q}) \\
&= \mathrm{Ext}^\bullet(\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_X(-D) \otimes \mathcal{L}^{-1}, \mathcal{O} - \mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_X(-D) \otimes \mathcal{L}^{-1}) \\
&= \mathrm{Ext}^\bullet(\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_X(-D) \otimes \mathcal{L}^{-1}, \mathcal{O}) - \mathrm{Ext}^\bullet(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}}) \\
&= \mathrm{Ext}^\bullet(\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_X(-D), \mathcal{O}) \otimes \mathcal{L} - \mathrm{Ext}^\bullet(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}}) \\
&= H^\bullet(X, \mathcal{O}_X(D)) \otimes \mathcal{L} - \mathrm{Ext}^\bullet(\mathcal{O}_{\mathcal{Z}} \otimes \mathcal{O}_X(-D), \mathcal{O}) \otimes \mathcal{L} - \mathrm{Ext}^\bullet(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}}) .
\end{aligned}$$

Two further calculations are needed. First,

$$\mathrm{Ext}^\bullet(I_Z, I_Z) = \mathrm{Ext}^0(I_Z, I_Z) - \mathrm{Ext}^1(I_Z, I_Z) + \mathrm{Ext}^2(I_Z, I_Z) = \mathbb{C} - TX^{[m]} + H^0(K_X)^\vee ,$$

where we have used that X is simply connected and Serre duality in the second equality. Second,

$$\begin{aligned}
\mathrm{Ext}^\bullet(\mathcal{O}_Z \otimes \mathcal{O}_X(-D), \mathcal{O}) &= \mathrm{Ext}^0(\mathcal{O}_Z \otimes \mathcal{O}_X(-D), \mathcal{O}) - \mathrm{Ext}^1(\mathcal{O}_Z \otimes \mathcal{O}_X(-D), \mathcal{O}) \\
&\qquad + \mathrm{Ext}^2(\mathcal{O}_Z \otimes \mathcal{O}_X(-D), \mathcal{O}) \\
&= H^0(K_X(-D) \otimes \mathcal{O}_Z)^\vee ,
\end{aligned}$$

where we used vanishing for dimension reasons and Serre duality. Substituting, we find

$$\begin{aligned}
\mathrm{Tan} - \mathrm{Obs} &= H^\bullet(X, \mathcal{O}_X(D)) \otimes \mathcal{L} - \left((K_X(-D))^{[m]}\right)^\vee \otimes \mathcal{L} + TX^{[m]} - \mathbb{C} - H^0(K_X)^\vee \\
&= T\,\mathbb{P} - H^1(X, \mathcal{O}_X(D)) \otimes \mathcal{L} + H^2(X, \mathcal{O}_X(D)) \otimes \mathcal{L} \\
&\qquad - \left((K_X(-D))^{[m]}\right)^\vee \otimes \mathcal{L} + TX^{[m]} - H^0(K_X)^\vee.
\end{aligned}$$

For the second equality, we have also used the Euler sequence

$$0 \to \mathcal{O} \to H^0(X, \mathcal{O}(D)) \otimes \mathcal{L} \to \mathrm{Tan}_{\mathbb{P}} \to 0 .$$

In conclusion, we see that the K-theory class of the obstruction bundle equals

$$H^1(X, \mathcal{O}_X(D)) \otimes \mathcal{L} - H^2(X, \mathcal{O}_X(D)) \otimes \mathcal{L} + \left((K_X(-D))^{[m]}\right)^\vee \otimes \mathcal{L} + H^0(K_X)^\vee.$$

After setting $M = K_X - D$, we can rewrite obstruction bundle as

$$\mathrm{Obs} = (H^1(M) - H^0(M) + M^{[m]})^\vee \otimes \mathcal{L} + H^0(K_X)^\vee. \tag{58}$$

By the definition of the virtual Euler characteristic,

$$\mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1, n, D)) = \int_{X^{[m]} \times \mathbb{P}} \mathsf{e}(\mathrm{Obs}) \, \frac{c(TX^{[m]})\, c(T\mathbb{P})}{c(\mathrm{Obs})} .$$

5.3. **Examples ($N = 1$).** We illustrate the calculations above by examples corresponding to several different geometries.

5.3.1. *Rational surfaces.* A rich theory is obtained when X is a rational surface. Since $H^0(K_X) = 0$ for rational surfaces, the obstruction bundle simplifies to

$$\mathrm{Obs} = (H^1(M) - H^0(M) + M^{[m]})^\vee \otimes \mathcal{L}.$$

Proof of Proposition 20. Let X be the blow-up of a rational surface at one point with exceptional divisor E. Take $D = E$ so that

$$\mathrm{Obs} = \left(M^{[m]}\right)^\vee.$$

Thus

$$\mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1, n, E)) = \int_{X^{[m]}} \mathsf{e}\left(\left(M^{[m]}\right)^\vee\right) \frac{c(TX^{[m]})}{c\left(\left(M^{[m]}\right)^\vee\right)}.$$

Such integrals have been computed in equation (40) of Corollary 38. We find

$$\sum_{n=1}^{\infty} q^{n-1} \mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1, n, E)) = \left((1-q)^2(1-2q)^{-1}\right)^{K_X^2+1}.$$

5.3.2. *K3 surfaces.* Let X be a $K3$ surface, and let D be a primitive big and nef curve class. In particular, we have

$$H^0(M) = H^1(M) = 0.$$

We write g for the genus of D. The obstruction bundle

$$\mathrm{Obs} = \left(M^{[m]}\right)^\vee \otimes \mathcal{L} + H^0(K_X)^\vee$$

has a trivial summand. As a result, all virtual invariants vanish.

A reduced obstruction bundle can be defined by removing the trivial factor. With the new obstruction theory, we find

$$\mathsf{e}^{\mathrm{red}}(\mathsf{Quot}_X(\mathbb{C}^1, n, D)) = \int_{X^{[m]} \times \mathbb{P}} \mathsf{e}\left(\left(M^{[m]}\right)^\vee \otimes \mathcal{L}\right) \frac{c(TX^{[m]})\, c(T\mathbb{P})}{c(\left(M^{[m]}\right)^\vee \otimes \mathcal{L})}, \tag{59}$$

for $M = \mathcal{O}_X(-D)$.

Proof of Theorem 21. Without the dual placed on the tautological bundle $M^{[m]}$, integrals similar to (59) also appear in Göttsche's conjecture and are computed by the Kawai-Yoshioka formula (11):

$$N_{g,n} = \int_{X^{[m]} \times \mathbb{P}} \mathsf{e}\left(D^{[m]} \otimes \mathcal{L}\right) \frac{c(TX^{[m]})\, c(T\mathbb{P})}{c(D^{[m]} \otimes \mathcal{L})}. \tag{60}$$

This was noted in [22, Section 4].

To prove the claim of Theorem 21,

$$\mathsf{e}^{\mathrm{red}}(\mathsf{Quot}_X(\mathbb{C}^1, n, D)) = N_{g,n}\,,$$

we will use formulas (59) and (60). Since $X^{[m]}$ is holomorphic symplectic, we can replace the tangent bundle in (59) with the isomorphic cotangent bundle. Thus, we must show

$$\int_{X^{[m]}\times\mathbb{P}} \mathsf{e}\left(\left(M^{[m]}\right)^{\vee}\otimes\mathcal{L}\right)\frac{c\left(\left(TX^{[m]}\right)^{\vee}\right)c(T\mathbb{P})}{c((M^{[m]})^{\vee}\otimes\mathcal{L})}$$
$$= \int_{X^{[m]}\times\mathbb{P}} \mathsf{e}\left(D^{[m]}\otimes\mathcal{L}\right)\frac{c(TX^{[m]})\,c(T\mathbb{P})}{c(D^{[m]}\otimes\mathcal{L})}\,.$$

After integrating out the hyperplane class on $\mathbb{P}$, we are led to the statement

$$\int_{X^{[m]}} \mathsf{P}\left(c_i\left(\left(M^{[m]}\right)^{\vee}\right), c_j\left(\left(TX^{[m]}\right)^{\vee}\right)\right) = \int_{X^{[m]}} \mathsf{P}\left(c_i(D^{[m]}), c_j(TX^{[m]})\right)$$

where P is a uniquely defined universal polynomial in the Chern classes of various tautological bundles on the Hilbert scheme $X^{[m]}$. After removing the duals (since $X^{[m]}$ is even dimensional), we must show

$$\int_{X^{[m]}} \mathsf{P}\left(c_i(M^{[m]}), c_j(TX^{[m]})\right) = \int_{X^{[m]}} \mathsf{P}\left(c_i(D^{[m]}), c_j(TX^{[m]})\right). \tag{61}$$

Equality (61) is then a consequence of [8, Theorem 4.1]. Expressions such as the ones in (61) are given by universal formulas in the Chern numbers. For the left hand side, these Chern numbers are

$$c_1(M)^2\,,\quad K_X^2\,,\quad c_1(M)\cdot K_X\,,\quad c_2(X)\,.$$

The right hand side is similar, with the relevant numbers being

$$c_1(D)^2\,,\quad K_X^2\,,\quad c_1(D)\cdot K_X\,,\quad c_2(X)\,.$$

Since X is a $K3$ surface, all the Chern numbers match, including

$$c_1(M)\cdot K_X = c_1(D)\cdot K_X = 0\,,$$

which may in general sign change. ☐

The case $D = 0$ is not covered by Theorem 21. However, in the $K3$ case, we can consider the reduced theory of the Hilbert scheme of points $X^{[n]}$ obtained by removing the canonical trivial factor from the obstruction bundle $\left(\mathcal{O}^{[n]}\right)^{\vee}$. The reduced virtual dimension is $n+1$, and the obstruction bundle equals

$$\mathrm{Obs} = \left(\mathcal{O}^{[n]} - \mathcal{O}\right)^{\vee} \to X^{[n]}\,.$$

While the question does not involve any curve classes, the calculation below makes use of Theorem 21 for curves of genus 1.

Proposition 40. *We have*

$$\sum_{n=1}^{\infty} q^n \mathsf{e}^{\mathrm{red}}(X^{[n]}) = \frac{24q}{(1-q)^2}\,.$$

Proof. We have

$$\mathsf{e}^{red}(X^{[n]}) = \int_{X^{[n]}} \mathsf{e}(\mathrm{Obs}) \cdot \frac{c(TX^{[n]})}{c(\mathrm{Obs})} = \int_{X^{[n]}} \mathsf{e}\left(\left(\mathcal{O}^{[n]} - \mathcal{O}\right)^{\vee}\right) \frac{c(TX^{[n]})}{c\left(\left(\mathcal{O}^{[n]} - \mathcal{O}\right)^{\vee}\right)}\,.$$

For $n > 0$, we will prove

$$\mathsf{e}^{\mathrm{red}}(X^{[n]}) = N_{1,n}\,. \tag{62}$$

We start by writing $\alpha_1, \ldots, \alpha_n$ for the roots of $\mathcal{O}^{[n]}$ with the convention that $\alpha_1 = 0$ corresponds to the trivial summand of the obstruction bundle. Then, claim (62) becomes

$$\int_{X^{[n]}} \prod_{i=2}^{n} \frac{-\alpha_i}{1-\alpha_i} \cdot c(TX^{[n]}) = N_{1,n}\,. \tag{63}$$

By deformation invariance, we may assume X is an elliptically fibered $K3$ with fiber class f. We apply Theorem 21 for the curve class $D = f$. The associated line bundle D has no higher cohomology, and the proof of Theorem 21 applies even though D is not big. We find

$$N_{1,n} = \int_{X^{[n]}\times\mathbb{P}^1} \mathsf{e}\left(\left(M^{[n]}\right)^{\vee} \otimes \mathcal{L}\right) \frac{c(TX^{[n]})\, c(T\mathbb{P}^1)}{c(\left(M^{[n]}\right)^{\vee} \otimes \mathcal{L})}\,,$$

where $M = \mathcal{O}_X(-f)$.

We write $\mu_1, \ldots, \mu_n$ for the roots of $M^{[n]}$, and let ζ be the hyperplane class on the projective line. The above integral becomes

$$\begin{aligned} N_{1,n} &= \int_{X^{[n]}\times\mathbb{P}^1} \prod_{i=1}^{n} \frac{\zeta-\mu_i}{1+\zeta-\mu_i} \cdot c(TX^{[n]})(1+2\zeta) \\ &= \int_{X^{[n]}} \left(2\prod_{i=1}^{n} \frac{-\mu_i}{1-\mu_i} + \sum_{i=1}^{n} \frac{1}{(1-\mu_i)^2} \prod_{j\neq i} \frac{-\mu_j}{1-\mu_j}\right) c(TX^{[n]})\,, \end{aligned}$$

where, in the second equality, we have integrated out the hyperplane class on $\mathbb{P}^1$. The resulting integral is a universal polynomial in the quantities

$$M^2\,, \quad M\cdot K_X\,, \quad K_X^2\,, \quad c_2(X)\,. \tag{64}$$

Indeed, the expression

$$2\prod_{i=1}^{n} \frac{-\mu_i}{1-\mu_i} + \sum_{i=1}^{n} \frac{1}{(1-\mu_i)^2} \prod_{j\neq i} \frac{-\mu_j}{1-\mu_j}$$

can be written in terms of the Chern classes of $M^{[n]}$. The claimed universality then follows from [8, Theorem 4.1].

Since the four numerical invariants (64) are the same if $M = -f$ or $M = 0$, we are free to replace the μ_i's by the α_i's without changing the answer. Therefore,

$$N_{1,n} = \int_{X^{[n]}} \left(2 \prod_{i=1}^{n} \frac{-\alpha_i}{1-\alpha_i} + \sum_{i=1}^{n} \frac{1}{(1-\alpha_i)^2} \prod_{j \neq i} \frac{-\alpha_j}{1-\alpha_j} \right) c(TX^{[n]}) \,.$$

Since $\alpha_1 = 0$, we obtain

$$N_{1,n} = \int_{X^{[n]}} \prod_{j \neq 1} \frac{-\alpha_j}{1-\alpha_j} \cdot c(TX^{[n]}) \,,$$

as claimed in (63).

Finally, using the Kawai-Yoshioka formula (11), we find

$$N_{1,n} = [q \cdot y^n] \left(\sqrt{y} - \frac{1}{\sqrt{y}} \right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2} = 24n \,,$$

for $n > 0$. □

5.3.3. *Surfaces of general type.* Let X be a nonsingular, simply connected, projective surface of general type with $p_g(X) > 0$. For $D = K_X$, The obstruction bundle (58) takes the form

$$\mathrm{Obs} = (\mathcal{O}^{[m]} - \mathcal{O})^\vee \otimes \mathcal{L} + H^0(K_X)^\vee \,.$$

Due to the presence of a trivial summand, the virtual Euler characteristic vanishes

$$\mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1, n, K_X)) = 0$$

for $m > 0$. The case

$$m = 0 \iff n = -K_X^2$$

is special, yielding the answer

$$\mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1, n, K_X)) = \int_{\mathbb{P}} \frac{1}{c(\mathcal{L})} = (-1)^{p_g+1} = (-1)^{\chi(\mathcal{O}_X)},$$

in agreement with [5, 7].

Proof of Proposition 22. Let D be an arbitrary effective curve class. To start, we take $N = 1$ and assume

$$D \neq 0 \,, \quad D \neq K_X \,,$$

since these cases have already been considered. Recall

$$\mathrm{Obs} = (H^1(M) - H^0(M) + M^{[m]})^\vee \otimes \mathcal{L} + H^0(K_X)^\vee.$$

If $M = K_X - D$ is not effective, then $H^0(M) = 0$. The virtual class is then forced to vanish by the trivial summand $H^0(K_X)^\vee$ of the obstruction bundle.

We may therefore assume M to be effective. By Serre duality,

$$\mathrm{rank}\, \mathrm{Obs} = h^1(D) - h^2(D) + m + p_g \,.$$

We also have

$$\left[\mathsf{Quot}_X(\mathbb{C}^1, n, D)\right]^{\mathrm{vir}} = \mathsf{e}(\mathrm{Obs}) \cap \left[\mathbb{P} \times X^{[m]}\right],$$

where

$$\dim \mathbb{P} = h^0(D) - 1\,.$$

We write $h \in A^1(\mathbb{P})$ for the hyperplane class and α_i for the Chern roots of $M^{[m]}$ on $X^{[m]}$. The virtual class then equals the degree $p_g + m + h^1(D) - h^2(D)$ part of

$$c(\mathrm{Obs}) = (1+h)^{h^1(D)-h^2(D)} \cdot \prod_{i=1}^{m} (1 + h - \alpha_i)\,. \tag{65}$$

The expression (65) contains terms of the form

$$h^k \cdot \text{symmetric polynomial of degree at most } m \text{ in the roots } \alpha_i\,,$$

where $k \leq h^0(D) - 1$ for dimension reasons. All terms therefore have degree bounded by

$$h^0(D) - 1 + m < h^1(D) - h^2(D) + m + p_g\,. \tag{66}$$

Consequently, the Euler class vanishes.

To justify inequality (66), we use the following chain of equivalences:

$$\begin{aligned} h^0(D) - 1 < h^1(D) - h^2(D) + p_g \quad &\Longleftrightarrow \quad \chi(D) < 1 + p_g = \chi(\mathcal{O}_X) \\ &\Longleftrightarrow \quad D \cdot (D - K_X) < 0 \\ &\Longleftrightarrow \quad D \cdot M > 0\,. \end{aligned}$$

The inequality $D \cdot M > 0$ holds since the pair (D, M) is a nontrivial effective splitting of K_X (the canonical class is 1-connected for minimal surfaces of general type [4, Proposition 6.1]). The proof of Proposition 22 for $N = 1$ is complete.

For $N > 1$, we use $\mathbb{C}^\star$-equivariant localization. The natural $\mathbb{C}^*$-action on $\mathsf{Quot}_X(\mathbb{C}^N, n, D)$ has fixed loci

$$\mathsf{F}[(n_1, D_1), \ldots, (n_N, D_N)]$$

indexed by all possible effective splittings

$$n_1 + \ldots + n_N = n\,, \quad D_1 + \ldots + D_N = D\,.$$

The corresponding subsheaves are

$$S = \bigoplus_{i=1}^{N} I_{Z_i} \hookrightarrow \mathbb{C}^N \otimes \mathcal{O}_X\,, \quad c_1(Z_i) = D_i\,, \quad \chi(\mathcal{O}_{Z_i}) = n_i\,.$$

The induced virtual class of

$$\mathsf{F}[(n_1, D_1), \ldots, (n_N, D_N)] = \mathsf{Quot}_X(\mathbb{C}^1, n_1, D_1) \times \cdots \times \mathsf{Quot}_X(\mathbb{C}^1, n_N, D_N)$$

is determined by the fixed part of

$$\mathrm{Ext}^\bullet(S, Q)^{\mathrm{fix}} = \bigoplus \mathrm{Ext}^\bullet(I_{Z_i}, \mathcal{O}_{Z_i})$$

and, therefore, splits over the factors. Using the case $N = 1$ already established, in order to obtain a nontrivial virtual fundamental class on the i^{th} factor, we must have

$$D_i = 0 \ \text{ or } \ D_i = K_X \quad \implies \quad D = \ell K_X \ \text{ for } \ 0 \le \ell \le N\,.$$

By the paragraph preceding the proof of Proposition 22, the choice $D_i = K_X$ forces Z_i to be supported only on canonical curves, without any point contributions. ☐

5.4. **Proof of Theorem 23.** The $N = 1$ case of Theorem 23 is a consequence of the calculations of Section 5.3.3. In the $N = 2$ case, Theorem 23 can be derived from Theorem 3: the localization contributions can be expressed as integrals over the symmetric product with 7 Segre factors.[11] However, we will treat all the cases $N \ge 1$ together using the strategy of the the proof Theorem 18.

Let X be a nonsingular, simply connected, minimal surface of general type admitting a nonsingular canonical curve $C \subset X$ of genus

$$g = K_X^2 + 1\,.$$

Let $0 \le \ell \le N$. Let

$$\mathsf{Z}^{\mathcal{E}}_{X,N,\ell K_X}(q) = \sum_{n \in \mathbb{Z}} q^n \mathsf{e}^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, n, \ell K_X))\,.$$

The formula of Theorem 23 is

$$\mathsf{Z}^{\mathcal{E}}_{X,N,\ell K_X}(q) = (-1)^{\ell\cdot\chi(\mathcal{O}_X)}\ q^{\ell(1-g)} \cdot \sum_{1 \le i_1 < \ldots < i_{N-\ell} \le N} \mathsf{A}(r_{i_1}, \ldots, r_{i_{N-\ell}})^{1-g}\,.$$

The sum is taken over all $\binom{N}{N-\ell}$ choices of $N-\ell$ distinct roots of the equation

$$z^N = q(z-1)^N\,.$$

Furthermore,

$$\mathsf{A}(x_1, \ldots, x_{N-\ell}) = \frac{(-1)^{\binom{N-\ell}{2}}}{N^{N-\ell}} \cdot \prod_{i=1}^{N-\ell} \frac{(1+x_i)^N(1-x_i)}{x_i^{N-1}} \cdot \prod_{i<j} \frac{(x_i-x_j)^2}{1-(x_i-x_j)^2}\,.$$

In case $\ell = N$, the formula is interpreted as

$$\mathsf{Z}^{\mathcal{E}}_{X,\,N,\,NK_X}(q) = (-1)^{N\cdot\chi(\mathcal{O}_X)}\, q^{N(1-g)}\,.$$

To prove the claimed evaluation, we consider the $\mathbb{C}^\star$-action on $\mathsf{Quot}_X(\mathbb{C}^N, n, \ell K_X)$ with weights $w_1, \ldots, w_N$ on the middle term of the sequence

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0\,.$$

We write

$$n = m + \ell(1-g)\,.$$

[11] We leave the argument to the intrepid reader.

For convenience, we set

$$k = N - \ell \,.$$

By the last sentence in the proof of Proposition 22, the contributing fixed loci correspond to kernels of the form

$$S = \bigoplus_{i=1}^{\ell} \mathcal{O}_X(-D_i) \oplus \bigoplus_{j=1}^{k} I_{Z_j} \hookrightarrow \mathbb{C}^N \otimes \mathcal{O}_X \,,$$

where $D_i \in |K_X|$ and Z_j is a 0-dimensional scheme of length m_j. Of course, we have

$$\sum_{j=1}^{k} m_j = m \,.$$

The weights $w_1, \ldots, w_N$ are distributed over the summands of S in $\binom{N}{k}$ possible ways, depending on the location of the curves and points. The fixed loci are therefore indexed by tuples $(m_1, \ldots, m_k)$ as well as choices of $\binom{N}{k}$ summands of $\mathbb{C}^N$. For a fixed partition $(m_1, \ldots, m_k)$, there are $\binom{N}{k}$ fixed loci all isomorphic to

$$\mathsf{F}[m_1, \ldots, m_k] = \left(\prod_{i=1}^{\ell} \mathbb{P} \right) \times \left(\prod_{j=1}^{k} X^{[m_j]} \right) \,.$$

Here, $\mathbb{P}$ denotes the linear series $|K_X|$. The obstruction bundle splits into obstruction bundles over the factors,

$$\mathrm{Obs} = \sum_{i=1}^{\ell} \mathrm{pr}_i^\star \left(H^0(K_X)^\vee - \mathcal{L} \right) + \left(\sum_{j=1}^{k} K_X^{[m_j]} \right)^\vee \,.$$

We therefore obtain

$$\begin{aligned} [\mathsf{F}[m_1, \ldots, m_k]]^{\mathrm{vir}} &= \mathsf{e}\left(\sum_{i=1}^{\ell} \mathrm{pr}_i^\star \left(H^0(K_X)^\vee - \mathcal{L} \right) + \left(\sum_{j=1}^{k} K_X^{[m_j]} \right)^\vee \right) \\ &= \prod_{i=1}^{\ell} \mathrm{pr}_i^\star \frac{1}{1 + c_1(\mathcal{L})} \cdot \prod_{j=1}^{k} \mathsf{e}\left(\left(K_X^{[m_j]} \right)^\vee \right) \\ &= (-1)^{\ell\chi} \cdot (-1)^m \iota_\star \left([\mathrm{pt}] \times \cdots \times [\mathrm{pt}] \times \left[C^{[m_1]} \times \cdots \times C^{[m_k]} \right] \right) . \end{aligned}$$

Here, $\chi = \chi(\mathcal{O}_X)$, and, for the canonical curve $C \subset X$, we have written

$$\iota : [\mathrm{pt}] \times \cdots \times [\mathrm{pt}] \times \left(C^{[m_1]} \times \cdots \times C^{[m_k]} \right) \hookrightarrow \left(\prod_{i=1}^{\ell} \mathbb{P} \right) \times \left(X^{[m_1]} \times \cdots \times X^{[m_k]} \right)$$

for the natural morphism.

We write

$$j : \mathsf{F}[m_1, \ldots, m_k] \hookrightarrow \mathsf{Quot}_X(\mathbb{C}^N, n, \ell K_X)$$

for the natural inclusion. The integral

$$\mathsf{e}^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^N,n,\ell K_X)\right)=\int_{[\mathsf{Quot}_X(\mathbb{C}^N,n,\ell K_X)]^{\mathrm{vir}}} c(T^{\mathrm{vir}}\mathsf{Quot}_X)$$

can be calculated by $\mathbb{C}^\star$-equivariant localization. Each fixed locus $\mathsf{F}=\mathsf{F}[m_1,\ldots,m_k]$ yields a contribution

$$\int_{[\mathsf{F}[m_1,\ldots,m_k]]^{\mathrm{vir}}}\frac{c(j^\star T^{\mathrm{vir}}\mathsf{Quot}_X)}{\mathsf{e}(\mathsf{N}^{\mathrm{vir}})}=(-1)^{m+\ell\chi}\int_{C^{[m_1]}\times\cdots\times C^{[m_k]}}\iota^\star\left(\frac{c(T^{\mathrm{vir}}\mathsf{F})\,c(\mathsf{N}^{\mathrm{vir}})}{\mathsf{e}(\mathsf{N}^{\mathrm{vir}})}\right). \tag{67}$$

We will analyze these contributions separately. We assume the weights $w_1,\ldots,w_\ell$ are distributed on the curve summands and the weights $w_{\ell+1},\ldots,w_N$ are distributed on the point summands. In other words, the kernels are

$$S=\bigoplus_{i=1}^{\ell}\mathcal{O}_X(-D_i)[w_i]\oplus\bigoplus_{j=1}^{k}I_{Z_j}[w_{j+\ell}]\,.$$

We will use the indices i,i' to refer to the curve summands, while the indices j,j' will be reserved for the point summands. We obtain

$$\begin{aligned}T^{\mathrm{vir}}\mathsf{F}&=\sum_{i=1}^{\ell}T\mathbb{P}+\sum_{j=1}^{k}TX^{[m_j]}-\mathrm{Obs}\\&=\sum_{i=1}^{\ell}T\mathbb{P}+\sum_{j=1}^{k}TX^{[m_j]}-\left(\sum_{i=1}^{\ell}\mathrm{pr}_i^\star\left(H^0(K_X)^\vee-\mathcal{L}\right)+\sum_{j=1}^{k}\left((K_X)^{[m_j]}\right)^\vee\right),\end{aligned}$$

which yields

$$\begin{aligned}\iota^\star T^{\mathrm{vir}}\mathsf{F}&=\sum_{i=1}^{\ell}\mathbb{C}^{p_g-1}+\sum_{j=1}^{k}\iota^\star TX^{[m_j]}-\left(\sum_{i=1}^{\ell}\mathbb{C}^{p_g-1}+\sum_{j=1}^{k}\iota^\star\left((K_X)^{[m_j]}\right)^\vee\right)\\&=\sum_{j=1}^{k}\left(TC^{[m_j]}+\Theta^{[m_j]}-\left(\Theta^{[m_j]}\right)^\vee\right)\\&=\sum_{j=1}^{k}\left(\left(K_C^{[m_j]}\right)^\vee+\Theta^{[m_j]}-\left(\Theta^{[m_j]}\right)^\vee\right).\end{aligned}$$

Here, $\Theta=\mathcal{O}_X(C)|_C$ is the theta characteristic. The last equality was shown in the proof of Theorem 18, see (39). There are no equivariant weights for $\iota^\star T^{\mathrm{vir}}\mathsf{F}$.

The virtual normal bundle splits into four terms

$$\mathsf{N}^{\mathrm{vir}}=\mathsf{N}_1+\mathsf{N}_2+\mathsf{N}_3+\mathsf{N}_4$$

where

$$\begin{aligned}
\mathsf{N}_1 &= \sum_{i=1}^{\ell}\sum_{j=1}^{k} \mathrm{Ext}^\bullet(\mathcal{O}_X(-D_i), \mathcal{O}_{Z_j})[w_{j+\ell} - w_i]\,, \\
\mathsf{N}_2 &= \sum_{i=1}^{\ell}\sum_{j=1}^{k} \mathrm{Ext}^\bullet(I_{Z_j}, \mathcal{O}_{D_i})[w_i - w_{j+\ell}]\,, \\
\mathsf{N}_3 &= \sum_{i=1}^{\ell}\sum_{i'\neq i} \mathrm{Ext}^\bullet(\mathcal{O}_X(-D_i), \mathcal{O}_{D_{i'}})[w_{i'} - w_i]\,, \\
\mathsf{N}_4 &= \sum_{j=1}^{k}\sum_{j'\neq j} \mathrm{Ext}^\bullet(\mathcal{I}_{Z_j}, \mathcal{O}_{Z_{j'}})[w_{j'+\ell} - w_{j+\ell}]\,.
\end{aligned}$$

We would normally include the tautological line bundle $\mathcal{L}$ in the expression of the subsheaf, but, since we are in the end restricting to a point via ι, there is no need.

We write N_1^{ij} for the ij-summand of N_1. We find

$$\iota^\star \mathsf{N}_1^{ij} \;=\; \iota^\star (K_X)^{[m_j]}[w_{j+\ell} - w_i] = \Theta^{[m_j]}[w_{j+\ell} - w_i]\,.$$

Similarly

$$\begin{aligned}
\iota^\star \mathsf{N}_2^{ij} &= \iota^\star \left(\mathrm{Ext}^\bullet(\mathcal{O} - \mathcal{O}_{Z_j}, \mathcal{O} - K_X^{-1})[w_i - w_{j+\ell}]\right) \\
&= \iota^\star \left(H^\bullet(\mathcal{O}) - H^\bullet(K_X^{-1}) - \left((K_X)^{[m_j]}\right)^\vee + \left((K_X^{\otimes 2})^{[m_j]}\right)^\vee \right) [w_i - w_{j+\ell}]\,,
\end{aligned}$$

where we have used, suppressing indices, that

$$\mathrm{Ext}^\bullet(\mathcal{O}_Z, \mathcal{O}) = \mathrm{Ext}^{2-\bullet}(\mathcal{O}, K_X \otimes \mathcal{O}_Z)^\vee = \left((K_X)^{[m]}\right)^\vee\,,$$

$$\mathrm{Ext}^\bullet(\mathcal{O}_Z, K_X^{-1}) = \mathrm{Ext}^{2-\bullet}(K_X^{-1}, K_X \otimes \mathcal{O}_Z)^\vee = \left((K_X^{\otimes 2})^{[m]}\right)^\vee\,.$$

Since

$$H^\bullet(\mathcal{O}) - H^\bullet(K_X^{-1}) = -\mathbb{C}^{g-1}\,,$$

we have

$$\iota^\star \mathsf{N}_2^{ij} = -\mathbb{C}^{g-1}[w_i - w_{j+\ell}] - \left(\Theta^{[m_j]}\right)^\vee [w_i - w_{j+\ell}] + \left(K_C^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\,.$$

For the third term of the virtual normal bundle,

$$\iota^\star \mathsf{N}_3^{ii'} = \left(H^\bullet(\mathcal{O}(D_i)) - H^\bullet(\mathcal{O}(D_i - D_i'))\right)[w_{i'} - w_i] = \left(H^\bullet(K_X) - H^\bullet(\mathcal{O}_X)\right)[w_{i'} - w_i] = 0.$$

For the fourth term, we have already computed in equation (35) of the proof of Lemma 34, for $j \neq j'$,

$$\iota^\star\left(\mathsf{N}_4^{jj'}\right) = \mathsf{T}_{jj'} + \mathsf{N}_{jj'}\,,$$

where

$$\begin{aligned}\mathsf{T}_{jj'} &= \operatorname{Ext}^\bullet_C(\mathcal{O}_{Z_j}, \mathcal{O}_{Z_{j'}} \otimes \Theta)[w_{j'+\ell} - w_{j+\ell}]\,,\\ \mathsf{N}_{jj'} &= \operatorname{Ext}^\bullet_C(\mathcal{I}_{Z_j}, \mathcal{O}_{Z_{j'}})[w_{j'+\ell} - w_{j+\ell}]\,.\end{aligned}$$

We also had observed there that, as a consequence of Serre duality,

$$\mathsf{e}(\mathsf{T}_{jj'} + \mathsf{T}_{j'j}) = 1\,.$$

Moreover, $\mathsf{N} = \sum_{j\neq j'} \mathsf{N}_{jj'}$ is identified with the normal bundle of the fixed locus

$$C^{[m_1]} \times \cdots \times C^{[m_k]} \hookrightarrow \mathsf{Quot}_C(\mathbb{C}^k, m)\,,$$

where the $\mathbb{C}^\star$-action has weights $w_{\ell+1}, \ldots, w_N$ on $\mathbb{C}^k$.

After collecting all terms, the fixed locus contribution becomes

$$(-1)^{m+\ell\chi} \cdot \int_{C^{[m_1]}\times\ldots\times C^{[m_k]}} \prod_{j=1}^{k} \frac{c\left(\left(K_C^{[m_j]}\right)^\vee\right) \cdot c(\Theta^{[m_j]})}{c\left(\left(\Theta^{[m_j]}\right)^\vee\right)} \cdot \prod_{i=1}^{\ell}\prod_{j=1}^{k} \frac{c(\Theta^{[m_j]}[w_{j+\ell} - w_i])}{\mathsf{e}(\Theta^{[m_j]}[w_{j+\ell} - w_i])}$$

$$\cdot \prod_{i=1}^{\ell}\prod_{j=1}^{k} \frac{\mathsf{e}\left(\left(\Theta^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\right)}{c\left(\left(\Theta^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\right)} \cdot \frac{c\left(\left(K_C^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\right)}{\mathsf{e}\left(\left(K_C^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\right)} \cdot \frac{(w_i - w_{j+\ell})^{g-1}}{(1 + w_i - w_{j+\ell})^{g-1}}$$

$$\cdot \prod_{1\leq j\neq j'\leq k} c(\mathsf{T}_{jj'})c(\mathsf{N}_{jj'}) \cdot \frac{1}{\mathsf{e}(\mathsf{N})}\,.$$

We note a cancellation between the Euler classes in the denominator of the second product and the numerator of the third product, yielding the answer

$$(-1)^{m+\ell\chi} \cdot (-1)^{\ell m} \cdot \int_{C^{[m_1}\times\ldots\times C^{[m_k]}} \prod_{j=1}^{k} \frac{c\left(\left(K_C^{[m_j]}\right)^\vee\right) \cdot c(\Theta^{[m_j]})}{c\left(\left(\Theta^{[m_j]}\right)^\vee\right)} \tag{68}$$

$$\cdot \prod_{i=1}^{\ell}\prod_{j=1}^{k} \frac{c(\Theta^{[m_j]}[w_{j+\ell} - w_i])}{c\left(\left(\Theta^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\right)} \cdot \frac{c\left(\left(K_C^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\right)}{\mathsf{e}\left(\left(K_C^{[m_j]}\right)^\vee [w_i - w_{j+\ell}]\right)} \cdot \frac{(w_i - w_{j+\ell})^{g-1}}{(1 + w_i - w_{j+\ell})^{g-1}}$$

$$\cdot \prod_{1\leq j\neq j'\leq k} c(\mathsf{T}_{jj'})c(\mathsf{N}_{jj'}) \cdot \frac{1}{\mathsf{e}(\mathsf{N})}\,.$$

Let $\mathsf{Contr}[m_1, \ldots, m_k] \in \mathbb{Q}((w))$ denote the integral thus obtained (without including the sign $(-1)^{(\ell+1)m+\ell\chi}$). We have

$$\mathsf{Z}^{\mathcal{E}}_{X,N,\ell K_X}(q) = \sum_{n\in\mathbb{Z}} q^n \mathsf{e}^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^N, \ell K_X, n)\right) = \sum \mathsf{Z}[m_1, \ldots, m_k]\,, \tag{69}$$

where

$$\mathsf{Z}[m_1,\ldots,m_k] = \sum_{m=0}^{\infty} (-1)^{(\ell+1)m+\ell\chi}\, q^{\ell(1-g)+m} \cdot \mathsf{Contr}[m_1,\ldots,m_k]\,.$$

As usual, the sum on the right in the equation (69) has $\binom{N}{k}$ terms depending on the placement of the weights (and is also over $m_1,\ldots,m_k$). We will set $w=0$ at the end.

We will transform the above contribution formulas into integrals over the Quot scheme $\mathsf{Quot}_C(\mathbb{C}^k,m)$. Recall, from the proof of Theorem 18, the virtual bundle

$$\mathcal{T}_m = \mathrm{Ext}^\bullet_C(Q, Q\otimes\Theta)$$

on $\mathsf{Quot}_C(\mathbb{C}^k,m)$. The tautological bundle

$$L^{[m]} \to \mathsf{Quot}_C(\mathbb{C}^k,m)$$

associated to a line bundle L on C was defined in Section 1.2. We define

$$\mathsf{Z}_{C,k}(q\,|\,w_1,\ldots,w_\ell\,|\,w_{\ell+1},\ldots,w_N) =$$

$$\sum_{m=0}^{\infty} q^m \int_{\mathsf{Quot}_C(\mathbb{C}^k,m)} c(T\mathsf{Quot}_C(\mathbb{C}^k,m))\cdot c(\mathcal{T}_m)\cdot \prod_{i=1}^{\ell} \frac{c(\Theta^{[m]}[-w_i])}{c\left(\left(\Theta^{[m]}\right)^\vee[w_i]\right)}\cdot \frac{c\left(\left(K_C^{[m]}\right)^\vee[w_i]\right)}{\mathsf{e}\left(\left(K_C^{[m]}\right)^\vee[w_i]\right)}\,.$$

In the integrand, twists by trivial bundles with nontrivial equivariant weights are included. We consider the function above as a $\mathbb{C}^\star$-equivariant integral given by the $\mathbb{C}^\star$-action on the Quot scheme with weights $w_{\ell+1},\ldots,w_N$. The function $\mathsf{Z}_{C,k}$ depends on q and on the weights w. By an algebraic cobordism argument, we see

$$\mathsf{Z}_{C,k} = \mathsf{A}^{1-g}$$

where

$$\mathsf{A} = \mathsf{A}(q\,|\,w_1,\ldots,w_\ell\,|\,w_{\ell+1},\ldots,w_N)$$

is a universal function which does not depend on the genus g of C.

We will apply $\mathbb{C}^\star$-equivariant localization to the integrals appearing in the formula for $\mathsf{Z}_{C,k}$. The result is related to (68): each integral in $\mathsf{Z}_{C,k}$ becomes a sum of contributions of the fixed loci

$$\iota : C^{[m_1]}\times\cdots\times C^{[m_k]} \hookrightarrow \mathsf{Quot}_C(\mathbb{C}^k,m)\,.$$

We note the restrictions

$$\iota^\star T\mathsf{Quot}_C(\mathbb{C}^k,m) = \sum_{j=1}^{k} TC^{[m_j]} + \sum_{j\neq j'} \mathsf{N}_{jj'}\,,$$

$$\iota^\star\mathcal{T}_m = \sum_{j\neq j'} \mathsf{T}_{jj'} + \sum_{j=1}^{k}(\Theta^{[m_j]} - (\Theta^{[m_j]})^\vee)\,.$$

Here, for $j = j'$, we have used

$$\mathrm{Ext}^\bullet(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Theta) = \Theta^{[m]} - \left(\Theta^{[m]}\right)^\vee .$$

Furthermore, the $\mathbb{C}^\star$-equivariant restrictions of the tautological bundles $\Theta^{[m]}$ on the Quot scheme to the fixed loci are given by

$$\iota^\star \Theta^{[m]}[-w_i] = \sum_{j=1}^{k} \Theta^{[m_j]}[w_{j+\ell} - w_i]\,,$$

$$\iota^\star \left(\Theta^{[m]}\right)^\vee [w_i] = \sum_{j=1}^{k} (\Theta^{[m_j]})^\vee [w_i - w_{j+\ell}]\,,$$

where the sign $-w_{j+\ell}$ on the second line appears because the dual was taken. Finally,

$$\iota^\star \left(K_C^{[m]}\right)^\vee [w_i] = \sum_{j=1}^{k} (K_C^{[m_j]})^\vee [w_i - w_{j+\ell}]\,.$$

The above $\mathbb{C}^\star$-equivariant localization terms of $\mathsf{Z}_{C,k}$ match expression (68) up to a common factor and signs. Summarizing, we find:

$$\mathsf{Z}^{\mathcal{E}}_{X,N,\ell K_X}(q) = (-1)^{\ell\chi}\, q^{\ell(1-g)} \cdot \sum \left(\prod_{i=1}^{\ell} \prod_{j=1}^{k} \frac{1 + w_i - w_{j+\ell}}{w_i - w_{j+\ell}} \right)^{1-g} \cdot \mathsf{A}((-1)^{\ell+1} q \,|\, w_1, \ldots, w_\ell \,|\, w_{\ell+1}, \ldots, w_N\,)^{1-g}\,.$$

We write

$$\widetilde{\mathsf{A}}(q \,|\, w_1, \ldots, w_\ell \,|\, w_{\ell+1}, \ldots, w_N) = \prod_{i=1}^{\ell} \prod_{j=1}^{k} \frac{1 + w_i - w_{j+\ell}}{w_i - w_{j+\ell}} \cdot \mathsf{A}((-1)^{\ell+1} q \,|\, w_1, \ldots, w_\ell \,|\, w_{\ell+1}, \ldots, w_N\,)\,,$$

so we have

$$\mathsf{Z}^{\mathcal{E}}_{X,N,\ell K_X}(q) = (-1)^{\ell\chi}\, q^{\ell(1-g)} \cdot \sum \widetilde{\mathsf{A}}(q \,|\, w_1, \ldots, w_\ell \,|\, w_{\ell+1}, \ldots, w_N)^{1-g}.$$

The last remaining step is to determine the function A. After specializing the curve $C = \mathbb{P}^1$, we have

$$\mathsf{A}(q \,|\, w_1, \ldots, w_\ell \,|\, w_{\ell+1}, \ldots, w_N) =$$

$$\sum_{m=0}^{\infty} q^m \int_{\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^k, m)} c(T\mathsf{Quot}_C(\mathbb{C}^k, m)) \cdot c(\mathcal{T}_m) \cdot \prod_{i=1}^{\ell} \frac{c(\Theta^{[m]}[-w_i])}{c\left(\left(\Theta^{[m]}\right)^\vee [w_i]\right)} \cdot \frac{c\left(\left(K_C^{[m]}\right)^\vee [w_i]\right)}{\mathsf{e}\left(\left(K_C^{[m]}\right)^\vee [w_i]\right)}\,.$$

All tautological structures in the above integral have been understood in the proof of Theorem 18. In fact, compared to the integrals which appear in the proof of Theorem 18, the only new terms are

$$\prod_{i=1}^{\ell}\prod_{j=1}^{k}\frac{c(\Theta^{[m_j]}[w_{j+\ell}-w_i])}{c\left(\left(\Theta^{[m_j]}\right)^{\vee}[w_i-w_{j+\ell}]\right)}\cdot\frac{c\left(\left(K_C^{[m_j]}\right)^{\vee}[w_i-w_{j+\ell}]\right)}{\mathsf{e}\left(\left(K_C^{[m_j]}\right)^{\vee}[w_i-w_{j+\ell}]\right)}$$

considered over the product

$$\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_k}\,. \tag{70}$$

As before, we write $h_1,\ldots,h_k$ for the hyperplane classes on the respective projective spaces in the product (70). Using Lemma 27, we obtain

$$c\left(\Theta^{[m]}[w]\right)=(1-h+w)^m\,,\quad c\left((\Theta^{[m]})^{\vee}[-w]\right)=(1+h-w)^m\,,$$

$$c\left(\left(K_C^{[m]}\right)^{\vee}[-w]\right)=\frac{(1+h-w)^{m+1}}{1-w}\,,\quad \mathsf{e}\left(\left(K_C^{[m]}\right)^{\vee}[-w]\right)=\frac{(h-w)^{m+1}}{-w}\,.$$

The new terms contribute the expression

$$\prod_{i=1}^{\ell}\prod_{j=1}^{k}\frac{(1-h_j+w_{j+\ell}-w_i)^{m_j}}{(h_j+w_i-w_{j+\ell})^{m_j}}\cdot\frac{1+h_j+w_i-w_{j+\ell}}{1+w_i-w_{j+\ell}}\cdot\frac{w_i-w_{j+\ell}}{h_j+w_i-w_{j+\ell}}\,.$$

Therefore, using (50), the contribution of the fixed locus of $\mathsf{Quot}^1_{\mathbb{P}}(\mathbb{C}^k,m)$ corresponding to the partition $(m_1,\ldots,m_k)$ equals

$$(-1)^{m(k-1)+\binom{k}{2}}\int_{\mathbb{P}^{m_1}\times\ldots\times\mathbb{P}^{m_k}}\Phi_1(h_1)^{m_1}\cdots\Phi_k(h_k)^{m_k}\cdot\Psi(h_1,\ldots,h_k)$$

where

$$\Phi_j(h_j)=\prod_{j'=1}^{k}(1-h_j+w_{j+\ell}-w_{j'+\ell})\cdot\prod_{j'\neq j}(h_j+w_{j'+\ell}-w_{j+\ell})^{-1}\cdot\prod_{i=1}^{\ell}\frac{1-h_j+w_{j+\ell}-w_i}{h_j+w_i-w_{j+\ell}}\,,$$

and

$$\Psi=\prod_{j'<j}(h_j-h_{j'}+w_{j'+\ell}-w_{j+\ell})^2\cdot\prod_{j,j'}(1+h_j+w_{j'+\ell}-w_{j+\ell})\cdot(1+h_j-h_{j'}+w_{j'+\ell}-w_{j+\ell})^{-1}$$

$$\cdot\prod_{j\neq j'}(h_j+w_{j'+\ell}-w_{j+\ell})^{-1}\cdot\prod_{i=1}^{\ell}\prod_{j=1}^{k}\frac{1+h_j+w_i-w_{j+\ell}}{h_j+w_i-w_{j+\ell}}\cdot\frac{w_i-w_{j+\ell}}{1+w_i-w_{j+\ell}}\,.$$

Only the products involving i are different from the expressions written in the proof of Theorem 18.

We now apply the Lagrange-Bürmann formula for the change of variables

$$t_j\quad=\quad\frac{h_j}{\Phi_j(h_j)}=\prod_{\alpha=1}^{N}\frac{h_j+w_\alpha-w_{j+\ell}}{1-h_j+w_{j+\ell}-w_\alpha}.$$

In the above product, the index α collects the terms in Φ_j corresponding to both i and j' into a uniform expression. We find

$$\mathsf{A} = (-1)^{\binom{k}{2}} \cdot \frac{\Psi}{K}(h_1, \ldots, h_k)$$

where

$$q(-1)^{k-1} = t_j = \prod_{\alpha=1}^{N} \frac{h_j + w_\alpha - w_{j+\ell}}{1 - h_j + w_{j+\ell} - w_\alpha} \,,$$

Let

$$\widetilde{\Psi} = \prod_{j'<j} (h_j - h_{j'} + w_{j'+\ell} - w_{j+\ell})^2 \cdot \prod_{j,j'} (1 + h_j + w_{j'+\ell} - w_{j+\ell}) \cdot (1 + h_j - h_{j'} + w_{j'+\ell} - w_{j+\ell})^{-1} \cdot$$

$$\prod_{j \neq j'} (h_j + w_{j'+\ell} - w_{j+\ell})^{-1} \cdot \prod_{i=1}^{\ell} \prod_{j=1}^{k} \frac{1 + h_j + w_i - w_{j+\ell}}{h_j + w_i - w_{j+\ell}} \,.$$

We find

$$\widetilde{\mathsf{A}}(q \,|\, w_1, \ldots, w_\ell \,|\, w_{\ell+1}, \ldots, w_N) = (-1)^{\binom{k}{2}} \frac{\widetilde{\Psi}}{K}(h_1, \ldots, h_k)$$

where, taking all signs into account, we have

$$q(-1)^{(k-1)+(\ell+1)} = \prod_{\alpha=1}^{N} \frac{h_j + w_\alpha - w_{j+\ell}}{1 - h_j + w_{j+\ell} - w_\alpha} \,.$$

In the limit $w \to 0$, the above equation becomes

$$q(-1)^N = h^N (1-h)^{-N} \,.$$

The limit is justified as in the proof of Theorem 18: we let $H_1, \ldots, H_N$ be the roots of the single equation

$$q(-1)^N = \prod_{\alpha=1}^{N} \frac{h + w_\alpha - w_1}{1 - h + w_1 - w_\alpha} \,,$$

and then we have

$$h_j = H_{j+\ell} + w_{j+\ell} - w_1 \,.$$

The final answer is a sum of $\binom{N}{k}$ terms corresponding to choices of subsets of k roots out of $H_1, \ldots, H_N$. Using the explicit expressions for $\tilde{\Psi}$ and K, the answer is seen to be symmetric in the H's and, therefore, expressible in terms of the elementary symmetric functions which are polynomials in w.

We find $\frac{\widetilde{\Psi}}{K}$ simplifies in the limit to the expression

$$\prod_{j<j'} (h_j - h_{j'})^2 \cdot (1 + h_j)^k \cdot \prod_{j,j'} (1 - (h_{j'} - h_j))^{-1} \cdot \prod_{j=1}^{k} h_j^{-(k-1)} \cdot \prod_{j=1}^{k} \frac{(1+h_j)^\ell}{h_j^\ell} \cdot \prod_{j=1}^{k} \frac{1 - h_j}{N}$$

where the last product comes from the K-term. Further simplification yields

$$\frac{1}{N^k}\prod_{j<j'}\frac{(h_j-h_{j'})^2}{1-(h_j-h_{j'})^2}\cdot\prod_{j=1}^{k}\frac{(1+h_j)^N\cdot(1-h_j)}{h_j^{N-1}},$$

which is precisely the formula stated in Theorem 23. $\square$

Appendix A. A combinatorial proof of Theorem 11

We present here a purely combinatorial argument for Theorem 11. For simplicity of notation, we consider trees whose edges are painted in only two colors denoted A and B. The generalization to several colors does not require additional ideas.

We write a for the total number of A edges, b for the number of B edges, and n for the number of vertices of a tree T. Clearly

$$a+b=n-1\,.$$

For each vertex v, we write a_v and b_v for the number of outgoing edges colored A and B respectively. Therefore,

$$\mathsf{wt}(T)=\frac{1}{(n-1)!}\prod_v a_v\,!\,b_v\,!\,.$$

We set

$$w_n(a,b)=\sum_T \mathsf{wt}(T)\,.$$

Let

$$t_n(a,b)=\frac{1}{n}\binom{2a+b}{a}\binom{a+2b}{b}.$$

The claim of Theorem 11 in the case of two colors is

$$w_n(a,b)=t_n(a,b)\,. \tag{71}$$

Define the generating series

$$\mathsf{W}(q\,|\,x,y)=\sum_{n=1}^{\infty}\sum_{a+b=n-1} w_n(a,b)\cdot x^a y^b q^n\,,$$

$$\mathsf{T}(q\,|\,x,y)=\sum_{n=1}^{\infty}\sum_{a+b=n-1} t_n(a,b)\cdot x^a y^b q^n\,.$$

By Lemmas 41 and 42 below, both W and T satisfy the cubic equation

$$\mathsf{Z}\cdot(1-x\mathsf{Z})\cdot(1-y\mathsf{Z})=q\,,\quad \mathsf{Z}|_{q=0}=0\,. \tag{72}$$

Since the solution of (72) is unique, we obtain

$$\mathsf{W}=\mathsf{T}$$

which implies (71) and completes the proof of Theorem 11.

Lemma 41. *We have*

$$\mathsf{T}\cdot(1-x\mathsf{T})\cdot(1-y\mathsf{T})=q\,.$$

Proof. The argument exactly follows the proof of Lemma 28. Set

$$f(t)=(1-xt)^{-1}(1-yt)^{-1}\,.$$

For $a+b=n-1$, we have

$$t_n(a,b)=\frac{(-1)^{n-1}}{n}\binom{-n}{a}\binom{-n}{b}\,.$$

Therefore,

$$\begin{aligned}\mathsf{T}(q) &= \sum_{n=1}^{\infty}\frac{q^n}{n}\cdot\left(\left[t^{n-1}\right](1-xt)^{-n}(1-yt)^{-n}\right)\\ &= \sum_{n=1}^{\infty}\frac{q^n}{n}\cdot\left(\left[t^{n-1}\right]f(t)^{n-1}\cdot f(t)\right)\,.\end{aligned}$$

Then,

$$\frac{d\mathsf{T}}{dq}=\sum_{n=1}^{\infty}q^{n-1}\cdot\left(\left[t^{n-1}\right]f(t)^{n-1}\cdot f(t)\right)=\frac{dt}{dq}\,,$$

where equation (16) was used above for the change of variables $q=\frac{t}{f(t)}$. Hence, we obtain

$$\mathsf{T}=t\,,$$

and the change of variables proves the Lemma. □

Lemma 42. *We have*

$$\mathsf{W}\cdot(1-x\mathsf{W})\cdot(1-y\mathsf{W})=q\,.$$

Proof. We will prove a recursion for $w_n(a,b)$ which implies the cubic equation of the Lemma. For convenience, we set $w_n(a,b)=0$ whenever $a+b\neq n+1$.

Fix a labelled 2-colored tree T with n vertices. Consider the vertex $\star$ with the *highest* label n. After removing the vertex $\star$ and all its incident edges from the tree T, we obtain disjoint subtrees $T_1,\ldots,T_\ell$. We set up the following notation:

- r and s denote the number of edges incident to the vertex $\star$ which are colored A and B respectively (where $\ell=r+s$),
- $n_1,\ldots,n_\ell$ are the number of vertices of the subtrees $T_1,\ldots,T_\ell$ respectively,
- $(a_1,b_1),\ldots,(a_\ell,b_\ell)$ are the numbers of edges of each color for subtrees $T_1,\ldots,T_\ell$.

The above quantities satisfy various constraints which are most easily expressed using partitions. We denote an ordered partitions by

$$\alpha^\bullet = (\alpha_1, \ldots, \alpha_\ell)\,,$$

and we write $|\alpha^\bullet|$ for the sum of parts. Then

$$|n^\bullet| = n-1\,, \quad |a^\bullet| + r = a \text{ (counting } A \text{ edges)}\,, \quad |b^\bullet| + s = b \text{ (counting } B \text{ edges)}\,.$$

The removal of the vertex $\star$ yields the following recursion:

$$w_n(a,b) = \sum \eta_{n^\bullet} \cdot w_{n_1}(a_1,b_1)\cdots w_{n_\ell}(a_\ell,b_\ell) \tag{73}$$

with the combinatorial factor

$$\eta_{n^\bullet} = \frac{r!}{\mathrm{Aut}(n_1,\ldots,n_r)} \cdot \frac{s!}{\mathrm{Aut}(n_{r+1},\ldots,n_\ell)}\,.$$

Indeed, the vertex $\star$ contributes $r!\,s!$ to the weight of T, while the other vertices are contained in one of the trees $T_1, \ldots, T_\ell$. Therefore

$$\mathsf{wt}(T) = \frac{1}{(n-1)!} r\,!\,s! \cdot \prod_{j=1}^{\ell} (n_j - 1)!\,\mathrm{wt}(T_j)\,.$$

After summing over all trees, we obtain

$$w_n(a,b) \quad = \quad \sum_T \mathsf{wt}(T) = \sum c_{n^\bullet} \cdot \frac{1}{(n-1)!} r\,!\,s! \cdot \prod_{j=1}^{\ell} (n_j-1)!\, w_{n_j}(a_j,b_j)\,.$$

The combinatorial factor

$$c_{n^\bullet} = (n_1 \cdots n_\ell) \cdot \binom{n-1}{n_1,\ldots,n_\ell} \cdot \frac{1}{\mathrm{Aut}(n_1,\ldots,n_r)} \cdot \frac{1}{\mathrm{Aut}(n_{r+1},\ldots,n_\ell)}$$

arises as follows

- the term $n_1 \cdots n_\ell$ counts all possible ways to attach the vertex $\star$ to one of the n_j vertices of the tree T_j, for $1 \le j \le \ell$,
- $\binom{n-1}{n_1,\ldots,n_\ell}$ counts all possible ways of distributing the labels $\{1, \ldots, n-1\}$ to the trees $T_1, \ldots, T_\ell$,
- the last two terms account for automorphisms.

Equation (73) then follows by collecting terms.

For notational convenience, we define the relabelling

$$n'_j = n_{j+r}\,, \quad a'_j = a_{j+r}\,, \quad b'_j = b_{j+r},\ \ 1 \le j \le s\,.$$

In the new notation, recursion (73) takes the form:

$$w_n(a,b) = \sum \frac{r!}{\mathrm{Aut}(n^\bullet)} \cdot \frac{s!}{\mathrm{Aut}(n'^\bullet)} \cdot \prod_{j=1}^{r} w_{n_j}(a_j,b_j) \cdot \prod_{j=1}^{s} w_{n'_j}(a'_j,b'_j)\,.$$

We define

$$\mathsf{W}_n = \sum_{a+b=n-1} w_n(a,b) \cdot x^a y^b$$

satisfying

$$\mathsf{W} = \sum_{n=1}^{\infty} \sum_{a+b=n-1} w_n(a,b) \cdot x^a y^b q^n = \sum_{n=1}^{\infty} q^n \mathsf{W}_n \, .$$

We compute

$$\begin{aligned}
\frac{\mathsf{W}}{q} &= \sum_{n=1}^{\infty} w_n(a,b) x^a y^b q^{n-1} \\
&= \sum \frac{r!}{\mathrm{Aut}(n^\bullet)} \cdot \frac{s!}{\mathrm{Aut}(n'^\bullet)} \cdot \prod_{j=1}^{r} w_{n_j}(a_j, b_j) \cdot \prod_{j=1}^{s} w_{n'_j}(a'_j, b'_j) \cdot x^{|a^\bullet| + |a'^\bullet|} y^{|b^\bullet| + |b'^\bullet|} x^r y^s q^{|n^\bullet| + |n'^\bullet|} \\
&= \sum \frac{r!}{\mathrm{Aut}(n^\bullet)} \cdot \frac{s!}{\mathrm{Aut}(n'^\bullet)} \cdot \prod_{j=1}^{r} \mathsf{W}_{n_j} \cdot \prod_{j=1}^{s} \mathsf{W}_{n'_j} \cdot x^r y^s q^{|n^\bullet| + |n'^\bullet|} \\
&= \left(\sum \frac{r!}{\mathrm{Aut}(n^\bullet)} \prod_{j=1}^{r} \mathsf{W}_{n_j} \cdot x^r q^{|n^\bullet|} \right) \cdot \left(\sum \frac{s!}{\mathrm{Aut}(n'^\bullet)} \prod_{j=1}^{s} \mathsf{W}_{n'_j} \cdot y^s q^{|n'^\bullet|} \right) \\
&= (1 - x\mathsf{W})^{-1} \cdot (1 - y\mathsf{W})^{-1} \, ,
\end{aligned}$$

where, on the third line, we have summed over the a's and b's.

For the last line, we have used the identity

$$\frac{1}{1 - x\mathsf{W}} = \sum \frac{r!}{\mathrm{Aut}(n^\bullet)} \prod_{j=1}^{r} \mathsf{W}_{n_j} \cdot x^r q^{|n^\bullet|} \tag{74}$$

which is easily derived from the Binomial Theorem. Indeed, after setting

$$\alpha_n = \mathsf{W}_n \cdot x q^n \, , \quad \alpha = \sum \alpha_n = x\mathsf{W} \, ,$$

equation (74) becomes

$$\frac{1}{1 - \alpha} = \sum \frac{r!}{\mathrm{Aut}(n^\bullet)} \prod_{j=1}^{r} \alpha_{n_j} \, ,$$

which is true since the two sides are different ways of expressing $\sum_r \alpha^r$. □

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