Log intersection theory of $\overline{\mathcal{M}}_{g,n}$

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* Photo: Sebastian Pociecha (unsplash)

I. Logarithmic intersection theory

What is log intersection theory?

Given any nonsingular variety X
with a normal crossings divisor $D \subset X$
we obtain a log scheme (X, D).

There are two related Chow constructions
lying over $CH^*(X)$

$$CH^*(X) \subset \log CH^*(X,D) \subset b\,CH^*(X)$$

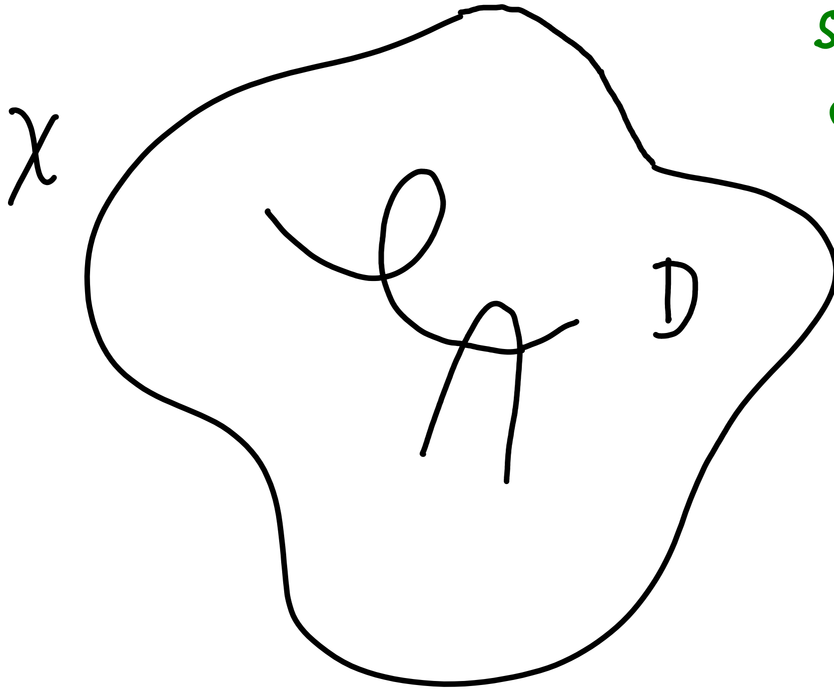
Shokurov

Our main example is the log scheme

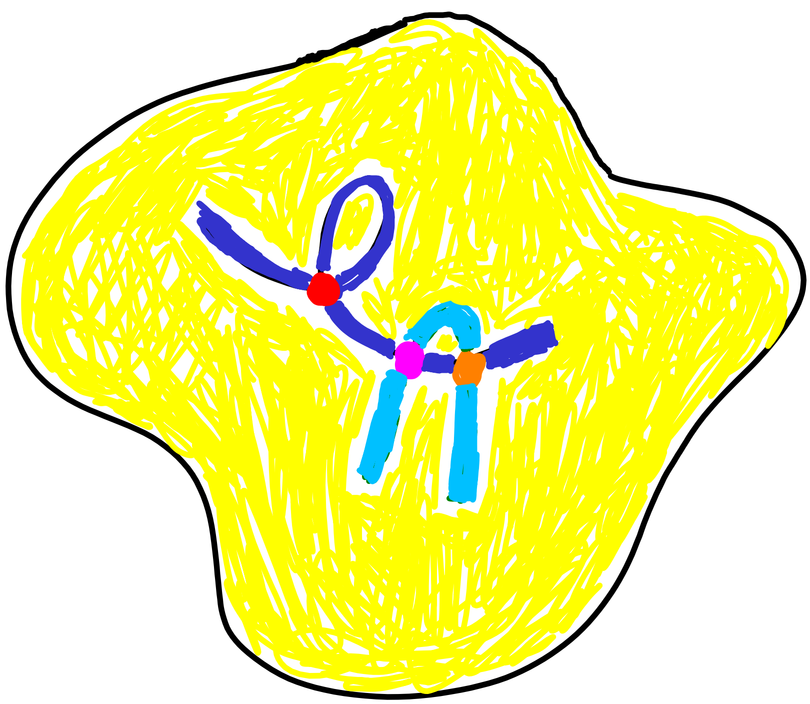
$$(\overline{M}_{g,n} , \Delta)$$

normal crossings divisor of nodal curves

Not assumed strict normal crossings



Basic Notion of stratification



Strata indicated by colors

A stratum $S \subset X$ is nonsingular and quasiprojective

$\overline{S} \subset X$ may be singular (mildly)

A simple blow-up of (X, D) is a blow up along a nonsingular stratum closure $\bar{S} \subsetneq X$.

$$\mathrm{Bl} : (\hat{X}, \hat{D}) \to (X, D)$$

blowup

strict transform of D union the exceptional divisor E

Define a category $\mathcal{B}(X, D)$

- Objects are $(\tilde{X}, \tilde{D}) \xrightarrow{\tilde{\phi}} (X, D)$ where $\tilde{\phi}$ is a composition of simple blowups

- Morphisms are commutative diagrams

$$\begin{tikzcd}
(\tilde{\tilde{X}}, \tilde{\tilde{D}}) \arrow[rr, "\gamma"] \arrow[dr, "\tilde{\tilde{\phi}}"'] & & (\tilde{X}, \tilde{D}) \arrow[dl, "\tilde{\phi}"] \\
& (X, D) &
\end{tikzcd}$$

γ is a composition of simple blowups

$$\boxed{\log CH^*(X, D) \overset{\text{def}}{=} \varinjlim_{(\tilde{X}, \tilde{D}) \in \mathcal{B}(X, D)} CH^*(\tilde{X})}$$

$bCH^*(X)$ has the same definition except that blowups along <u>all</u> nonsingular varieties are allowed.

Exercise : $bCH^*(X)$ is generated by divisors

[Molcho - Schmitt - P 2020]

Hint : Suppose $\alpha \in CH^*(Y)$, $Y \to X$ blowup

Since Y is nonsingular, $CH^*(Y)$ is generated by $c_k(E)$ where $E \to Y$ is a vector bundle. $\exists\ Z \xrightarrow{f} Y$ blowup where $f^*(E)$ has a filtration with subquotients given by line bundles. $\square$

The main point here for us:

Let (V,Δ) be a nonsingular projective variety with a normal crossing divisor

We know $\overline{M}_{g,A}(V/\Delta,\beta)$ moduli of log stable maps carries virtual fundamental class

Li, Ruan
Jun Li
Abramovich, Chen
Gross, Siebert

Standard log GW theory is defined by push forward along

$$\overline{M}_{g,A}(V/\Delta,\beta) \xrightarrow{\pi} \overline{M}_{g,n},$$

$$\pi_* \left[\overline{\mathcal{M}}_{g,A}(V/\Delta, \beta) \right]^{vir} \in CH^*(\overline{\mathcal{M}}_{g,n})$$

But in fact a much more subtle refined theory exists:

log boundary $\partial \overline{\mathcal{M}}_{g,n}$

$$\left[\overline{\mathcal{M}}_{g,A}(V/\Delta, \beta) \right]^{vir}_{log} \in \log CH^*(\overline{\mathcal{M}}_{g,n})$$

Some history, past and future:

- first constructions for the double ramification cycle — Holmes, Marcus-Wise
- in the above generality, almost all of the necessary steps in Ranganathan's paper Log GW via Expansions
- a full construction in an upcoming paper by Herr-Molcho-P-Wise

Basic questions:

(A) Does the degeneration formula (normally pushed forward to $CH^*(\overline{M}_{g,n})$) lift to $\log CH^*(\overline{M}_{g,n})$?

Ranganathan

(B) Is there a useful log CohFT ?

Holmes, Spelier

(C) Can virtual localization be lifted to $\log CH^*(\overline{M}_{g,n})$?

Graber

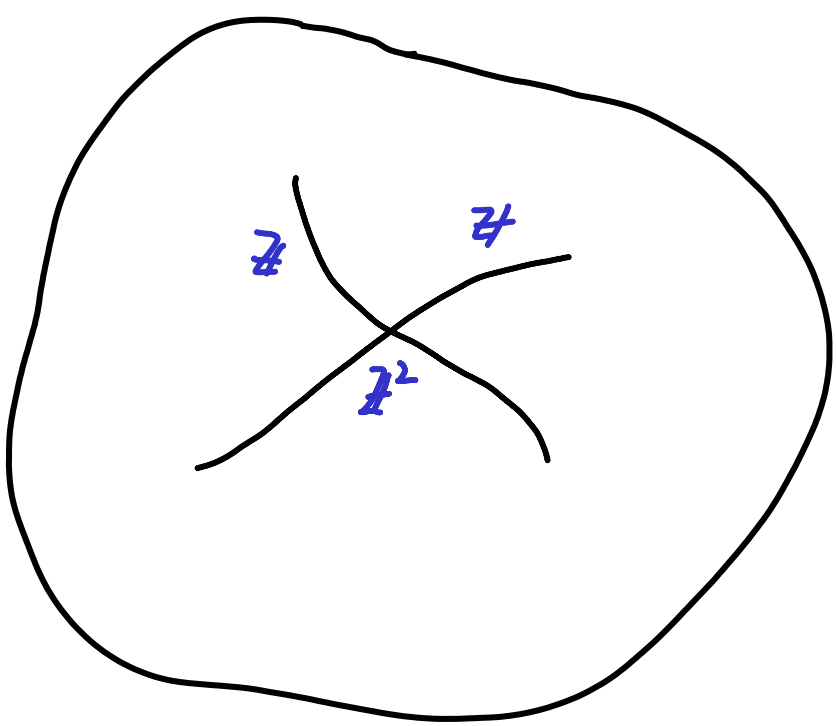
Answers are going to be Yes.

II. How to study $\log \mathrm{Ch}(X, D)$?

Language of Piecewise polys on the Cone Complex

Ranganathan
Molcho-P-Schmitt
MR, Holmes-Schwarz

$(X, D) \rightsquigarrow C(X, D)$ Cone Complex



open Statum codim r

$\updownarrow$

$\mathbb{R}^r \cong \mathbb{Z}^r \otimes_{\mathbb{Z}} \mathbb{R}$

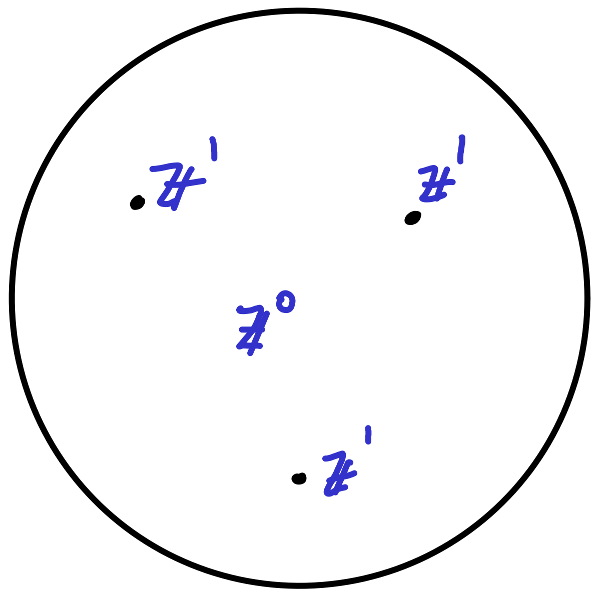
face given by

$\mathbb{R}^r_{\geq 0}$

When strata closures meet $\Rightarrow$ inclusion of faces of $C(X,D)$

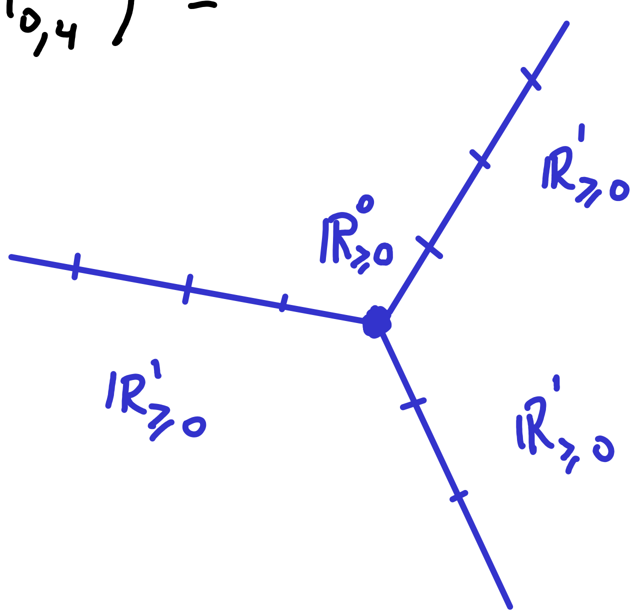
Simple example:

$(\overline{M}_{0,4}, \partial\overline{M}_{0,4})$



Cone complex is

$$C(\overline{M}_{0,4}, \partial\overline{M}_{0,4}) =$$



Algebra $PP(X, D)$ ← Piecewise polys on subdivisions

$C(X, D)$

Theorem: $PP(X, D) \xrightarrow{\Phi} \log CH^*(X, D)$

This is how to think about classes

Proposed by Ranganathan, see Molcho-P-Schmitt

$\mathrm{Image}\big(PP(X, D)\big) \subset \log CH^*(X, D)$

These are tautological classes in $\log CH^*$

Definition $\log R^*(X, D) = \mathrm{Image}\big(PP(X, D)\big)$

Holmes - Molcho - P - Pixton - Ranganathan - Schmitt

Theorem: $PP(\bar{M}_{0,n}, \partial\bar{M}_{0,n}) \to \log CH^*(\bar{M}_{0,n}, \partial\bar{M}_{0,n})$

is surjective with $\ker = WDVV$

In other words:

$$\frac{PP(\bar{M}_{0,n}, \partial\bar{M}_{0,n})}{\langle WDVV \rangle} \cong \log CH^*(\bar{M}_{0,n}, \partial\bar{M}_{0,n})$$

Proof:

- Use Keel's presentation of $CH^*(\bar{M}_{0,n})$
- Study Blow ups
- Use a new tubular property of the boundary geometry of $\bar{M}_{0,n}$

[**Theorem:** for (X, D) toric, — Brion, Payne, Fulton

$$PP(X, D) / \langle \text{Div rels} \rangle \cong \log CH^*(X, D)$$

III. log DR : $(\mathbb{P}^1 / 0 \vee \infty)$

Abel-Jacobi theory

Let $A = (a_1, \dots, a_n)$ with $a_i \in \mathbb{Z}$ and $\sum_{i=1}^{n} a_i = 0$

Let $\mathrm{Jac}_0 \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$ be the universal Jacobian of multidegree 0 line bundles.

We have a rational map

$$AJ_A : \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathrm{Jac}_0$$

defined on nonsingular curves by

$$(C, p_1, \dots, p_n) \mapsto \mathcal{O}_C\left(\sum a_i p_i\right)$$

$$\begin{tikzcd} \mathrm{Jac}_0 \arrow[d, "\pi"] \\ \overline{\mathcal{M}}_{g,n} \arrow[u, bend left, dashed, "\mathrm{AJ}_A"] \arrow[u, bend right, "\text{zero section with image } Z"'] \end{tikzcd}$$

We would like to define a locus in $\overline{\mathcal{M}}_{g,n}$ which corresponds to the condition

$$\text{“}\mathcal{O}_C\left(\sum a_i p_i\right) \cong \mathcal{O}_C\text{”}$$

$\boxed{\text{Not a closed condition}}$

Abel-Jacobi locus where there exists a function

$$f : (C, p_1, \dots, p_n) \to \mathbb{P}^1$$

with zeros and poles given by $A = (a_1, \dots, a_n)$

We would like to define the locus by

$$\text{“}\mathrm{AJ}_A^{-1}(Z) \subset \overline{\mathcal{M}}_{g,n}\text{”}$$

$\boxed{\text{Not a closed subvariety}}$

Holmes
Marcus-Wise

Partially

Idea is to resolve AJ_A via log blow-ups of $\overline{\mathcal{M}}_{g,n}$

$$
\begin{tikzcd}
\widetilde{\mathcal{M}} \arrow[r, dashed, "\widetilde{AJ}_A"] \arrow[dr] & \mathcal{J}ac_0 \arrow[d, "\pi"] \\
& \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left, "AJ_A"] \arrow[u, bend right, "\text{zero section with image } Z"']
\end{tikzcd}
$$

Where $\widetilde{AJ}_A\big|_{\mathcal{U}}^{-1}(Z) \subset \widetilde{\mathcal{M}}$ is a closed subvariety

using open set $\mathcal{U} \subset \widetilde{\mathcal{M}}$ of definition

Of course the log blow-up

$$\widetilde{\mathcal{M}} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

is not canonical.

But the resulting cycle class

$$\widetilde{AJ}_A\big|_{\mathcal{U}}^{*}[Z] \quad \text{supported on} \quad \widetilde{AJ}_A\big|_{\mathcal{U}}^{-1}(Z)$$

defines a canonical log cycle class

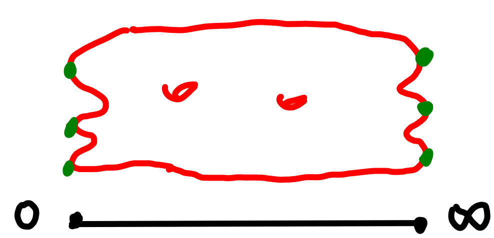
$$DR_{g,A}^{\log} \in \log CH^g(\overline{\mathcal{M}}_{g,n})$$

which pushes-forward to the usual

$$DR_{g,A} \in CH^g(\overline{\mathcal{M}}_{g,n})$$

defined via the Gromov-Witten theory

of $\mathbb{P}^1$.



In fact $DR^{\log}_{g,A}$ is more natural

than $DR_{g,A}$ from several perspectives.

Example: Let $A = (a_1, \ldots, a_n)$

and $B = (b_1, \ldots, b_n)$ $\sum a_i = \sum b_i = 0$

given any $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in SL_2(\mathbb{Z})$

We obtain new vectors

$$MA = m_{11} A + m_{21} B$$

$$MB = m_{12} A + m_{22} B$$

SL-invariance also for more vectors

Theorem (Holmes - Pixton - Schmitt 2017)

$$DR^{\log}_{g,A} \cdot DR^{\log}_{g,B} = DR^{\log}_{g,MA} \cdot DR^{\log}_{g,MB}$$

in $\log CH^{g}(\bar{\mathcal{M}}_{g,n}, \partial \bar{\mathcal{M}}_{g,n})$

Computation (Buryak-Rossi 2019):

$$\int_{\overline{\mathcal{M}}_{g,3}} \pi_* \left(DR^{\log}_{g,A} \cdot DR^{\log}_{g,B} \cdot DR^{\log}_{g,C} \right)$$

$$\parallel$$

$$\frac{\delta^{2g}}{2^{3g}\, g!\, (2g+1)!!}$$

Later derivations by Bousseau, Ranganathan

by left multiplication

What is δ ? Must be an SL_3-invariant

of the 3x3 matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$.

Can't be det (since det = 0).

δ = GCD of all 2x2 minors of

Sign doesn't matter!

A difficulty in studying $DR^{\log}_{g,A}$ is knowing how much to blow-up.

But there is an almost perfect solution for this via stability conditions.

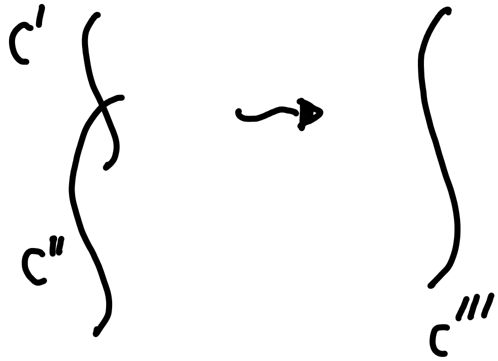
A stability condition θ of type (g,n) is a rule which assigns a rational number to every irreducible component of every stable curve of $\overline{M}_{g,n}$ satisfying

(i) deformation invariance

(ii) compatibility with smoothing of nodes

(iii) $\theta(C) = 0$ for nonsingular $(C, p_1, \ldots, p_n)$

Compatibility with smoothing of nodes:



$$\Theta(C') + \Theta(C'') = \Theta(C''')$$

Once we have Θ

$\Rightarrow$ moduli stack of degree 0 line bundles* on stable curves

$$\begin{array}{c} Pic^{\Theta} \\ \pi \downarrow \\ \overline{M}_{g,n} \end{array}$$

* standard caveat concerning possible singularities at nodes

Studied for over 30 years:

Caporaso, P, Kass-Pagani, Abreu-Pacini

Esteves, Melo, Viviani

follow the conventions here

A review: $\begin{array}{c} L \\ \downarrow \\ C \end{array}$ is θ-stable iff

intersection with complement

$$-\frac{E(\Gamma,\Gamma^c)}{2} + \theta(\Gamma) < \deg L|_{\Gamma} < \frac{E(\Gamma,\Gamma^c)}{2} + \theta(\Gamma)$$

for all proper subcurves $\Gamma \subset C$

We choose θ to be nondegenerate ← No strictly semistable issues

and small ← trivial bundle is stable

and revisit the Abel-Jacobi diagram:

$$\begin{tikzcd}
 & \mathrm{Jac}_0 \arrow[d, "\pi"] \\
\widetilde{\mathcal{M}} \arrow[ur, dashed, "\widetilde{AJ}_A"] \arrow[r] & \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left, "AJ_A"] \arrow[u, bend right, swap, "\text{zero section}"]
\end{tikzcd}$$

zero section with image Z

$$\begin{tikzcd} \mathrm{Pic}^{\Theta} \arrow[d, "\pi"'] \\ \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

determines a canonical blow-up

$$\begin{tikzcd} & \mathcal{L}^{\Theta} \arrow[dl] \\ \mathcal{C}^{\Theta} \arrow[d] & \\ \mathrm{Pic}^{\Theta} & \end{tikzcd}$$

universal curve and line bundle

$$\begin{tikzcd} & \mathrm{Pic}^{\Theta} \arrow[d, "\pi"] \\ \overline{\mathcal{M}}_{g,n}^{\Theta} \arrow[ur, "\widetilde{AJ}_A"] \arrow[r] & \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left, "AJ_A"] \arrow[u, bend right, "\text{zero section}"'] \end{tikzcd}$$

zero section with image Z

explicit combinatorial subdivision of the cone complex

$C(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$

Bae-Holmes-P-Schmitt-Schwarz

Theorem: Universal DR applied to

$$\begin{tikzcd} & \mathcal{L}^{\Theta} \arrow[dl] \\ \mathcal{C}^{\Theta} \arrow[d] & \\ \overline{\mathcal{M}}_{g,n} & \end{tikzcd}$$

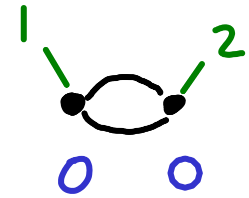
[HMPPS]

yields $DR^{\log}_{g,A}$

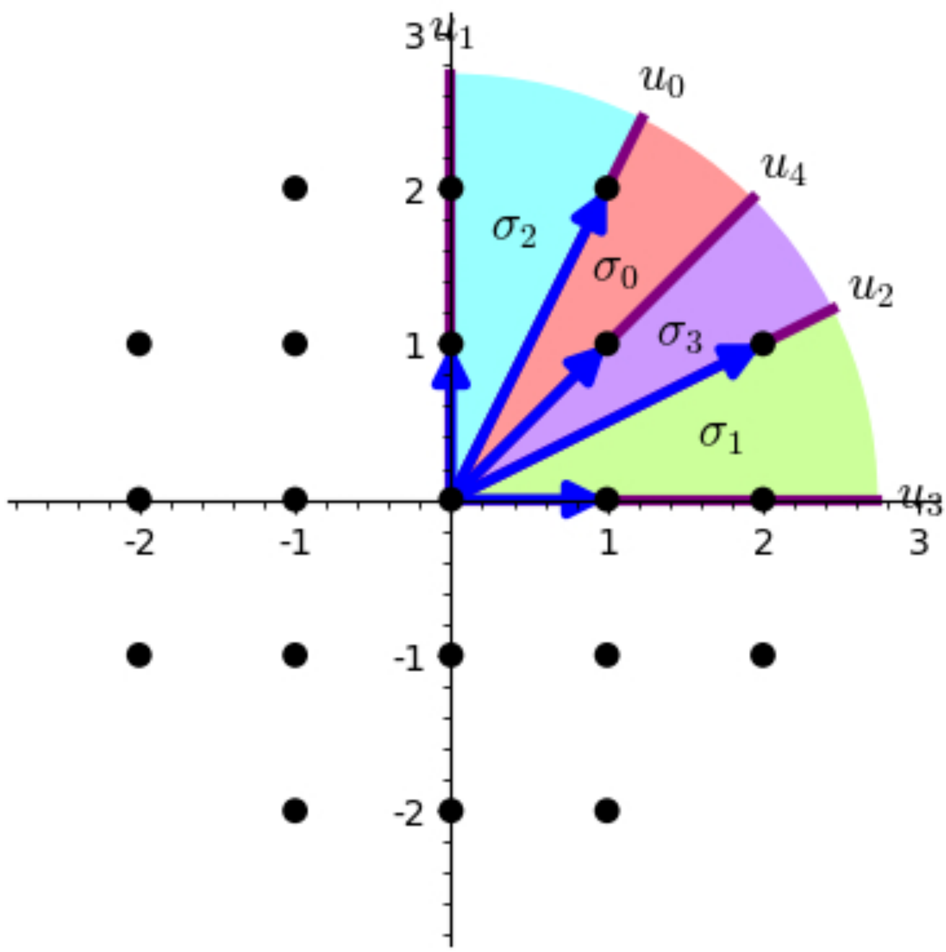
Proof: Uses criterion of Holmes-Schwarz.

Example of shattering the cone complex:

- $\overline{\mathcal{M}}_{1,A}$, $A=(3,-3)$

- Cone $\mathbb{R}^2_{\geqslant 0}$ corresponding to 

- Stability condition with least shattering



Holmes
Schmitt

Logtaut (Sage package for Admcycles)

Final step in the calculation of $\mathrm{DR}^{\log}_{g,A}$ is to express the output of the Universal DR formula in $\log R^*_+(\overline{M}_{g,n}, \partial\overline{M}_{g,n})$.

including ψ, Kappa

Holmes Schmitt

The answer is explicit (and has even been coded in Sage) but I will explain it

Schematically

$$\log R^*_+(\overline{M}_{g,n}, \partial\overline{M}_{g,n})$$

$$\in$$

$$\mathrm{DR}^{\log}_{g,A} = \left[\exp\left(-\tfrac{1}{2}\left(\eta + \Phi(f_2)\right)\right) \cdot \Phi(f_1) \right]_g$$

Codim grade

$$\eta = -\sum a_i^2 \psi_i$$

explicit PP on cone complex of $\overline{M}^{\Theta}_{g,n}$

Main Theorem of Holmes-Molcho-P-Pixton-Schmitt from Holmes-Molcho-P-Pixton-Schmitt :

• The definition of f_1 requires a sum over weightings: for a positive integer r, an *admissible weighting mod* r on $\widehat{\Gamma}$ is a flow w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

$$\mathrm{div}(w) = D \in (\mathbb{Z}/r\mathbb{Z})^{V(\widehat{\Gamma})}\,.$$

We define

$$\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)} = \sum_w r^{-h_1(\Gamma)} \prod_{e\in E(\widehat{\Gamma})} \exp\left(\frac{\overline{w}(\vec{e})\cdot\overline{w}(\overleftarrow{e})}{2}\widehat{\ell}_e\right) \in \mathbb{Q}[[\widehat{\ell}_e : e \in E(\widehat{\Gamma})]], \quad (24)$$

where the sum runs over admissible weightings w mod r. Inside the exponential, $\overline{w}(\vec{e})$ and $\overline{w}(\overleftarrow{e})$ denote the unique representative of $w(\vec{e}) \in \mathbb{Z}/r\mathbb{Z}$ and $w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ in $\{0,\ldots,r-1\}$.

As in [25, Appendix], one shows that in each fixed degree in the variables $\widehat{\ell}_e$, the element $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$ is polynomial in r for sufficiently large r. We denote by $\mathrm{Cont}_{(\widehat{\Gamma},D,I)}$ the polynomial in the variables $\widehat{\ell}_e$ obtained by substituting $r=0$ into the polynomial expression for $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$. We define

$$f_1\big|_{\sigma_I} = \mathrm{Cont}_{(\widehat{\Gamma},D,I)}\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \in \mathbb{Q}[[\ell_e : e \in E(\Gamma)]]\,, \quad (25)$$

where we use the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$ associated to σ_I from Claim 2. We claim that these functions fit together to give a well-defined strict piecewise formal power series f_1 on $\widetilde{\Sigma}_\theta$.

• To define f_2 on $\widetilde{\Sigma}_\theta$, we fix a vertex $v_0 \in V(\widehat{\Gamma})$. For every length assignment $\widehat{\ell}$ in the cone τ_I and any vertex $v \in V(\widehat{\Gamma})$, let $\gamma_{v_0\to v}$ be a path from v_0 to v in $\widehat{\Gamma}$. We define

$$\alpha(v) = \sum_{\vec{e}\in\gamma_{v_0\to v}} I(\vec{e})\cdot\widehat{\ell}_e\,, \quad (26)$$

where the sum is over the oriented edges $\vec{e}$ constituting the path $\gamma_{v_0\to v}$. The defining equations of τ_I imply that for $\widehat{\ell}\in\tau_I$ the expression (26) is independent of the chosen path $\gamma_{v_0\to v}$. We define

$$f_2 = \sum_{v\in V(\widehat{\Gamma})} (D+\deg_{k,A})(v)\cdot\alpha(v)\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \in \mathbb{Q}[\ell_e : e\in E(\Gamma)]\,. \quad (27)$$

The substitution of variables $\widehat{\ell} = \widehat{\ell}(\ell)$, which give the inverse of the isomorphism $\tau_I \to \sigma_I$ and thus have image in τ_I, ensure that the expression is independent of the choice of the paths $\gamma_{v_0\to v}$. The expression is independent of the base vertex v_0. since the divisor $D+\deg_{k,A}$ has total degree 0 on $\widehat{\Gamma}$.

IV. Further directions

(after Kumaran, Molcho, Ranganathan)

- using log DR and product formula in log Chow

$\Downarrow$

- Computation of log CH virtual class of $\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1 / \Delta$

full toric boundary

Birational invariance

Abramovich–Wise

$\Downarrow$

- Computation of log CH virtual class of X / Δ

full toric case

$\Downarrow$

- The above concerns the entire log CH virtual class. But in upcoming work of Kumaran-Ranganathan insertions are added in the toric case.

A goal: Control the log CH virtual class in the toric case formally by combining genus 0 data and log DR.

(we have such control in usual GW theory)

A wild hope would be that such a formulation would generalize past the toric case.

Example of $\mathbb{P}^3$ with anticanonical divisor D

Can try to approach using target log DR

↓

★ $\mathbb{P}^3 / D_4$ quartic K3

? $\mathbb{P}^3 / D_3 \cup D_1$ cubic surface + plane

? $\mathbb{P}^3 / D_2 \cup D_2$ two quadrics

? $\mathbb{P}^3 / D_2 \cup D_1 \cup D_1$ quadric + 2 planes

★ $\mathbb{P}^3 / D_1 \cup D_1 \cup D_1 \cup D_1$ 4 planes

↑

fits the path above by Kumaran-Molcho-Ranganathan

The End