

SEGRE CLASSES AND HILBERT SCHEMES OF POINTS

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ABSTRACT. We prove a closed formula for the integrals of the top Segre classes of tautological bundles over the Hilbert schemes of points of a $K3$ surface X . We derive relations among the Segre classes via equivariant localization of the virtual fundamental classes of Quot schemes on X . The resulting recursions are then solved explicitly. The formula proves the K -trivial case of a conjecture of M. Lehn from 1999.

The relations determining the Segre classes fit into a much wider theory. By localizing the virtual classes of certain relative Quot schemes on surfaces, we obtain new systems of relations among tautological classes on moduli spaces of surfaces and their relative Hilbert schemes of points. For the moduli of $K3$ surfaces, we produce relations intertwining the κ classes and the Noether-Lefschetz loci. Conjectures are proposed.

0. INTRODUCTION

0.1. **Segre classes.** Let (S, H) be a pair consisting of a nonsingular projective surface S and a line bundle

$$H \rightarrow S.$$

The *degree* of the pair (S, H) is defined via the intersection product on S ,

$$H \cdot H = \int_S H^2 \in \mathbb{Z}.$$

The Hilbert scheme of points $S^{[n]}$ carries a tautological rank n vector bundle $H^{[n]}$ whose fiber over $\zeta \in S^{[n]}$ is given by

$$\zeta \mapsto H^0(H \otimes \mathcal{O}_\zeta).$$

The top Segre class

$$N_{S,H,n} = \int_{S^{[n]}} s_{2n}(H^{[n]})$$

appeared first in the algebraic study of Donaldson invariants via the moduli space of rank 2 bundles on S [26]. Such Segre classes play a basic role in the Donaldson-Thomas counting of sheaves (often entering via the obstruction theory). A classical interpretation of $N_{S,H,n}$ is also available. If $|H|$ is a linear system of dimension $3n - 2$ which induces a map

$$S \rightarrow \mathbb{P}^{3n-2},$$

$N_{S,H,n}$ counts the n -chords of dimension $n - 2$ to the image of S .

The main result of the paper is the calculation of the top Segre classes for all pairs (X, H) in the $K3$ case.

Theorem 1. *If (X, H) is a nonsingular K3 surface of degree 2ℓ , then*

$$\int_{X^{[n]}} s_{2n}(H^{[n]}) = 2^n \binom{\ell - 2n + 2}{n}.$$

0.2. Lehn's conjecture. Let S be a nonsingular projective surface. The Segre class $N_{S,H,n}$ can be expressed as a polynomial of degree n in the four variables

$$H^2, H \cdot K_S, K_S^2, c_2(S),$$

see [24] for a proof. Furthermore, the form

$$(1) \quad \sum_{n=0}^{\infty} N_{S,H,n} z^n = \exp \left(H^2 \cdot A_1(z) + (H \cdot K_S) \cdot A_2(z) + K_S^2 \cdot A_3(z) + c_2(S) \cdot A_4(z) \right)$$

in terms of four universal power series A_1, A_2, A_3, A_4 was proven in [7]. The formulas for the four power series were explicitly conjectured by M. Lehn in 1999.

Conjecture 1 (Lehn [14]). *We have*

$$(2) \quad \sum_{n=0}^{\infty} N_{S,H,n} z^n = \frac{(1-w)^a (1-2w)^b}{(1-6w+6w^2)^c}$$

for the change of variable

$$z = \frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3}$$

and constants

$$a = H \cdot K_S - 2K_S^2, \quad b = (H - K_S)^2 + 3\chi(\mathcal{O}_S), \quad c = \frac{1}{2}H(H - K_S) + \chi(\mathcal{O}_S).$$

As usual in the study of the Hilbert scheme of points, Theorem 1 determines two of the power series in (1). Specifically, Theorem 1 implies

$$(3) \quad A_1 \left(\frac{1}{2}t(1+t)^2 \right) = \frac{1}{2} \log(1+t),$$

$$(4) \quad A_4 \left(\frac{1}{2}t(1+t)^2 \right) = \frac{1}{8} \log(1+t) - \frac{1}{24} \log(1+3t).$$

The evaluation of A_1 and A_4 proves Lehn's conjecture for all surfaces with numerically trivial canonical bundle.

Corollary 1. *If (A, H) is an abelian or bielliptic surface of degree 2ℓ , then*

$$\int_{A^{[n]}} s_{2n}(H^{[n]}) = \frac{2^n \ell}{n} \binom{\ell - 2n - 1}{n-1}.$$

Corollary 2. *If (E, H) is an Enriques surface of degree 2ℓ , then*

$$\left(\sum_{n=0}^{\infty} N_{E,H,n} z^n \right)^2 = \sum_{n=0}^{\infty} 2^n \binom{2\ell - 2n + 2}{n} z^n.$$

0.3. Strategy of the proof. The intersection theory of the Hilbert scheme of points can be approached via the inductive recursions set up in [7] or via the Nakajima calculus [14, 19]. By these methods, the integration of tautological classes is reduced to a combinatorial problem. Another strategy is to prove an equivariant version of Lehn’s conjecture for the Hilbert scheme of points of \mathbb{C}^2 via appropriately weighted sums over partitions. However, we do not know how to prove Theorem 1 along these lines.¹

Let (X, H) be a nonsingular projective $K3$ surface. We consider integrals over the Quot scheme $\mathcal{Q}_{H, \chi}(\mathbb{C}^2)$ parametrizing quotients

$$\mathbb{C}^2 \otimes \mathcal{O}_X \rightarrow F \rightarrow 0$$

where F is a rank 0 coherent sheaf satisfying

$$c_1(F) = H \quad \text{and} \quad \chi(F) = \chi.$$

The Quot scheme admits a reduced virtual class, and the integrals

$$\int_{[\mathcal{Q}_{H, \chi}(\mathbb{C}^2)]^{\text{red}}} \gamma \cdot 0^k$$

vanish for all $k > 0$ and for all choices of cohomology classes γ . Here, the notation 0 stands for the first Chern class of the trivial line bundle

$$c_1(\mathcal{O}) = 0 \in A^1(\mathcal{Q}_{H, \chi}(\mathbb{C}^2)).$$

Virtual localization [9] with respect to a \mathbb{C}^* action applied to the above integrals yields linear recursions between the expressions

$$\int_{X^{[n]}} c_{n-i}(H^{[n]}) s_{n+i}(H^{[n]}).$$

The linear recursions are trivial for all but finitely many values of $H^2 = 2\ell$. The nontrivial recursions can be solved to show that the top Segre integrals vanish for the values

$$2n - 2 \leq \ell \leq 3n - 3.$$

These vanishings determine the intersections up to an ambiguity given by the leading term of a polynomial which we can calculate explicitly.

0.4. The moduli space of $K3$ surfaces. The relations used to prove Theorem 1 fit into a wider program aimed at studying the tautological rings of the moduli space of surfaces. Consider the relative Quot scheme $\mathcal{Q}_{H, \chi}^{\text{rel}}(\mathbb{C}^2)$ over a family of smooth surfaces. We evaluate on the base, via equivariant localization, the vanishing pushforwards

$$\pi_* \left(\gamma \cdot 0^k \cap \left[\mathcal{Q}_{H, \chi}^{\text{rel}}(\mathbb{C}^2) \right]^{\text{vir}} \right),$$

¹The parallel problem in dimension 1, the calculation of the Segre classes of tautological bundles over Hilbert schemes of points of nonsingular *curves*, has been solved in [5, 13, 27].

for all $k > 0$ and various choices of cohomology classes γ . The Segre classes of the tautological bundles over the relative Hilbert schemes of points appear naturally in the localization output. In cohomological degree zero on the base, the resulting equations lead to Lehn's formulas above. In higher cohomological degree, the analysis of the localization output is increasingly harder, and the calculations are more intricate. They give rise to new and rich systems of relations among tautological classes on moduli spaces of surfaces and their relative Hilbert schemes of points.

We illustrate this program in Section 4 by concrete examples for the moduli of quasipolarized $K3$ surfaces. We obtain in this fashion relations intertwining the κ classes and the Noether-Lefschetz loci. These calculations point to general conjectures.

0.5. Plan of the paper. Section 1 concerns localization on Quot schemes. Foundational aspects of the virtual classes of Quot schemes on surfaces S are discussed in Section 1.1. The virtual localization formula for the \mathbb{C}^* action on the Quot schemes of $K3$ surfaces is presented in Section 1.2. Explicit localization relations are derived in Section 1.3.

Theorem 1 is proven in Section 2 by solving the recursion relations of Section 1.3. In Section 3, the connections between Theorem 1 and Lehn's conjecture are explained (and Corollaries 1 and 2 are proven). An application to elliptically fibered surfaces is given in Corollary 3 of Section 3.

In Section 4, we discuss the tautological classes of the moduli of $K3$ surfaces. In particular, we write down relations in the tautological ring and formulate conjectures.

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1. LOCALIZATION ON THE QUOT SCHEME

1.1. Geometric setup. Let S be a nonsingular projective surface equipped with a divisor class H . We consider the Quot scheme $\mathcal{Q}_{H,\chi}(\mathbb{C}^r)$ parametrizing short exact sequences

$$0 \rightarrow E \rightarrow \mathbb{C}^r \otimes \mathcal{O}_S \rightarrow F \rightarrow 0$$

where F is a rank 0 coherent sheaf satisfying

$$c_1(F) = H \quad \text{and} \quad \chi(F) = \chi.$$

With the exception of the Hilbert scheme of points

$$S^{[n]} = \mathcal{Q}_{0,\chi}(\mathbb{C}^1)$$

the intersection theory of Quot schemes over surfaces has not been extensively studied. Rank 1 calculation can be found in [6], and higher rank calculations over del Pezzo surfaces were considered in [23].

In comparison, the intersection theory of the Quot scheme of a *curve* may be pursued in a virtual sense for a fixed curve [15] or by letting the curve vary via the moduli space of stable quotients [16]. The relations in the tautological ring of the moduli of curves [11, 21, 22] via virtual localization on the moduli stable quotients are parallel to the relations we introduce in Section 4.

Fundamental to our study is the following result (which we will use here only in the $r = 2$ case).

Lemma 1. *The Quot scheme $\mathcal{Q}_{H,\chi}(\mathbb{C}^r)$ admits a canonical perfect obstruction theory of virtual dimension $r\chi + H^2$.*

Proof. Since the details are similar to the curve case [15], we only discuss the main points. The obstruction theory of the Quot scheme is governed by the groups $\text{Ext}^i(E, F)$. We claim the vanishing

$$\text{Ext}^2(E, F) = \text{Ext}^0(F, E \otimes K_S)^\vee = 0.$$

Indeed, since E is a subsheaf of $\mathbb{C}^r \otimes \mathcal{O}_S$, the latter group injects

$$\text{Ext}^0(F, E \otimes K_S) \hookrightarrow \text{Ext}^0(F, \mathbb{C}^r \otimes K_S) = 0,$$

where the last vanishing follows since F is torsion. As a consequence, the difference

$$(5) \quad \text{Ext}^0(E, F) - \text{Ext}^1(E, F) = \chi(E, F) = r\chi + H^2$$

is constant.

Since the higher obstructions vanish, the moduli space $\mathcal{Q}_{H,\chi}(\mathbb{C}^r)$ carries a virtual fundamental class of dimension (5). \square

Let (X, H) be a primitively polarized $K3$ surface of degree 2ℓ and Picard rank 1. Let

$$H^2 = 2\ell \quad \text{and} \quad n = \chi + \ell.$$

In the $K3$ case, we show the Quot scheme admits a reduced virtual fundamental class.

Lemma 2. *For a $K3$ surface (X, H) there is a natural surjective map*

$$\text{Ext}^1(E, F) \rightarrow \mathbb{C},$$

and a reduced virtual fundamental class $[\mathcal{Q}_{H,\chi}(\mathbb{C}^r)]^{red}$ of dimension $r\chi + 2\ell + 1$.

Proof. The argument is standard. Indeed, the defining short exact sequence

$$0 \rightarrow E \rightarrow \mathbb{C}^r \otimes \mathcal{O}_X \rightarrow F \rightarrow 0$$

induces a natural morphism

$$\mathrm{Ext}^1(E, F) \rightarrow \mathrm{Ext}^2(F, F) \xrightarrow{\mathrm{Trace}} H^2(\mathcal{O}_X) = \mathbb{C}.$$

To prove surjectivity of the composition, it suffices to show

$$\mathrm{Ext}^1(E, F) \rightarrow \mathrm{Ext}^2(F, F)$$

is surjective, since the trace is surjective. The cokernel of the map is identified with

$$\mathrm{Ext}^2(\mathbb{C}^r, F) = H^2(F) \otimes \mathbb{C}^r$$

which vanishes since F has 1-dimensional support. The reduced virtual dimension equals

$$\chi(E, F) + 1 = \chi(\mathbb{C}^r, F) - \chi(F, F) + 1 = r\chi + H^2 + 1 = r\chi + 2\ell + 1,$$

where the summand 1 on the left is due to the reduction. \square

In the $r = 2$ case, the virtual dimension formula specializes to

$$2(\chi + \ell) + 1 = 2n + 1.$$

1.2. Virtual localization. We continue to work with a primitively polarized $K3$ surface (X, H) of degree 2ℓ and Picard rank 1.

We study the virtual intersection theory of the Quot scheme $\mathcal{Q}_{H,\chi}(\mathbb{C}^2)$ via the equivariant localization. To this end, consider the diagonal \mathbb{C}^\star action,

$$\mathbb{C}^2 = \mathbb{C}[0] + \mathbb{C}[1],$$

with weights 0 and 1 on the two summands. A torus action on $\mathcal{Q}_{H,\chi}(\mathbb{C}^2)$ is canonically induced. For a top degree Chow class α over the Quot scheme, the virtual localization formula of [9] reads:

$$(6) \quad \int_{[\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\mathrm{vir}}} \alpha = \sum_{\mathbb{F}} \int_{[\mathbb{F}]^{\mathrm{vir}}} \frac{\tilde{\alpha}|_{\mathbb{F}}}{e(\mathbf{N}_{\mathbb{F}}^{\mathrm{vir}})}.$$

On the right side, $\tilde{\alpha}$ is any lift of α to equivariant Chow. In addition,

- the \mathbb{F} 's are the torus fixed loci of $\mathcal{Q}_{H,\chi}(\mathbb{C}^2)$,
- $\mathbf{N}_{\mathbb{F}}^{\mathrm{vir}}$ are the virtual normal complexes of the fixed loci,
- $e(\cdot)$ stands for the equivariant Euler class.

For nonsingular projective varieties (endowed with the trivial perfect obstruction theory), equation (6) specializes to the standard Atiyah-Bott localization formula [1].

We turn now to the analysis of the fixed point loci. The torus fixed quotients come from kernels which split into rank 1 factors

$$0 \rightarrow E = E_1 \oplus E_2 \hookrightarrow \mathcal{O}_X \oplus \mathcal{O}_X \rightarrow F = F_1 \oplus F_2 \rightarrow 0.$$

Since $c_1(F) = H$, we have

$$-c_1(E_1) - c_1(E_2) = H.$$

The two sections show that E_1^\vee and E_2^\vee are effective line bundles. Since the curve class H is irreducible, the kernel must split as

$$E_1 = \mathcal{I}_Z, \quad E_2 = \mathcal{I}_W \otimes \mathcal{O}_X(-C) \quad \text{or} \quad E_1 = \mathcal{I}_Z \otimes \mathcal{O}_X(-C), \quad E_2 = \mathcal{I}_W,$$

where C is a curve in the linear series $|H|$ and

$$Z, W \subset X$$

are 0-dimensional subschemes of lengths z and w respectively. The condition $\chi(F) = \chi$ is equivalent to

$$z + w = \ell + \chi = n.$$

For each value of z and w , we obtain two distinct fixed loci, both isomorphic to the product

$$X^{[z]} \times X^{[w]} \times \mathbb{P}$$

where the projective space

$$\mathbb{P} = |H|$$

denotes the linear series in which the curve C varies.

We now turn to the virtual class and the virtual normal bundle of the fixed loci. Except in the two cases

- $(z, w) = (0, n)$ and $E_1 \oplus E_2 = \mathcal{O}_X \oplus \mathcal{I}_W \otimes \mathcal{O}_X(-C)$,
- $(z, w) = (n, 0)$ and $E_1 \oplus E_2 = \mathcal{I}_Z \otimes \mathcal{O}_X(-C) \oplus \mathcal{O}_X$,

the reduced obstruction bundle has an *additional* trivial summand which forces the localization contribution in (6) to vanish. The non-reduced obstruction bundle is obtained by considering the fixed part of $\text{Ext}^1(E, F)$ which splits as

$$\text{Ext}^1(E_1, F_1) + \text{Ext}^1(E_2, F_2).$$

By the proof of Lemma 2, if F_1 and F_2 are both non-zero, we have two surjective maps

$$\text{Ext}^1(E_1, F_1) \rightarrow \mathbb{C}, \quad \text{Ext}^1(E_2, F_2) \rightarrow \mathbb{C}.$$

Since the reduced virtual class is obtained by reducing only one trivial factor, a trivial factor still remains.

We now analyze the two surviving fixed components F^+ and F^- associated to the splittings

$$E_1 \oplus E_2 = \mathcal{O}_X \oplus \mathcal{I}_W \otimes \mathcal{O}_X(-C) \quad \text{and} \quad E_1 \oplus E_2 = \mathcal{I}_Z \otimes \mathcal{O}_X(-C) \oplus \mathcal{O}_X$$

respectively. The isomorphisms

$$F^+ \simeq F^- \simeq X^{[n]} \times \mathbb{P}$$

are immediate.

Let \mathcal{L} be the hyperplane class on \mathbb{P} ,

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}}(1).$$

The line bundle associated to the universal curve $\mathcal{C} \hookrightarrow X \times \mathbb{P}$ of the linear series $|H|$ is

$$\mathcal{O}(\mathcal{C}) = H \otimes \mathcal{L}.$$

On the universal surface of the fixed locus

$$F^+ \times X \rightarrow F^+,$$

the universal subsheaves are

$$\mathcal{E}_1 = \mathcal{O}, \quad \mathcal{E}_2 = \mathcal{I}_W \otimes \mathcal{O}(-C) = \mathcal{I}_W \otimes H^{-1} \otimes \mathcal{L}^{-1}.$$

To find the reduced virtual class of F^+ , we study the fixed part of the reduced obstruction theory:

$$\begin{aligned} (\text{Tan} - \text{Obs})^{\text{fix}} &= \text{Ext}^\bullet(\mathcal{E}_2, \mathcal{F}_2) + \mathbb{C} \\ &= \text{Ext}^\bullet(\mathcal{I}_W \otimes H^{-1} \otimes \mathcal{L}^{-1}, \mathbb{C} - \mathcal{I}_W \otimes H^{-1} \otimes \mathcal{L}^{-1}) + \mathbb{C} \\ &= \text{Ext}^\bullet(\mathcal{I}_W \otimes H^{-1} \otimes \mathcal{L}^{-1}, \mathbb{C}) - \text{Ext}^\bullet(\mathcal{I}_W, \mathcal{I}_W) + \mathbb{C} \\ &= \text{Ext}^\bullet(\mathcal{I}_W \otimes H^{-1}, \mathbb{C}) \otimes \mathcal{L} - \text{Ext}^\bullet(\mathcal{I}_W, \mathcal{I}_W) + \mathbb{C} \\ &= H^0(X, H) \otimes \mathcal{L} - \text{Ext}^\bullet(\mathcal{O}_W \otimes H^{-1}, \mathbb{C}) \otimes \mathcal{L} - \text{Ext}^\bullet(\mathcal{I}_W, \mathcal{I}_W) + \mathbb{C}. \end{aligned}$$

The latter expression simplifies using the following four observations:

- (i) $\text{Ext}^1(\mathcal{I}_W, \mathcal{I}_W) \simeq \text{Ext}^0(\mathcal{I}_W, \mathcal{O}_W)$ is the tangent space to $X^{[n]}$ at W ,
- (ii) $\text{Ext}^2(\mathcal{I}_W, \mathcal{I}_W) = \text{Ext}^0(\mathcal{I}_W, \mathcal{I}_W) = \mathbb{C}$ by Serre duality,
- (iii) $\text{Ext}^0(\mathcal{O}_W \otimes H^{-1}, \mathbb{C}) = \text{Ext}^1(\mathcal{O}_W \otimes H^{-1}, \mathbb{C}) = 0$ since W is of dimension 0,
- (iv) $\text{Ext}^2(\mathcal{O}_W \otimes H^{-1}, \mathbb{C}) = H^0(H^{-1} \otimes \mathcal{O}_W)^\vee$ by Serre duality.

Applying (i-iv), we find

$$\begin{aligned} (\text{Tan} - \text{Obs})^{\text{fix}} &= \mathbb{C}^{\ell+2} \otimes \mathcal{L} - \mathcal{L} \otimes \left((H^{-1})^{[n]} \right)^\vee + \text{Tan}_{X^{[n]}} - \mathbb{C} \\ &= \text{Tan}_{\mathbb{P}} - \mathcal{L} \otimes \left((H^{-1})^{[n]} \right)^\vee + \text{Tan}_{X^{[n]}} \end{aligned}$$

where we have also used the Euler sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathbb{C}^{\ell+2} \otimes \mathcal{L} \rightarrow \text{Tan}_{\mathbb{P}} \rightarrow 0.$$

Therefore, over the fixed locus F^+ , we have the obstruction bundle

$$\text{Obs} = \mathcal{L} \otimes \left((H^{-1})^{[n]} \right)^\vee.$$

As explained in Proposition 5.6 of [2], the virtual class of the fixed locus F^+ is obtained by taking the Euler class of the locally free Obs:

$$[F^+]^{\text{vir}} = e \left(\mathcal{L} \otimes \left((H^{-1})^{[n]} \right)^\vee \right).$$

The analysis of the fixed part of the reduced obstruction theory for F^- is identical. We find that the virtual class of the fixed locus is

$$[F^-]^{\text{vir}} = e \left(\mathcal{L} \otimes \left((H^{-1})^{[n]} \right)^\vee \right)$$

just as for F^+ .

Turning to the normal bundle, we study the moving part of the reduced obstruction theory on F^+ :

$$\begin{aligned} \mathbf{N}^+ &= \text{Ext}^\bullet(E_1, F_2) + \text{Ext}^\bullet(E_2, F_1) \\ &= \text{Ext}^\bullet(\mathcal{O}_X, F_2) \\ &= H^\bullet(F_2) \\ &= H^\bullet(\mathcal{O}_X) - H^\bullet(E_2) \\ &= \mathbb{C} + \mathbb{C} - H^\bullet(E_2). \end{aligned}$$

In families, $H^\bullet(E_2)$ equals

$$H^\bullet(H^{-1} \otimes \mathcal{L}^{-1} \otimes \mathcal{I}_{\mathcal{W}}) = \mathbb{C}^{\ell+2} \otimes \mathcal{L}^{-1} - H^\bullet(H^{-1} \otimes \mathcal{O}_{\mathcal{W}}) \otimes \mathcal{L}^{-1}.$$

Hence, we find

$$\mathbf{N}^+ = \left(\mathbb{C} + \mathbb{C} + \mathcal{L}^{-1} \otimes (H^{-1})^{[n]} - \mathcal{L}^{-1} \otimes \mathbb{C}^{\ell+2} \right) [1],$$

where the [1] indicates the torus weight. Similarly,

$$\mathbf{N}^- = \left(\mathbb{C} + \mathbb{C} + \mathcal{L}^{-1} \otimes (H^{-1})^{[n]} - \mathcal{L}^{-1} \otimes \mathbb{C}^{\ell+2} \right) [-1],$$

The normal complexes \mathbf{N}^\pm have virtual rank $n - \ell$. Taking into account the opposite equivariant weights for the two fixed loci, we write the inverses of the Euler classes of the virtual normal bundles over the two fixed loci as

$$\begin{aligned} \frac{1}{e(\mathbf{N}^+)} &= \frac{(1 - \zeta)^{\ell+2}}{c_+((H^{-1})^{[n]} \otimes \mathcal{L}^{-1})}, \\ \frac{1}{e(\mathbf{N}^-)} &= (-1)^{n-\ell} \cdot \frac{(1 + \zeta)^{\ell+2}}{c_-((H^{-1})^{[n]} \otimes \mathcal{L}^{-1})}. \end{aligned}$$

where ζ is the hyperplane class on \mathbb{P} . We have used the Chern class notation

$$c_+ = \sum_{i \geq 0} c_i \quad \text{and} \quad c_- = \sum_{i \geq 0} (-1)^i c_i.$$

1.3. The calculation. Let ζ also denote the pull-back of the hyperplane class under the support morphism

$$\mathcal{Q}_{H,\chi}(\mathbb{C}^2) \rightarrow \mathbb{P}, \quad [0 \rightarrow E \rightarrow \mathbb{C}^2 \otimes \mathcal{O}_X \rightarrow F \rightarrow 0] \mapsto \text{supp } F.$$

Consider the integrals

$$0 = I_k = \frac{(-1)^n}{2} \int_{[\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\text{red}}} \zeta^{2k-1} \cdot 0^{2n+2-2k}, \quad 1 \leq k \leq n,$$

against the reduced virtual class

$$[\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\text{red}} \in A_{2n+1}([\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\text{red}}).$$

The notation 0 stands for the first Chern class of the trivial line bundle

$$c_1(\mathcal{O}) = 0 \in A^1(\mathcal{Q}_{H,\chi}(\mathbb{C}^2)).$$

We will express the vanishing integrals I_k using the virtual localization formula (6). This requires specifying equivariant lifts of the classes involved. First, the reduced virtual class admits a canonical lift to the \mathbb{C}^\star -equivariant cycle theory. The action of \mathbb{C}^\star on ζ is trivial. Finally, we lift the \mathbb{C}^\star action on 0 as the equivariant first Chern class of the representation [1].

By examining the overall signs, the two non-vanishing fixed loci F^+ and F^- can be seen to have identical contribution. We obtain:

$$\begin{aligned} I_k &= \int_{X^{[n]} \times \mathbb{P}} \zeta^{2k-1} \frac{(1-\zeta)^{\ell+2} \mathbf{e}((H^{-1})^{[n]} \otimes \mathcal{L}^{-1})}{c((H^{-1})^{[n]} \otimes \mathcal{L}^{-1})} \\ &= \int_{X^{[n]} \times \mathbb{P}} \zeta^{2k-1} (1-\zeta)^{\ell+2} \left(\sum_{i=0}^n c_{n-i}((H^{-1})^{[n]}) (-\zeta)^i \right) \left(\sum_{j \geq 0} \frac{s_j((H^{-1})^{[n]})}{(1-\zeta)^{n+j}} \right) \\ &= \sum_{i,j \geq 0} \int_{X^{[n]} \times \mathbb{P}} (-1)^i \cdot \zeta^{2k+i-1} (1-\zeta)^{\ell+2-n-j} \cdot c_{n-i}((H^{-1})^{[n]}) \cdot s_j((H^{-1})^{[n]}) \\ &= (-1)^\ell \sum_{i=0}^n \binom{\ell+2-i-2n}{\ell+2-i-2k} \int_{X^{[n]}} c_{n-i}((H^{-1})^{[n]}) \cdot s_{n+i}((H^{-1})^{[n]}). \end{aligned}$$

Setting

$$\alpha_i = \int_{X^{[n]}} c_{n-i}((H^{-1})^{[n]}) \cdot s_{n+i}((H^{-1})^{[n]}) \quad \text{for } 0 \leq i \leq n,$$

we have proven the following result.

Lemma 3. *For all $1 \leq k \leq n$, we have*

$$\sum_{i=0}^n \binom{\ell+2-i-2n}{\ell+2-i-2k} \alpha_i = 0.$$

In the statement above, we use the follow standard conventions for the binomial:

- $\binom{0}{0} = 1$,
- $\binom{a}{b} = 0$ for $0 \leq a < b$,
- $\binom{a}{b} = 0$ for $b < 0$,
- $\binom{-a}{b} = \frac{-a(-a-1)\cdots(-a-b+1)}{b!}$ for $a > 0, b \geq 0$.

Since

$$c((H^{-1})^{[n]}) \cdot s((H^{-1})^{[n]}) = 1,$$

we obtain additionally

$$\alpha_0 + \alpha_1 + \dots + \alpha_n = 0.$$

2. PROOF OF THEOREM 1

To prove Theorem 1 for H^{-1} , we must show

$$\alpha_n = 2^n \binom{\ell - 2n + 2}{n}.$$

For the argument below, we will regard α_n as a degree n polynomial in ℓ , as shown in [24]. The proof will be obtained by combining the following two statements:

- (i) the polynomial $\alpha_n(\ell)$ has roots $\ell = 2n - 2, 2n - 1, \dots, 3n - 3$,
- (ii) the leading term of $\alpha_n(\ell)$ is

$$\alpha_n(\ell) = \frac{2^n}{n!} \ell^n + \dots$$

Proof of (i). We will use Lemma 3 for the values

$$(7) \quad \ell = 2n - 2, \dots, 3n - 3.$$

In fact, the Lemma yields trivial relations for $\ell > 3n - 3$. We fix a value of ℓ in the sequence (7), and show that for this value

$$\alpha_n(\ell) = 0.$$

All α 's in the argument below will be evaluated at this fixed value of ℓ , but for simplicity of notation, we will not indicate this explicitly. For convenience, write

$$\ell = 2n - 2 + p, \quad 0 \leq p \leq n - 1,$$

and relabel

$$\beta_j = \alpha_{j+p}, \quad 1 \leq j \leq n - p.$$

We will show that

$$\beta_{n-p} = 0.$$

Lemma 3 written for $k \mapsto n - k$ yields the relations

$$\sum_{i=0}^n \binom{p-i}{p-i+2k} \alpha_i = 0$$

for all $0 \leq k \leq n-1$. When $k=0$, the binomials corresponding to $i \leq p$ are equal to 1, while those for $i > p$ are 0. We obtain

$$(8) \quad \alpha_0 + \dots + \alpha_p = 0 \implies \beta_1 + \dots + \beta_{n-p} = 0.$$

When $1 \leq k \leq n-1$, the binomials with $i \leq p$ are zero, so the only possible non-zero binomial numbers are obtained for $i > p$. Writing $i = p + j$, the above identity becomes

$$(9) \quad \sum_{j=1}^{n-p} \binom{-j}{2k-j} \beta_j = 0.$$

Note that

$$\binom{-j}{2k-j} = (-1)^j \binom{2k-1}{j-1}.$$

We collect all the equations (8) and (9) for values $k \leq n-p-1$ derived above. Using the notation of the Lemma 4 below, the system of equations in β_j 's thus obtained can be written as

$$A_{n-p} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_{n-p} \end{bmatrix} = 0.$$

By Lemma 4, A_{n-p} is invertible. Hence,

$$\beta_{n-p} = 0$$

as claimed.

Lemma 4. *The $n \times n$ matrix A_n with entries*

$$a_{ij} = (-1)^{j+1} \binom{2i-1}{j}, \quad 0 \leq i, j \leq n-1$$

is invertible.

Proof. Let A be the $n \times n$ matrix with entries

$$a_{ij} = \binom{m_i}{j}, \quad 0 \leq i, j \leq n-1.$$

If the m_i are distinct, we will prove the invertibility of A via a standard Vandermonde argument. Since A_n is obtained (up to the signs $(-1)^{j+1}$ which only affect the sign of the determinant) by the specialization

$$m_i = 2i - 1,$$

the Lemma follows.

If two m 's are equal, then the matrix A has two equal rows, so $\det A$ vanishes. Therefore, $\det A$ is divisible by

$$\prod_{i < j} (m_i - m_j).$$

Since A has degree $\frac{n(n-1)}{2}$ in the m 's, it suffices to show that the leading term is non-zero. If the leading term were zero, then $\det A = 0$ for all choices of m_i . However, for $m_i = i$ the matrix A is triangular with 1's on the diagonal. \square

Proof of (ii). For the series $A_1(z)$ in (1), we write

$$A_1(z) = a_{11}z + a_{12}z^2 + a_{13}z^3 + \dots$$

By evaluating the coefficient of z in (1), we obtain

$$2\ell a_{11} = \int_X s_2(H) = 2\ell \implies a_{11} = 1.$$

The coefficient of z^n in (1) yields

$$\begin{aligned} N_{X,H,n} &= \frac{(2\ell a_{11})^n}{n!} + \text{lower order terms in } \ell \\ &= \frac{2^n}{n!} \ell^n + \dots \end{aligned}$$

Statements (i) and (ii) imply Theorem 1 for primitively polarized $K3$ surfaces (X, H) of degree 2ℓ and Picard rank 1. By (1), Theorem 1 then holds for all pairs (X, H) where X is a nonsingular projective $K3$ surface. \square

Remark 1. It is natural to consider the remaining integrals

$$\int_{X^{[n]}} c_{n-i}(H^{[n]}) \cdot s_{n+i}(H^{[n]}).$$

The case $i = n$ was the subject of Lehn's conjecture. On the other hand, $i = 0$ corresponds to the curve calculations of [5, 13, 27]:

$$\int_{X^{[n]}} c_n(H^{[n]}) \cdot s_n(H^{[n]}) = \int_{C^{[n]}} s_n((H|_C)^{[n]}) = (-4)^n \binom{\frac{\ell}{2}}{n},$$

where C is a smooth curve in the linear series $|H|$. The intermediate cases are unknown. However, when $i = n - 1$, the methods of this work show that

$$\int_{X^{[n]}} c_1(H^{[n]}) \cdot s_{2n-1}(H^{[n]}) = -2^n \binom{\ell - 2n + 2}{n - 1} \cdot \left(\ell - \frac{3n - 3}{2} \right).$$

Crucially, the proof of (i) above also establishes that α_{n-1} , viewed as a degree n polynomial in ℓ , has $n - 1$ roots at

$$\ell = 2n - 2, 2n - 3, \dots, 3n - 4.$$

Therefore, α_{n-1} is determined up to an ambiguity of two coefficients. These can be read off from the asymptotics

$$\alpha_{n-1} = -\frac{1}{(n-1)!} \cdot (2\ell)^n + \frac{(5n-3)}{(n-2)!} \cdot (2\ell)^{n-1} + \dots$$

To prove this fact, it is more useful to consider the full Chern and Segre polynomials in the variables x and y

$$c_x = 1 + c_1x + c_2x^2 + \dots, \quad s_y = 1 + s_1y + s_2y^2 + \dots$$

In the $K3$ case, we derive using [7] the exponential form

$$\sum_{n=0}^{\infty} z^n \int_{X^{[n]}} c_x(H^{[n]}) \cdot s_y(H^{[n]}) = \exp(2\ell \cdot A(z) + B(z))$$

for power series

$$A(z) = a_1z + a_2z^2 + \dots, \quad B(z) = b_1z + b_2z^2 + \dots$$

whose coefficients are polynomials in x and y . Evaluating the coefficient of z^n , we obtain the asymptotics

$$\int_{X^{[n]}} c_x(H^{[n]}) \cdot s_y(H^{[n]}) = \frac{a_1^n}{n!} \cdot (2\ell)^n + \left(\frac{a_1^{n-1}b_1}{(n-1)!} + \frac{a_1^{n-2}a_2}{(n-2)!} \right) \cdot (2\ell)^{n-1} + \dots$$

When $n = 1$ this becomes

$$\int_X c_x(H) \cdot s_y(H) = 2y(y-x) \cdot \ell \implies a_1 = y(y-x), \quad b_1 = 0,$$

while for $n = 2$, we find

$$a_2 = y^2(y-x)(2x-5y).$$

Therefore,

$$\int_{X^{[n]}} c_x(H^{[n]}) \cdot s_y(H^{[n]}) = \frac{y^n(y-x)^n}{n!} \cdot (2\ell)^n + \frac{y^n(y-x)^{n-1}(2x-5y)}{(n-2)!} \cdot (2\ell)^{n-1} + \dots$$

Finally, we isolate the integral α_{n-1} by considering the coefficient of xy^{2n-1} .

3. LEHN'S CONJECTURE FOR K -TRIVIAL SURFACES

We verify here several statements made in Section 0 concerning Lehn's conjecture:

- (i) Theorem 1 is equivalent to Conjecture 1 for $K3$ surfaces.
- (ii) Conjecture 1 for $K3$ surfaces implies equations (3) and (4).
- (iii) Corollaries 1 and 2 hold.

Proof of (i). Let (X, H) be a $K3$ surface of degree 2ℓ . We must prove

$$\sum_{n=0}^{\infty} N_{X,H,n} z^n = \frac{(1-2w)^{2\ell+6}}{(1-6w+6w^2)^{\ell+2}}$$

for $N_{X,H,n} = 2^n \binom{\ell-2n+2}{n}$ given by Theorem 1 and the change of variables

$$z = \frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3}.$$

The following Lemma applied to

$$x = 2z, \quad y = 2(w-w^2)$$

completes the proof of (i).

Lemma 5. *After the change of variable*

$$x = \frac{y(1-2y)^2}{(1-3y)^3},$$

we have

$$\sum_{n=0}^{\infty} x^n \binom{\ell-2n+2}{n} = \frac{(1-2y)^{\ell+3}}{(1-3y)^{\ell+2}}.$$

Proof. We carry out a further change of variables

$$\frac{1-2y}{1-3y} = 1+t \implies x = t(1+t)^2.$$

We must show

$$\sum_{n=0}^{\infty} x^n \binom{\ell-2n+2}{n} = \frac{(1+t)^{\ell+3}}{1+3t}.$$

Denote the left side by $f_{\ell}(x)$. Pascal's identity shows the recursion

$$(10) \quad f_{\ell+1}(x) = f_{\ell}(x) + x f_{\ell-2}(x).$$

The corresponding recursion for the right side $\frac{(1+t)^{\ell+3}}{1+3t}$ is easily verified. We prove the Lemma by induction on ℓ .

We first verify the base case consisting of three consecutive integers, for instance

$$\ell = -4, -3, -2.$$

When $\ell = -4$, we must show

$$\sum_{n=0}^{\infty} x^n \binom{-2n-2}{n} = \frac{1}{(1+t)(1+3t)} = \left(\frac{dx}{dt}\right)^{-1} = \frac{dt}{dx}$$

which may be rewritten as

$$(11) \quad t = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \binom{-2n-2}{n},$$

where $t = g(x)$ is the solution of the equation

$$t(1+t)^2 = x,$$

valid in a neighborhood of $x = 0$ and with $g(0) = 0$. The Taylor expansion (11) for g is equivalent to the value of the n^{th} -derivative

$$\frac{d^n g}{dx^n}(0) = (n-1)! \binom{-2n}{n-1}.$$

The n^{th} -derivative of the inverse function g at 0 is determined by the Schur-Jabotinsky Theorem [10]: $g^{(n)}(0)$ equals

$$(n-1)! \cdot \text{coefficient of } t^{-1} \text{ in the expansion of } x^{-n}.$$

Since

$$x^{-n} = t^{-n}(1+t)^{-2n},$$

the above coefficient has value $\binom{-2n}{n-1}$ as required.

Using the identity

$$\frac{1+t}{1+3t} = (1+t) - \frac{3}{(1+t)(1+3t)} \cdot t(1+t)^2 = (1+t) - 3f_{-4}(x) \cdot x$$

and equation (11), we obtain

$$\begin{aligned} \frac{1+t}{1+3t} &= 1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \binom{-2n-2}{n} - 3 \sum_{n=0}^{\infty} x^{n+1} \binom{-2n-2}{n} \\ &= 1 + \sum_{n=0}^{\infty} x^{n+1} \binom{-2n-2}{n+1} \\ &= f_{-2}(x), \end{aligned}$$

which verifies the Lemma for $\ell = -2$.

For $n \geq 1$, we have the identity

$$\binom{-2n-1}{n} = \frac{3}{2} \binom{-2n}{n}.$$

Taking into account corrections coming from $n = 0$, we obtain

$$\sum_{n=0}^{\infty} x^n \binom{-2n-1}{n} = \frac{3}{2} \sum_{n=0}^{\infty} x^n \binom{-2n}{n} - \frac{1}{2}.$$

Equivalently,

$$f_{-3}(x) = \frac{3}{2}f_{-2}(x) - \frac{1}{2} = \frac{3}{2} \cdot \frac{1+t}{1+3t} - \frac{1}{2} = \frac{1}{1+3t},$$

which verifies the remaining base case. \square

Proof of (ii). We combine equations (1) and (2) for a $K3$ surface (X, H) of degree 2ℓ to obtain

$$\exp(2\ell A_1(z) + 24A_4(z)) = \frac{(1-2w)^{2\ell+6}}{(1-6w+6w^2)^{\ell+2}}.$$

Setting $\ell = 0$ yields

$$\exp(24A_4(z)) = \frac{(1-2w)^6}{(1-6w+6w^2)^2}$$

and thus

$$\exp(2\ell A_1(z)) = \frac{(1-2w)^{2\ell}}{(1-6w+6w^2)^\ell} \implies A_1(z) = \frac{1}{2} \log \left(\frac{1-4w+4w^2}{1-6w+6w^2} \right).$$

Setting as before

$$z = \frac{1}{2}t(1+t)^2 \implies \frac{1-4w+4w^2}{1-6w+6w^2} = 1+t,$$

we obtain equation (3). Equation (4) follows by the same method.

Proof of (iii). Let (A, H) be an abelian or bielliptic surface of degree 2ℓ . Equation (1) and the proof of (ii) yield

$$\sum_{n=0}^{\infty} N_{A,H,n} z^n = \exp(2\ell A_1(z)) = \frac{(1-2w)^{2\ell}}{(1-6w+6w^2)^\ell}.$$

Corollary 1 is equivalent to the identity

$$\sum_{n=0}^{\infty} z^n \frac{2^n \ell}{n} \binom{\ell-2n-1}{n-1} = \frac{(1-2w)^{2\ell}}{(1-6w+6w^2)^\ell}.$$

After the change of variables $2z = x$ and $2(w-w^2) = y$, we obtain

$$(12) \quad \sum_{n=0}^{\infty} x^n \frac{\ell}{n} \binom{\ell-2n-1}{n-1} = \left(\frac{1-2y}{1-3y} \right)^\ell$$

in the notation of Lemma 5. We have

$$\frac{dy}{dx} = \left(\frac{dx}{dy} \right)^{-1} = \frac{(1-3y)^4}{1-2y} \implies d \left(\frac{1-2y}{1-3y} \right) = \frac{(1-3y)^2}{1-2y} dx.$$

Writing Lemma 5 for $\ell \mapsto \ell-5$, we obtain

$$\sum_{n=0}^{\infty} x^n \binom{\ell-2n-3}{n} = \frac{(1-2y)^{\ell-2}}{(1-3y)^{\ell-3}}$$

and therefore also

$$\sum_{n=0}^{\infty} x^n \binom{\ell-2n-3}{n} dx = \frac{(1-2y)^{\ell-1}}{(1-3y)^{\ell-1}} d \left(\frac{1-2y}{1-3y} \right).$$

After integration, we obtain

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \binom{\ell-2n-3}{n} = \frac{1}{\ell} \frac{(1-2y)^\ell}{(1-3y)^\ell} - \frac{1}{\ell}$$

which yields (12) after the shift $n \rightarrow n-1$. This proves Corollary 1.

Let (E, H) be an Enriques surface of degree 2ℓ . Equation (1) yields

$$\sum_{n=0}^{\infty} N_{E,H,n} z^n = \exp(2\ell A_1(z) + 12A_4(z)) = \frac{(1-2w)^{2\ell+3}}{(1-6w+6w^2)^{\ell+1}}.$$

Using the change of variables $2z = x$ and $2(w-w^2) = y$ as above, we obtain

$$\sum_{n=0}^{\infty} \frac{N_{E,H,n}}{2^n} x^n = \frac{(1-2y)^{\ell+\frac{3}{2}}}{(1-3y)^{\ell+1}}.$$

In particular, by Lemma 5,

$$(13) \quad \left(\sum_{n=0}^{\infty} \frac{N_{E,H,n}}{2^n} x^n \right)^2 = \sum_{n=0}^{\infty} x^n \binom{2\ell-2n+2}{n} = \sum_{n=0}^{\infty} \frac{N_{X,H,n}}{2^n} x^n,$$

where the (X, H) is the $K3$ double covering E with the divisor class determined by pull-back. The proof of Corollary 2 is complete.

Corollary 3. *If Y is a minimal elliptic surface and $H = mf$ is a multiple of the fiber class, then*

$$\sum_n \frac{N_{Y,H,n}}{2^n} x^n = \left(\sum_{n=0}^{\infty} x^n \binom{-2n+2}{n} \right)^{\chi(\mathcal{O}_Y)/2}.$$

Proof. Equation (1) yields

$$\sum_n N_{Y,H,n} z^n = \exp(12\chi(\mathcal{O}_Y) \cdot A_4(z)) = \left(\frac{(1-2w)^3}{1-6w+6w^2} \right)^{\chi(\mathcal{O}_Y)}.$$

The Corollary follows via the change of variables $2z = x$ and $2(w - w^2) = y$ and Lemma 5. \square

4. MODULI OF $K3$ SURFACES

4.1. Tautological classes. Let $\mathcal{M}_{2\ell}$ be the moduli space of quasi-polarized $K3$ surfaces (X, H) of degree 2ℓ . The Noether-Lefschetz loci define classes in the Chow ring $A^*(\mathcal{M}_{2\ell}, \mathbb{Q})$. Let

$$NL^*(\mathcal{M}_{2\ell}) \subset A^*(\mathcal{M}_{2\ell}, \mathbb{Q})$$

be the subalgebra generated by Noether-Lefschetz loci (of all codimensions). A basic result conjectured in [18] and proven in [3] is the isomorphism

$$NL^1(\mathcal{M}_{2\ell}) = A^1(\mathcal{M}_{2\ell}, \mathbb{Q}).$$

Another method of constructing classes in $A^*(\mathcal{M}_{2\ell}, \mathbb{Q})$ is the following. Let

$$\pi : \mathcal{X} \rightarrow \mathcal{M}_{2\ell}$$

be the universal surface. Let

$$\mathcal{H} \rightarrow \mathcal{X} \quad \text{and} \quad \mathcal{T}_\pi \rightarrow \mathcal{X}$$

the universal quasi-polarization (canonical up to a twist by an element of $\text{Pic}(\mathcal{M}_{2\ell})$) and the relative tangent bundle. The κ classes are defined by

$$\kappa_{a,b} = \pi_* \left(c_1(\mathcal{H})^a \cdot c_2(\mathcal{T}_\pi)^b \right) \in A^{a+2b-2}(\mathcal{M}_{2\ell}, \mathbb{Q}).$$

There is no need to include a κ index for the first Chern class of \mathcal{T}_π since

$$c_1(\mathcal{T}_\pi) = -\pi^* \lambda$$

where $\lambda = c_1(\mathbb{E})$ is the first Chern class of the Hodge line bundle

$$\mathbb{E} \rightarrow \mathcal{M}_{2\ell}$$

with fiber $H^0(X, K_X)$ over the moduli point $(X, H) \in \mathcal{M}_{2\ell}$. The Hodge class λ is known to be supported on Noether-Lefschetz divisors.² The ring generated by λ has been determined in [8].

On the Noether-Lefschetz locus $\mathcal{M}_\Lambda \subset \mathcal{M}_{2\ell}$ corresponding to the Picard lattice Λ , richer κ classes may be defined using *all* the powers of *all* the universal line bundles associated to Λ . If H_1, \dots, H_r is a basis of Λ and

$$\mathcal{H}_1, \dots, \mathcal{H}_r \rightarrow \mathcal{X}_\Lambda$$

are the associated universal bundles (canonical up to a twist by an element of $\text{Pic}(\mathcal{M}_\Lambda)$) over the universal surface

$$\mathcal{X}_\Lambda \rightarrow \mathcal{M}_\Lambda,$$

we define the κ^Λ classes by

$$\kappa_{a_1, \dots, a_r, b} = \pi_* \left(c_1(\mathcal{H}_1)^{a_1} \cdots c_1(\mathcal{H}_r)^{a_r} \cdot c_2(\mathcal{T}_\pi)^b \right) \in \mathbf{A}^{\sum_i a_i + 2b - 2}(\mathcal{M}_\Lambda).$$

We define the *tautological ring* of the moduli space of $K3$ surfaces,

$$\mathbf{R}^*(\mathcal{M}_{2\ell}) \subset \mathbf{A}^*(\mathcal{M}_{2\ell}, \mathbb{Q}),$$

to be the subring generated by the push-forwards from the Noether-Lefschetz loci of all monomials in the κ classes. By definition,

$$\mathbf{NL}^*(\mathcal{M}_{2\ell}) \subset \mathbf{R}^*(\mathcal{M}_{2\ell}).$$

4.2. Relations. Consider the universal surface

$$\pi : \mathcal{X} \rightarrow \mathcal{M}_{2\ell},$$

and the π -relative Quot scheme $\mathcal{Q}_{dH, \chi}^\pi(\mathbb{C}^r)$ parametrizing quotients

$$\mathbb{C}^r \otimes \mathcal{O}_X \rightarrow F \rightarrow 0$$

where F is a rank 0 coherent sheaf on (X, H) satisfying

$$c_1(F) = dH \quad \text{and} \quad \chi(F) = \chi.$$

Just as in Section 1.1, the π -relative Quot scheme admits a reduced virtual class via the π -relative obstruction theory. The push-forwards

$$(14) \quad \pi_* \left(\gamma \cdot 0^k \cap [\mathcal{Q}_{dH, \chi}^\pi(\mathbb{C}^r)]^{\text{red}} \right) \in \mathbf{A}^*(\mathcal{M}_{2\ell}, \mathbb{Q})$$

vanish for all $k > 0$ and for all choices of cohomology classes γ . There is a natural torus action on $\mathcal{Q}_{dH, \chi}^\pi(\mathbb{C}^r)$ via the torus action on \mathbb{C}^r . Virtual localization [9] applied to the push-forwards (14) yields relations in $\mathbf{R}^*(\mathcal{M}_{2\ell})$. The Noether-Lefschetz loci appear naturally when the curve class dH splits into non-multiples of H .

²In [18], λ is considered a degenerate Noether-Lefschetz divisor. See [17, Theorem 3.1] to express λ in terms of proper Noether-Lefschetz divisors.

There is no difficulty in writing the resulting relations in $R^*(\mathcal{M}_{2\ell})$ in terms of push-forwards from the π -relative Hilbert schemes of points. To show the relations are *non-trivial* requires non-vanishing results in the intersection theory of the π -relative Hilbert scheme. A thorough study of the relations in $R^*(\mathcal{M}_{2\ell})$ obtained by localizing the virtual class of π -relative Quot schemes will appear elsewhere.

4.3. Virtual localization. We illustrate the program proposed above with several examples. For simplicity, we will only consider the case $d = 1$. The relative Quot scheme

$$p : \mathcal{Q}_{H,\chi}^\pi(\mathbb{C}^2) \rightarrow \mathcal{M}_{2\ell}$$

has relative reduced virtual dimension equal to $2n + 1$ where

$$n = \chi + \ell.$$

As before, there is a support morphism

$$\mathcal{Q}_{H,\chi}^\pi(\mathbb{C}^2) \rightarrow \mathbb{P}(\pi_*\mathcal{H}), \quad [0 \rightarrow E \rightarrow \mathbb{C}^2 \otimes \mathcal{O}_X \rightarrow F \rightarrow 0] \mapsto \text{supp } F.$$

Consider the codimension 1 vanishing push-forwards

$$\frac{(-1)^n}{2} \cdot p_* \left(0^{2n+2-2k} \cdot \zeta^{2k} \cap [\mathcal{Q}_{H,\chi}^\pi(\mathbb{C}^2)]^{\text{red}} \right) \in \mathbb{A}^1(\mathcal{M}_{2\ell}, \mathbb{Q}),$$

for $0 \leq k \leq n$. We evaluate these expressions via equivariant localization for the torus action induced from the splitting

$$\mathbb{C}^2 = \mathbb{C}[0] + \mathbb{C}[1].$$

Over the generic point (X, H) of the moduli space, the fixed loci over the Quot scheme correspond to kernels of the form

$$E = I_Z + I_W \otimes \mathcal{O}_X(-C)$$

where

$$\ell(Z) = z, \quad \ell(W) = w, \quad z + w = n$$

and C is a curve in the linear series $|H|$. The weights are distributed in two possible ways over the summands.

There are additional fixed loci over surfaces in the Noether-Lefschetz divisors corresponding to a nontrivial-splitting $[H] = [C_1] + [C_2]$ so that

$$E = I_Z \otimes \mathcal{O}_X(-C_1) + I_W \otimes \mathcal{O}_X(-C_2).$$

The curve classes C_1, C_2 are effective, so they intersect H nonnegatively. Since

$$C_1 \cdot H + C_2 \cdot H = 2\ell, \quad \ell(Z) + \ell(W) = n - C_1 \cdot C_2,$$

there are finitely many such Noether-Lefschetz divisors and fixed loci over them. We will address them later.

For now, to describe the *generic* fixed loci and their contributions, we introduce the following notation:

- Over the moduli space $\mathcal{M}_{2\ell}$, the bundle

$$\mathbb{V} = \pi_* \mathcal{H}$$

has rank $\ell + 2$. We write

$$\mathbb{P} = \mathbb{P}(\mathbb{V}) \quad \text{and} \quad \mathcal{L} = \mathcal{O}_{\mathbb{P}}(1)$$

for the associated relative linear series $|\mathcal{H}|$ over the moduli space and the corresponding tautological line bundle respectively.

- Over the relative Hilbert scheme $\mathcal{X}^{[z]} \rightarrow \mathcal{M}_{2\ell}$, we introduce the rank z bundle

$$\mathcal{H}^{[z]} = \text{pr}_{2*}(\text{pr}_1^* \mathcal{H} \otimes \mathcal{O}_{\mathcal{Z}}),$$

obtained as the push forward from the universal surface of the twisted universal subscheme

$$\mathcal{Z} \subset \mathcal{X} \times_{\mathcal{M}} \mathcal{X}^{[z]} \rightarrow \mathcal{X}^{[z]}.$$

The fixed loci are

$$\mathbf{A}[z, w] = \mathcal{X}^{[z]} \times_{\mathcal{M}} \left(\mathcal{X}^{[w]} \times_{\mathcal{M}} \mathbb{P} \right).$$

Over the universal surface

$$\mathbf{A}[z, w] \times_{\mathcal{M}} \mathcal{X},$$

the universal subsheaf equals

$$\mathcal{E} = I_{\mathcal{Z}} \oplus I_{\mathcal{W}} \otimes \mathcal{L}^{-1} \otimes \mathcal{H}^{-1}.$$

The non-reduced obstruction bundle over $\mathbf{A}[z, w]$ is found by considering the fixed part of the virtual tangent bundle:

$$\text{Ext}^\bullet(\mathcal{E}, \mathcal{F})^{\text{fix}} = \text{Ext}^\bullet(I_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}}) + \text{Ext}^\bullet(I_{\mathcal{W}} \otimes \mathcal{L}^{-1} \otimes \mathcal{H}^{-1}, \mathcal{O}_X - I_{\mathcal{W}} \otimes \mathcal{L}^{-1} \otimes \mathcal{H}^{-1}).$$

To calculate this bundle explicitly, we follow the method of Section 1.2 for a *fixed* surface. The only difference is that in the use of relative duality, the relative canonical bundle will yield a copy of the Hodge class

$$K_{\mathcal{X}/\mathcal{M}} = \pi^* \mathbb{E}.$$

We only record here the final answer (for $z \neq 0$ and $w \neq 0$):

$$\text{Obs}^{\text{red}} = \mathbb{E}^\vee + \mathbb{E}^\vee \otimes \left(\mathcal{O}^{[z]} \right)^\vee + \mathbb{E}^\vee \otimes \mathcal{L} \otimes \left((\mathcal{H}^{-1})^{[w]} \right)^\vee.$$

Just as in the case of a fixed surface where almost all fixed loci have a trivial summand, the fixed loci here keep an additional copy of the Hodge bundle. The corresponding contribution

$$\text{Cont } \mathbf{A}[z, w] = m(z, w) \cdot \lambda$$

is a multiple of the Hodge class. In our examples below, $m(z, w)$ can be found via a calculation over the Hilbert scheme of a fixed surface. (A separate calculation shows that the same conclusion also holds for $w = 0$.)

There are however two exceptions, where the formulas above must be modified. These correspond to the two fixed loci F^+ and F^- where the splitting of the kernel is

$$E = \mathcal{O}_X[0] \oplus I_W \otimes \mathcal{O}_X(-C)[1] \quad \text{or} \quad E = I_Z \otimes \mathcal{O}_X(-C)[0] \oplus \mathcal{O}_X[1]$$

for a curve C in the linear series $|H|$. We have

$$F^+ \simeq F^- \simeq \mathcal{X}^{[n]} \times_{\mathcal{M}} \mathbb{P}.$$

The following hold true:

- The obstruction bundle equals

$$\text{Obs}^{\text{red}} = \mathbb{E}^\vee \otimes \mathcal{L} \otimes \left((\mathcal{H}^{-1})^{[n]} \right)^\vee,$$

so that

$$[F^+]^{\text{vir}} = [F^-]^{\text{vir}} = (-1)^n \cdot e \left(\mathbb{E} \otimes \mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]} \right).$$

- The virtual normal bundle is

$$\mathbf{N}^\pm = \left(\mathbb{C} + \mathbb{E}^\vee + \mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]} - \mathcal{L}^{-1} \otimes \mathbb{V}^\vee \otimes \mathbb{E}^\vee \right) [\pm 1].$$

The formulas of Section 1.2 are special cases.

Therefore,

$$\begin{aligned} \frac{1}{e(\mathbf{N}^+)} &= \frac{c_-(\mathcal{L} \otimes \mathbb{V} \otimes \mathbb{E})}{1 - \lambda} \cdot \frac{1}{c_+(\mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]})}, \\ \frac{1}{e(\mathbf{N}^-)} &= (-1)^{n-\ell} \cdot \frac{c_+(\mathcal{L} \otimes \mathbb{V} \otimes \mathbb{E})}{1 + \lambda} \cdot \frac{1}{c_-(\mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]})}. \end{aligned}$$

The localization contributions of the fixed loci F^+ and F^- are equal and take the form

$$q_\star \left(\zeta^{2k} \cdot e \left(\mathbb{E} \otimes \mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]} \right) \cdot \frac{c_-(\mathcal{L} \otimes \mathbb{V} \otimes \mathbb{E})}{1 - \lambda} \cdot \frac{1}{c_+(\mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]})} \right),$$

where

$$q : \mathcal{X}^{[n]} \times_{\mathcal{M}} \mathbb{P} \rightarrow \mathcal{M}_{2\ell}$$

is the projection. This expression can be expanded as in Section 1.3.

The following push-forwards from the relative Hilbert scheme are central to the calculation:

$$\gamma_i = \text{pr}_\star \left(s_{n+i+1} \left((\mathcal{H}^{-1})^{[n]} \right) \cdot c_{n-i} \left((\mathcal{H}^{-1})^{[n]} \right) \right) \in \mathbf{A}^1(\mathcal{M}_{2\ell}), \quad 0 \leq i \leq n.$$

For the sum of contributions of fixed loci dominating $\mathcal{M}_{2\ell}$, we obtain

$$(-1)^{\ell+1} \cdot \left(\sum_{i=0}^n \binom{\ell+1-2n-i}{\ell+1-2k-i} \cdot \gamma_i + a_k \cdot c_1(\mathbb{V}) + b_k \cdot \lambda \right),$$

for constants a_k and b_k that depend on ℓ, k, n . For instance,

$$a_k = \sum_{i=0}^n \binom{\ell + 1 - 2n - i}{\ell + 2 - 2k - i} \alpha_i.$$

The expression thus obtained must be in the span of the Noether-Lefschetz divisors. Similar relations can be written down in higher codimension.

The class

$$\gamma_n = \text{pr}_* \left(s_{2n+1} \left((\mathcal{H}^{-1})^{[n]} \right) \right)$$

is the codimension one analogue of the Segre integrals which are the subject of Lehn's conjecture. While there are procedures to calculate γ_n , we do not yet have a closed form expression. Nonetheless, by the recursions of [7], it can be seen that

$$\gamma_n \in \mathbf{R}^*(\mathcal{M}_{2\ell})$$

is a combination of the classes $\kappa_{3,0}, \kappa_{1,1}, \lambda$ with coefficients which are polynomials in ℓ .

4.4. Examples. We present here explicit relations involving κ classes and Noether-Lefschetz divisors. As explained in the introduction of [18], there are two essentially equivalent ways of thinking of the Noether-Lefschetz divisors. Our convention here is that we specify the *Picard class*. In other words, we consider surfaces for which there exists a class β with given lattice

$$\begin{bmatrix} H^2 & \beta \cdot H \\ \beta \cdot H & \beta^2 \end{bmatrix}.$$

4.4.1. *Degree two.* The simplest case is of K3 surfaces of degree $H^2 = 2$. Relevant to the discussion are the following Noether-Lefschetz divisors:

- (i) The divisor \mathcal{P} corresponding to the lattice

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

- (ii) The reduced divisor \mathcal{S} corresponding to the lattice

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Here, the linear series $|H|$ fails to be ample along the (-2) -curve. The divisor \mathcal{S} contains \mathcal{P} as a component.

By work of O'Grady [20], the following isomorphism holds

$$A^1(\mathcal{M}_2, \mathbb{Q}) = \mathbb{Q} \cdot [\mathcal{P}] + \mathbb{Q} \cdot [\mathcal{S}].$$

However, there are more tautological classes to consider. In addition to the Hodge class λ , we also have

$$\kappa_{3,0} = \pi_*(c_1(\mathcal{H})^3) \quad \text{and} \quad \kappa_{1,1} = \pi_*(c_1(\mathcal{H}) \cdot c_2(\mathcal{T}_\pi)).$$

While the classes $\kappa_{3,0}$ and $\kappa_{1,1}$ are canonically defined only up to a choice of a quasi-polarization, the difference

$$\kappa_{3,0} - 4\kappa_{1,1}$$

is independent of choices. There are two relations connecting the four tautological classes

$$\kappa_{3,0} - 4\kappa_{1,1}, \lambda, [\mathcal{P}], [\mathcal{S}].$$

Proposition 1. *We have*

$$\begin{aligned} \kappa_{1,1} - 4\kappa_{3,0} - 18\lambda + 12[\mathcal{P}] &= 0, \\ \kappa_{1,1} - 4\kappa_{3,0} + \frac{9}{2}\lambda - \frac{24}{5}[\mathcal{P}] - \frac{3}{10}[\mathcal{S}] &= 0. \end{aligned}$$

Proof. The relations are obtained by considering the following numerics:

- (i) For the first relation, we use the relative Quot scheme of short exact sequences

$$0 \rightarrow E \rightarrow \mathbb{C}^2 \otimes \mathcal{O}_X \rightarrow F \rightarrow 0$$

where

$$c_1(F) = H, \quad \chi(F) = 0.$$

The relative virtual dimension equals 3, and we evaluate the pushforward

$$p_*(0^4 \cap [\mathcal{Q}_{H,0}^\pi(\mathbb{C}^2)]^{\text{red}}) \in \mathbf{A}^1(\mathcal{M}_2).$$

- (ii) For the second relation, we use the numerics

$$c_1(F) = H, \quad \chi(F) = 1.$$

The relative virtual dimension equals 5, and we consider the pushforward

$$p_*(0^6 \cap [\mathcal{Q}_{H,1}^\pi(\mathbb{C}^2)]^{\text{red}}) \in \mathbf{A}^1(\mathcal{M}_2).$$

We provide details only for case (i) since (ii) is parallel.

- Over a generic surface (X, H) , the total contribution can be found by the method of Section 4.3. The answer is

$$2\gamma_1 - 4c_1(\mathbb{V}) = 2\kappa_{3,0} - 4c_1(\mathbb{V}) = \frac{4\kappa_{3,0} - \kappa_{1,1}}{3} + 6\lambda.$$

Grothendieck-Riemann-Roch was used to express

$$c_1(\mathbb{V}) = -\frac{3\lambda}{2} + \frac{\kappa_{1,1}}{12} + \frac{\kappa_{3,0}}{6}.$$

- Over the locus \mathcal{P} , the classes $H_1 = \beta$, $H_2 = H - \beta$ are effective with

$$H_1 \cdot H_2 = 1, \quad H_1^2 = 0, \quad H_2^2 = 0.$$

The torus fixed kernels are

$$E = \mathcal{O}(-C_1) + \mathcal{O}(-C_2)$$

for curves C_1 and C_2 in the linear series $|H_1|$ and $|H_2|$. There are two fixed loci, with identical contributions, obtained by switching the weights. The multiplicity of the divisor \mathcal{P} in the final calculation equals -2 over each fixed locus.

The -2 can be confirmed by working over a fixed surface. Each fixed locus is isomorphic to

$$|H_1| \times |H_2| = \mathbb{P}^1 \times \mathbb{P}^1.$$

The fixed part of the tangent obstruction theory equals

$$\mathrm{Ext}^\bullet(\mathcal{O}(-C_1), \mathcal{O}_{C_1}) + \mathrm{Ext}^\bullet(\mathcal{O}(-C_2), \mathcal{O}_{C_2}).$$

The Hodge factor of the reduced theory cancels the additional Hodge class coming from the normal bundle of the divisor \mathcal{P} in the moduli stack, see [20]. The virtual class of the fixed locus agrees with the usual fundamental class.

Similarly, the normal bundle is found by considering the moving part

$$\mathbf{N} = \mathrm{Ext}^\bullet(E, F)^{\mathrm{mov}} = \mathbf{N}_1[1] + \mathbf{N}_2[-1]$$

where

$$\mathbf{N}_1 = \mathrm{Ext}^\bullet(\mathcal{O}(-C_1), \mathcal{O}_{C_2}) = H^\bullet(\mathcal{O}(C_1)|_{C_2}) = H^\bullet(\mathcal{O}(C_1)) - H^\bullet(\mathcal{O}(C_1 - C_2))$$

$$\mathbf{N}_2 = \mathrm{Ext}^\bullet(\mathcal{O}(-C_2), \mathcal{O}_{C_1}) = H^\bullet(\mathcal{O}(C_2)|_{C_1}) = H^\bullet(\mathcal{O}(C_2)) - H^\bullet(\mathcal{O}(C_2 - C_1)).$$

In families, these formulas also include the the tautological bundles \mathcal{L}_1 and \mathcal{L}_2 over the linear series. We have

$$\mathbf{N}_1 = H^\bullet(H_1) \otimes \mathcal{L}_1 - H^\bullet(H_1 \otimes H_2^{-1}) \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^{-1} = \mathcal{L}_1 \otimes \mathbb{C}^2 - \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$$

and analogously

$$\mathbf{N}_2 = \mathcal{L}_2 \otimes \mathbb{C}^2 - \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}.$$

The final multiplicity becomes

$$\int_{\mathbb{P}^1 \times \mathbb{P}^1} \frac{1}{\mathbf{e}(\mathbf{N})} = \int_{\mathbb{P}^1 \times \mathbb{P}^1} \frac{(1 + \zeta_1 - \zeta_2) \cdot (-1 - \zeta_1 + \zeta_2)}{(1 + \zeta_1)^2 (-1 + \zeta_2)^2} = -2.$$

□

Remark 2. We can write down relations in higher codimension as well. For instance, evaluating the pushforward

$$p_*(0^4 \cdot \zeta \cap [\mathcal{Q}_{H,0}^\pi(\mathbb{C}^2)]^{\mathrm{red}}) \in \mathbf{A}^2(\mathcal{M}_2)$$

yields the identity

$$11\kappa_{4,0} - \kappa_{2,1} - 3\kappa_{3,0}^2 - 4\kappa_{3,0} \cdot [\mathcal{P}] + 6\kappa_{3,0} \cdot \lambda - 8\lambda \cdot [\mathcal{P}] + 4\lambda^2 = 0.$$

4.4.2. *Degree four.* Our next example concerns the case of quartic $K3$ surfaces with $H^2 = 4$. We single out three Noether-Lefschetz divisors:

- (i) The reduced divisor \mathcal{P}_1 corresponding to the lattice

$$\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}.$$

- (ii) The reduced divisor \mathcal{P}_2 underlying the Noether-Lefschetz locus corresponding to the lattice

$$\begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}.$$

The divisor \mathcal{P}_2 is not irreducible, since it receives contributions from hyperelliptic locus and from \mathcal{P}_1 .

- (iii) The reduced divisor \mathcal{S} corresponding to the lattice

$$\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Here, the linear series $|H|$ fails to be ample along the (-2) -curve.

A $K3$ surface in the complement of the union of the three divisors \mathcal{P}_1 , \mathcal{P}_2 , \mathcal{S} is necessarily a nonsingular quartic in \mathbb{P}^3 . Again by work of [20], we have an isomorphism

$$\mathbf{A}^1(\mathcal{M}_4, \mathbb{Q}) = \mathbb{Q} \cdot [\mathcal{P}_1] + \mathbb{Q} \cdot [\mathcal{P}_2] + \mathbb{Q} \cdot [\mathcal{S}].$$

The class

$$\kappa_{1,1} - 2\kappa_{3,0}$$

is independent of the choice of polarization. The following relations are obtained:

Proposition 2. *We have*

$$10(\kappa_{1,1} - 2\kappa_{3,0}) - 38\lambda + 8[\mathcal{P}_1] - 2[\mathcal{P}_2] - 3[\mathcal{S}] = 0,$$

$$4(\kappa_{1,1} - 2\kappa_{3,0}) - 26\lambda + 16[\mathcal{P}_1] + 2[\mathcal{P}_2] - [\mathcal{S}] = 0.$$

The proof of Proposition 2 uses the relative Quot scheme of short exact sequences

$$0 \rightarrow E \rightarrow \mathbb{C}^2 \otimes \mathcal{O}_X \rightarrow F \rightarrow 0$$

where the numerics are chosen so that

$$c_1(F) = H, \quad \chi(F) = 0.$$

The relative virtual dimension equals 5. The first relation is obtained by evaluating the pushforward

$$p_*(0^6 \cap [\mathcal{Q}_{H,0}^\pi(\mathbb{C}^2)]^{\text{red}}) \in \mathbf{A}^1(\mathcal{M}_4),$$

while second relation is obtained by evaluating

$$p_*(0^4 \cdot \zeta^2 \cap [\mathcal{Q}_{H,0}^\pi(\mathbb{C}^2)]^{\text{red}}) \in \mathbf{A}^1(\mathcal{M}_4).$$

The details are similar to the proof of Proposition 1.

4.4.3. *Degree 6.* Consider the reduced Noether-Lefschetz divisors

$$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{S}$$

corresponding to the lattices

$$\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}.$$

The divisors \mathcal{P}_2 and \mathcal{P}_3 are not irreducible, since they contain \mathcal{P}_1 as a component. It is shown in [20] that the Picard rank is 4. We expect two relations between the invariant combination of the κ 's, the Hodge class and the 4 boundary divisors:

Proposition 3. *We have*

$$\begin{aligned} (-3\kappa_{1,1} + 4\kappa_{3,0}) + 21\lambda - 16[\mathcal{P}_1] - [\mathcal{P}_2] + [\mathcal{S}] &= 0, \\ -11(3\kappa_{1,1} - 4\kappa_{3,0}) + 280\lambda - 230[\mathcal{P}_1] - 38[\mathcal{P}_2] - 2[\mathcal{P}_3] + 10[\mathcal{S}] &= 0. \end{aligned}$$

These relations can be found by integrating over $\mathcal{Q}_{H,-1}^\pi(\mathbb{C}^2)$ and over $\mathcal{Q}_{H,0}^\pi(\mathbb{C}^2)$.

4.4.4. *Degree 8.* In this case, the Picard rank is 4, see [20]. We expect two relations between the invariant combination of the κ 's, the Hodge class and the 4 boundary divisors

$$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{S}$$

corresponding to the lattices

$$\begin{bmatrix} 8 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix}.$$

Integrating over $\mathcal{Q}_{H,-2}^\pi(\mathbb{C}^2)$ and $\mathcal{Q}_{H,-1}^\pi(\mathbb{C}^2)$ we find:

Proposition 4.

$$\begin{aligned} \frac{8}{3}(\kappa_{1,1} - \kappa_{3,0}) - 24\lambda + 24[\mathcal{P}_1] + 4[\mathcal{P}_2] - [\mathcal{S}] &= 0, \\ -\frac{128}{3}(\kappa_{1,1} - \kappa_{3,0}) + 424\lambda - 440[\mathcal{P}_1] - 92[\mathcal{P}_2] - 8[\mathcal{P}_3] + 15[\mathcal{S}] &= 0. \end{aligned}$$

4.5. **Conjecture.** The main conjecture suggested by the abundance of relations obtained by localizing the virtual class of π -relative Quot schemes is the following.

Conjecture 2. *For all $\ell \geq 1$, we have $\mathrm{NL}^*(\mathcal{M}_{2\ell}) = \mathrm{R}^*(\mathcal{M}_{2\ell})$.*

If true, Conjecture 2 would lead to a much simpler picture of the additive structure of $\mathrm{R}^*(\mathcal{M}_{2\ell})$ since good approaches to the span of the Noether-Lefschetz classes are available [4, 12]. We further speculate that the relations obtained by localizing the virtual class of π -relative Quot schemes are sufficient to prove Conjecture 2. We have checked this in small degree and small codimension.

REFERENCES

- [1] M. Atiyah, R. Bott, *The moment map and equivariant cohomology*, *Topology* 23 (1984), 1 – 28.
- [2] K. Behrend, B. Fantechi, *The intrinsic normal cone*, *Invent. Math.* 128 (1997), 45 – 88.
- [3] N. Bergeron, Z. Li, J. Millson, C. Moeglin, *The Noether-Lefschetz conjecture and generalizations*, [arXiv:1412.3774](https://arxiv.org/abs/1412.3774).
- [4] R. E. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, *Duke Math. J.* 97 (1999), 219 – 233.
- [5] E. Cotterill, *Geometry of curves with exceptional secant planes: linear series along the general curve*, *Math. Zeit.* 267 (2011), 549 – 582.
- [6] M. Dürr, A. Kabanov, C. Okonek, *Poincare invariants*, *Topology* 46 (2007), 225 – 294.
- [7] G. Ellingsrud, L. Göttsche, M. Lehn, *On the cobordism class of the Hilbert scheme of a surface*, *J. Algebraic Geom.* 10 (2001), 81 – 100.
- [8] G. van der Geer, T. Katsura, *Note on tautological classes of moduli of K3 surfaces*, *Mosc. Math. J.* 5 (2005), 775 – 779.
- [9] T. Graber, R. Pandharipande, *Localization of virtual classes*, *Invent. Math.* 135 (1999), 487 – 518.
- [10] E. Jabotinsky, *Representations of functions by matrices*, *Proc. Amer. Math. Soc.* 4 (1953), 546 – 553.
- [11] F. Janda, *Tautological relations in moduli spaces of weighted pointed curves*, [arXiv:1306.6580](https://arxiv.org/abs/1306.6580).
- [12] S. Kudla, J. Millson, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, *Pub. IHES* 71 (1990), 121 – 172.
- [13] P. Le Barz, *Sur une formule de Castelnuovo pour les espaces multiseccants*, *Boll. Unione Mat. Ital. Sez. B* 10 (2007), 381 – 387.
- [14] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, *Invent. Math.* 136 (1999), 157 – 207.
- [15] A. Marian, D. Oprea, *Virtual intersections on the Quot scheme and Vafa-Intriligator formulas*, *Duke Mathematical Journal* 136 (2007), 81 – 131.
- [16] A. Marian, D. Oprea, R. Pandharipande, *The moduli space of stable quotients*, *Geometry and Topology* 15 (2011), 1651 – 1706.
- [17] D. Maulik, *Supersingular K3 surfaces for large primes*, *Duke Math. J.* 163 (2014), 2357 – 2425.
- [18] D. Maulik, R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, A celebration of algebraic geometry, 469 – 507, *Clay Math. Proc.*, 18, Amer. Math. Soc., Providence, RI, 2013.
- [19] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, *Ann. Math.* 145 (1997), 379 – 388.
- [20] K. O’Grady, *Moduli of abelian and K3 surfaces*, Ph.D. Thesis, Brown University, 1986.
- [21] R. Pandharipande, *The kappa ring of the moduli space of curves of compact type*, *Acta Math.* 208 (2012), 335 – 388.
- [22] A. Pixton, R. Pandharipande, *Relations in the tautological ring of the moduli space of curves*, [arXiv:1301.4561](https://arxiv.org/abs/1301.4561).
- [23] D. Schultheis, *Virtual invariants of Quot schemes over del Pezzo surfaces*, Ph.D. Thesis, UC San Diego (2012).
- [24] A. S. Tikhomirov, *Standard bundles on a Hilbert scheme of points on a surface*, Algebraic geometry and its applications, Yaroslavl’ 1992, Aspects of Mathematics, Vol. E25, Vieweg Verlag, 1994.
- [25] A. S. Tikhomirov, T. L. Troshina, *Top Segre class of a standard vector bundle \mathcal{E}_D^4 on the Hilbert scheme $\text{Hilb}^4(S)$ of a surface S* , Algebraic geometry and its applications, Yaroslavl’ 1992, Aspects of Mathematics, Vol. E25, Vieweg Verlag, 1994.
- [26] A. N. Tyurin, *The Spin-polynomial invariants of the smooth structures of algebraic surfaces*, *Iz. An SSSR* 57 (1993), 279 – 371.
- [27] Z. Wang, *Tautological integrals on Hilbert schemes of points on curves*, [arXiv:1506.08405](https://arxiv.org/abs/1506.08405).

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