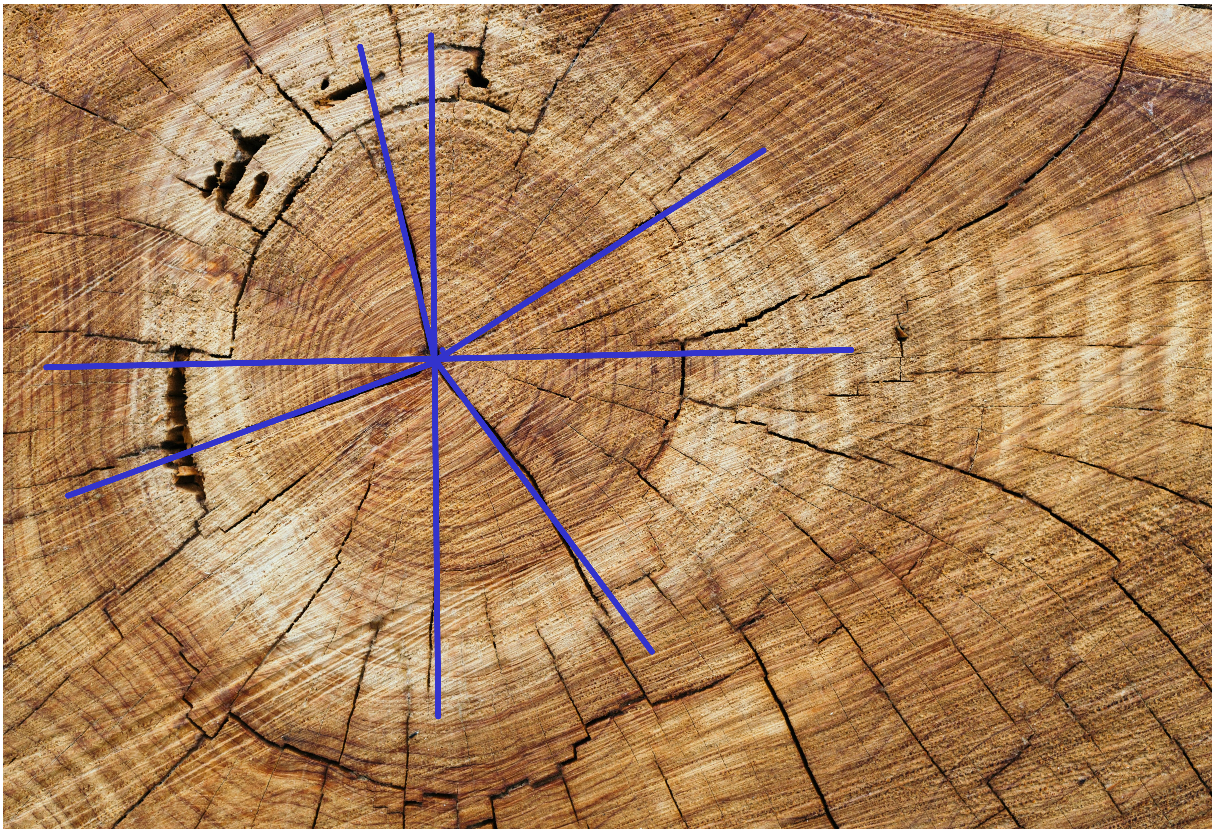
Log intersection theory of $\overline{M}_{g,n}$



Tel Aviv Geometry Seminar

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Photo: Sarah Worth (Unsplash)

$\overline{\mathcal{M}}_{g,n}$ Deligne-Mumford moduli space of stable curves.

orbifold of $\dim_{\mathbb{C}}$ $3g-3+n$

Standard theories

$$RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$$

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset CH^*(\overline{\mathcal{M}}_{g,n})$$

We will however consider

$$\log CH^*(\overline{\mathcal{M}}_{g,n})$$

often inspires fear and loathing, but dont worry

What is log intersection theory?

Given any nonsingular variety X
with a normal crossings divisor $D \subset X$
we obtain a log scheme (X, D).

There are two related Chow constructions
lying over $CH^*(X)$

$$CH^*(X) \subset \log CH^*(X,D) \subset b\,CH^*(X)$$

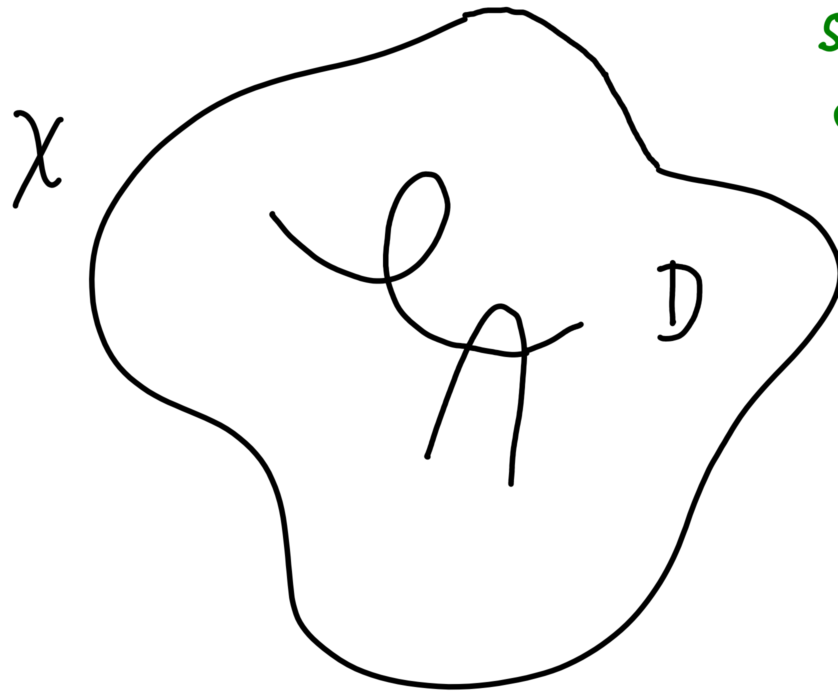
Shokurov

Our main example is the log scheme

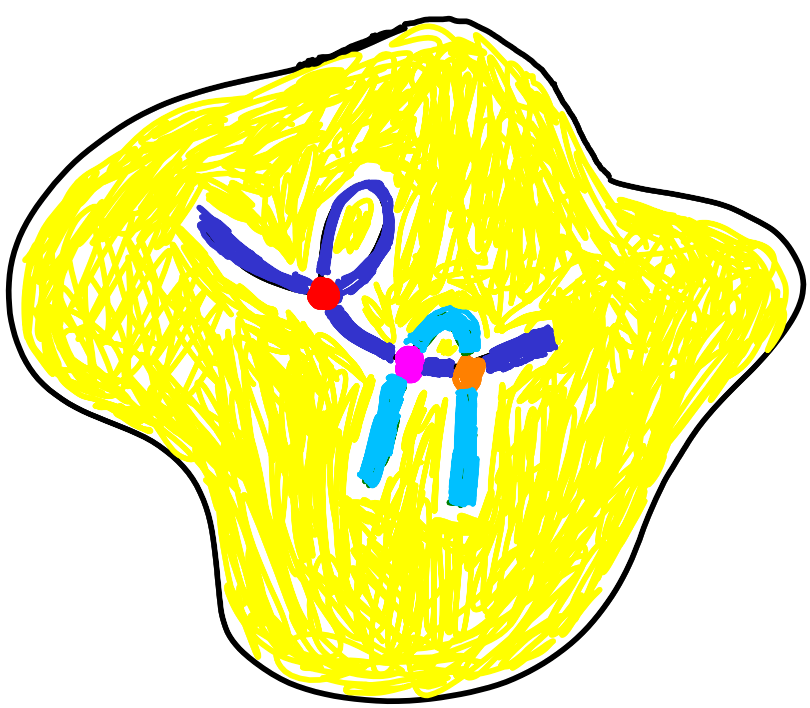
$$\left(\overline{\mathcal{M}}_{g,n} \, , \, \partial \overline{\mathcal{M}}_{g,n} \right)$$

normal crossings
divisor of
nodal curves

Not assumed strict normal crossings



Basic Notion of stratification



Strata indicated by colors

A stratum $S \subset X$ is nonsingular and quasiprojective

$\bar{S} \subset X$ may be singular (mildly)

A simple blow-up of (X, D) is a blow up along a nonsingular stratum closure $\bar{S} \subsetneq X$.

$$Bl: (\hat{X}, \hat{D}) \to (X, D)$$

blowup

strict transform of D union the exceptional divisor E

Define a category $B(X, D)$

- Objects are $(\tilde{X}, \tilde{D}) \xrightarrow{\tilde{\phi}} (X, D)$ where $\tilde{\phi}$ is a composition of simple blowups

- Morphisms are commutative diagrams

$$\begin{tikzcd}
(\tilde{\tilde{X}}, \tilde{\tilde{D}}) \arrow[rr, "\gamma"] \arrow[dr, "\tilde{\tilde{\phi}}"'] & & (\tilde{X}, \tilde{D}) \arrow[dl, "\tilde{\phi}"] \\
& (X, D) &
\end{tikzcd}$$

γ is a composition of simple blowups

$$\log CH^*(X, D)$$

$$\| \text{ definition}$$

$$\varinjlim_{(\tilde{X}, \tilde{D}) \in B(X,D)} CH^*(\tilde{X})$$

maps are pullbacks

The idea is to simultaneously study the cycle theory of log compactifications of $U = X - D$.

$$(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}) \longleftarrow$$ nonsingular with normal crossings

$$\log CH^*(\overline{\mathcal{M}}_{g,n}) \overset{\text{Def}}{=} \log CH^*(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$$

<u>Questions</u>:

(i) why study?

(ii) Even if you wanted to, how?

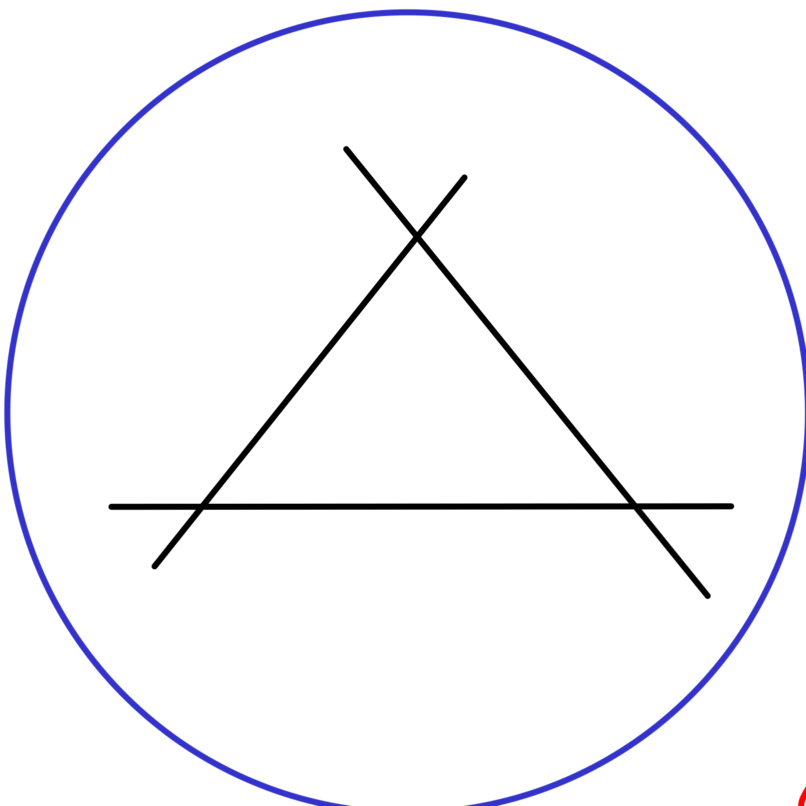
(iii) Are there any results?

- Motivations from toric geometry

$(X, \partial X)$

nonsingular toric variety

toric boundary



$(\mathbb{P}^2, \partial \mathbb{P}^2)$

$$\log CH^*(X, \partial X) = ?$$

Complete results

Brion, Payne, Fulton, ...

- Motivations from log GW theory:

Let (X, D) be a variety with a normal crossing divisor

Log scheme

Abramovich-Chen-Gross-Siebert

There is a theory of log stable maps, virtual classes

Abramovich-Marcus-Wise

Herr-Molcho-P-Wise

$\Downarrow$

Classes in $\log CH^*(\bar{M}_{g,n})$ Graber-Ranganathan, Spelier

just as usual GW theory $\Rightarrow$ classes in $CH^*(\bar{M}_{g,n})$

- Motivations from Abel-Jacobi theory

Resolving the Abel-Jacobi map, D. Holmes

Double ramification cycle Marcus-Wise

How to study $\log CH(X, D)$?

Language of Piecewise polys on the Cone Complex

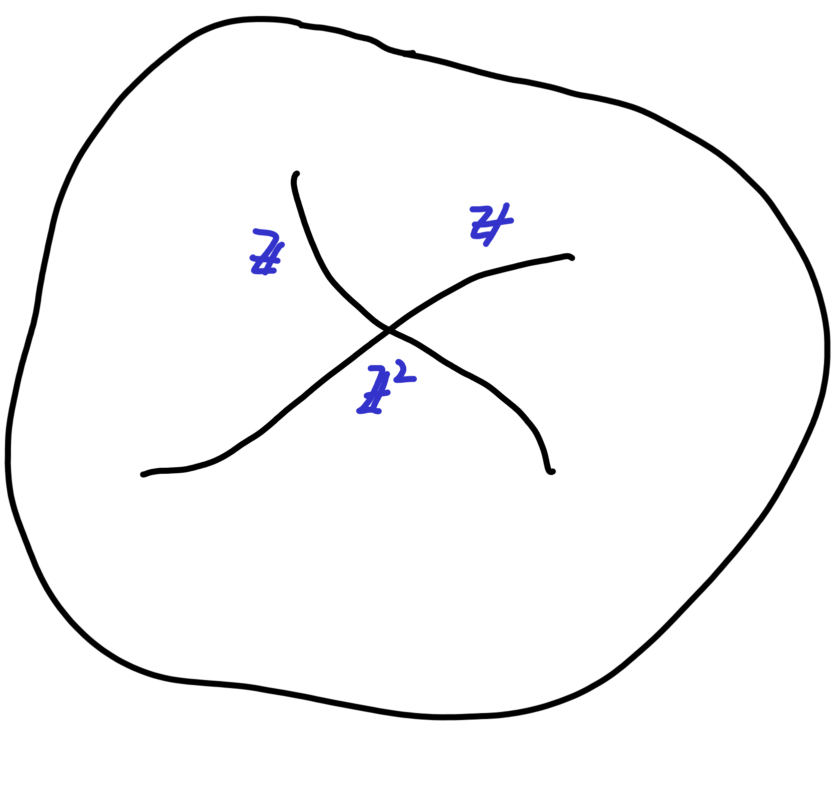
Ranganathan

Molcho-P-Schmitt

MR, Holmes-Schwarz

$$(X, D) \rightsquigarrow C(X, D)$$

Cone Complex



open Stratum codim r

$$\updownarrow$$

$$\mathbb{R}^r \cong \mathbb{Z}^r \otimes_{\mathbb{Z}} \mathbb{R}$$

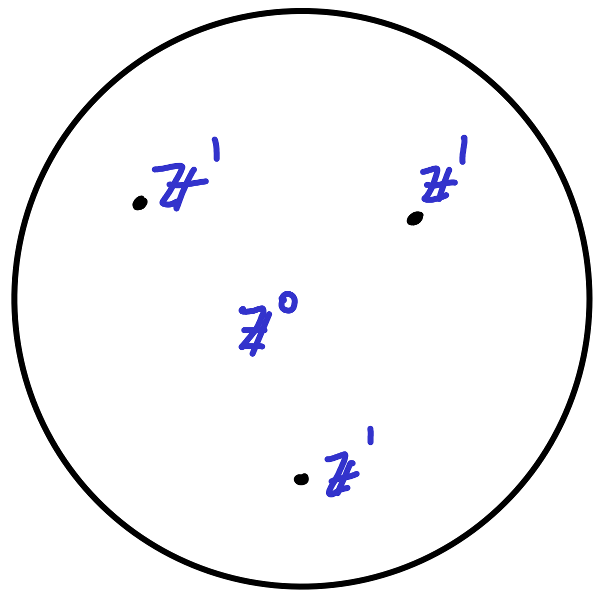
face given by

$$\mathbb{R}^r_{\geq 0}$$

When strata closures meet $\Rightarrow$ inclusion of faces of $C(X,D)$

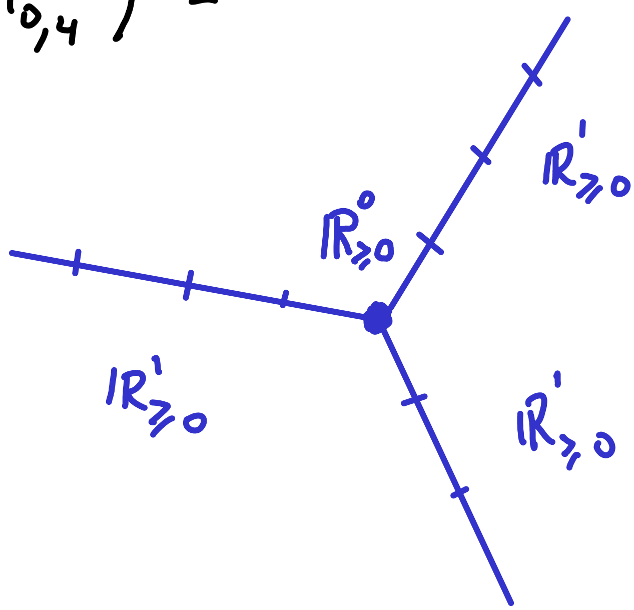
Simple example:

$$(\bar{M}_{0,4}, \partial \bar{M}_{0,4})$$



Cone complex is

$$C(\bar{M}_{0,4}, \partial \bar{M}_{0,4}) =$$



Algebra $PP(X, D)$ ← Piecewise polys on subdivisions

$C(X, D)$

Theorem: $PP(X, D) \xrightarrow{\Phi} \log CH^*(X, D)$

This is how to think about classes → Proposed by Ranganathan, see Molcho-P-Schmitt

$$\mathrm{Image}\big(PP(X, D)\big) \subset \log CH^*(X, D)$$

These are tautological classes in $\log CH^*$

Definition $\log R^*(X, D) = \mathrm{Image}\big(PP(X, D)\big)$

Holmes - Molcho - P - Pixton - Schmitt

Theorem*: $PP(\overline{M}_{0,n}, \partial\overline{M}_{0,n}) \to \log CH^*(\overline{M}_{0,n}, \partial\overline{M}_{0,n})$

is surjective with $\ker = \mathrm{WDVV}$

In other words:

$$\frac{PP(\overline{M}_{0,n}, \partial\overline{M}_{0,n})}{\langle \mathrm{WDVV} \rangle} \cong \log CH^*(\overline{M}_{0,n}, \partial\overline{M}_{0,n})$$

Proof:
- Use Keel's presentation of $CH^*(\overline{M}_{0,n})$
- Study Blow ups
- Use a new tubular property of the boundary geometry of $\overline{M}_{0,n}$

Theorem: for (X, D) toric, Brion, Payne, Fulton

$$PP(X, D) / \langle \text{Div rels} \rangle \cong \log CH^*(X, D)$$

Question: What is the kernel of

$$PP(\bar{\mathcal{M}}_{1,n}, \partial\bar{\mathcal{M}}_{1,n}) \to \log CH^*(\bar{\mathcal{M}}_{1,n}, \partial\bar{\mathcal{M}}_{1,n}) \ ?$$

Can the kernel be completely explained using WDVV and Getzler equations?

$\boxed{\text{My view: YES}}$

Lots of following questions

as an example

Claim: $\log R^{3g-3+n}(\bar{\mathcal{M}}_{g,n}, \partial\bar{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$

Doesnt change with more classes

Another definition of the tautological ring:

$\log R^*_L(\bar{\mathcal{M}}_{g,n}, \partial\bar{\mathcal{M}}_{g,n})$ where we include systematically all tautological classes from the moduli of curves

Systematically here means: at every stage of the blow-up, also add all tautological classes defined on the center.

Claim: We can write a set of relations $\log \mathcal{P}$ in $\log R_L^*(\bar{M}_{g,n}, \partial\bar{M}_{g,n})$ which is implied by Pixton's set $\mathcal{P}$ for the moduli space of curves.

Perhaps the set $\log \mathcal{P}$ generates all relations in $\log R_L^*(\bar{M}_{g,n}, \partial\bar{M}_{g,n})$?

Theorem*: Pixton's conjecture that P generates all relations in all the tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$

$$\Downarrow$$

Log P generates all relations in all the log tautological rings $\log R^*_L(\overline{\mathcal{M}}_{g,n})$

A different sort of result:

Theorem [Molcho-P-Schmitt]

$$\lambda_g \in \log R^g(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

no L

Top chern class of the Hodge bundle on $\overline{\mathcal{M}}_g$

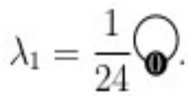
$$\begin{array}{ccc} H^0(C,\omega_C) & \subset & \mathbb{E}_g \\ \downarrow & & \downarrow \\ [C] & \in & \overline{\mathcal{M}}_g \end{array}$$

Proof: use Pixton's DR formula

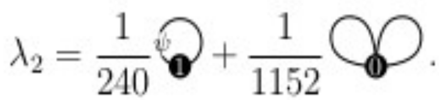
Pixton's formula has no kappa classes:

Diagrams from JPPZ

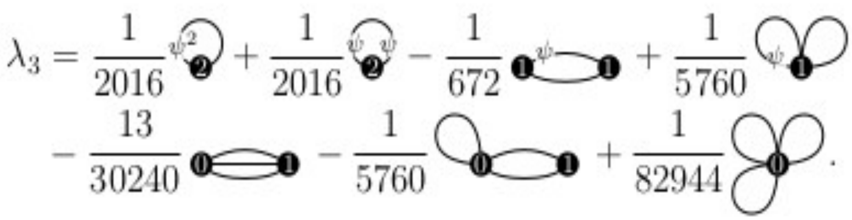
Genus 1.



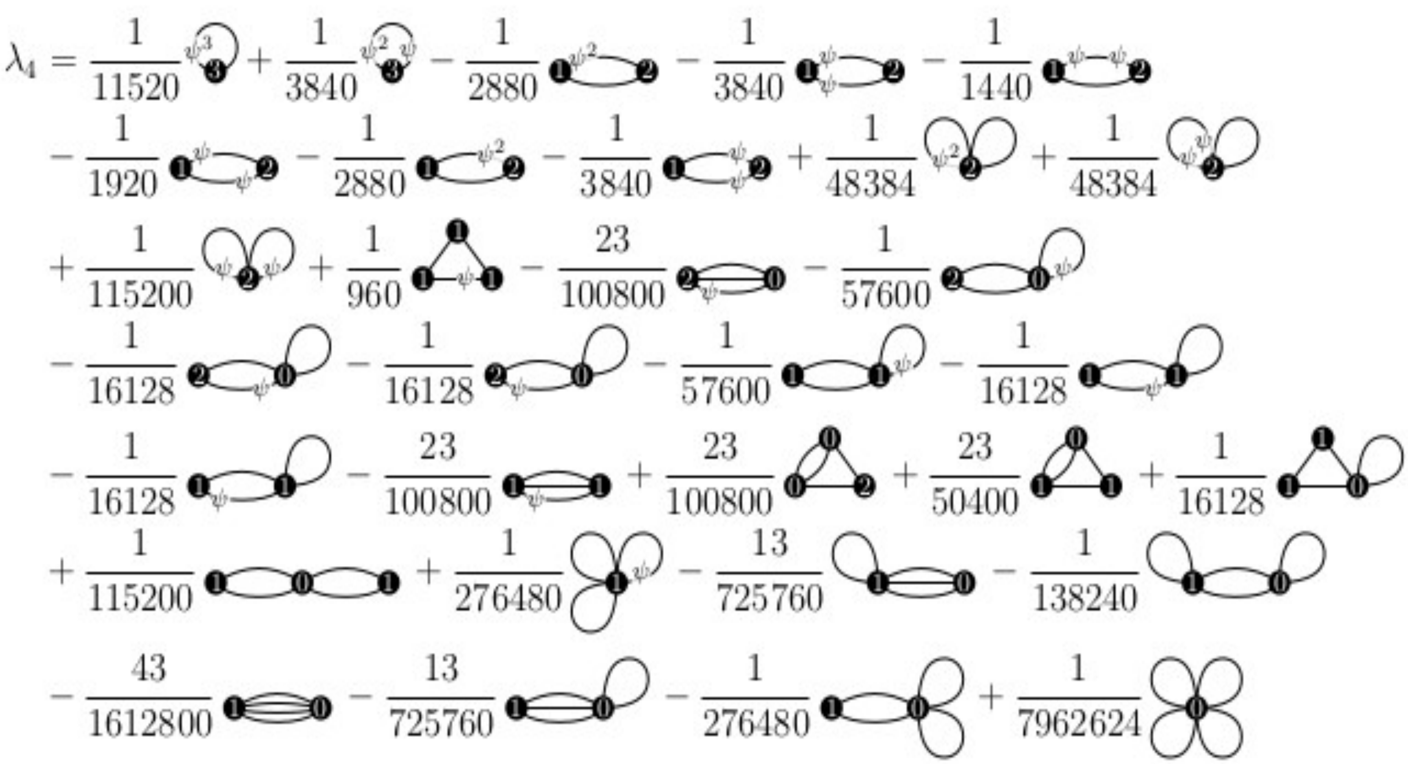
Genus 2.



Genus 3.



Genus 4.



Artist: Felix

All of the discussion so far may be viewed as an introduction to the main topic of the lecture: Abel-Jacobi theory

Let $A = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{Z}$ and $\sum_{i=1}^{n} a_i = 0$

Let $\mathrm{Jac}_0 \xrightarrow{\pi} \overline{M}_{g,n}$ be the universal Jacobian of multidegree 0 line bundles.

We have a rational map

$$AJ_A : \overline{M}_{g,n} \dashrightarrow \mathrm{Jac}_0$$

defined on nonsingular curves by

$$(C, p_1, \ldots, p_n) \mapsto \mathcal{O}_C\left(\sum a_i p_i\right)$$

$$\begin{tikzcd} \mathrm{Jac}_0 \arrow[d, "\pi"] \\ \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left=60, "AJ_A"] \arrow[u, bend right=60, "\text{zero section with image } Z"'] \end{tikzcd}$$

We would like to define a locus in $\overline{\mathcal{M}}_{g,n}$ which corresponds to the condition

$$\text{“} \mathcal{O}_C\left(\sum a_i p_i\right) \cong \mathcal{O}_C \text{”}$$

Not a closed condition

Abel-Jacobi locus where there exists a function

$$f : (C, p_1, \dots, p_n) \to \mathbb{P}^1$$

with zeros and poles given by $A = (a_1, \dots, a_n)$

We would like to define the locus by

$$\text{“} AJ_A^{-1}(Z) \subset \overline{\mathcal{M}}_{g,n} \text{”}$$

Not a closed subvariety

Holmes
Marcus-Wise

Idea is to (Partially) resolve AJ_A via log blow-ups of $\bar{\mathcal{M}}_{g,n}$

$$\begin{tikzcd}
 & \mathrm{Jac}_0 \arrow[d, "\pi"] \\
\tilde{\mathcal{M}} \arrow[ur, dashed, "\widetilde{AJ}_A"] \arrow[r] & \bar{\mathcal{M}}_{g,n} \arrow[u, bend left=50, dashed, "AJ_A"] \arrow[u, bend right=50, "\text{zero section with image } Z"']
\end{tikzcd}$$

Where $\widetilde{AJ}_A\big|_{\mathcal{U}}^{-1}(Z) \subset \tilde{\mathcal{M}}$ is a closed subvariety

using open set $\mathcal{U} \subset \tilde{\mathcal{M}}$ of definition

Of course the log blow-up

$$\tilde{\mathcal{M}} \longrightarrow \bar{\mathcal{M}}_{g,n}$$

is not canonical.

But the resulting cycle class

$$\widetilde{AJ}_A\big|_{\mathcal{U}}^{\star}[Z] \quad \text{supported on} \quad \widetilde{AJ}_A\big|_{\mathcal{U}}^{-1}(Z)$$

defines a canonical log cycle class

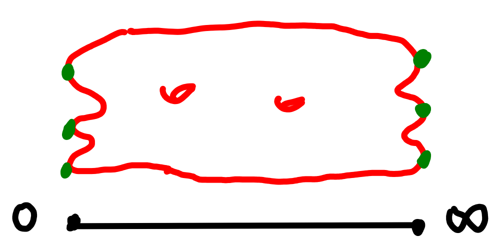
$$DR_{g,A}^{\log} \in \log CH^g(\bar{\mathcal{M}}_{g,n})$$

which pushes-forward to the usual

$$DR_{g,A} \in CH^g(\bar{\mathcal{M}}_{g,n})$$

defined via the Gromov-Witten theory

of $\mathbb{P}^1$.



In fact $DR^{\log}_{g,A}$ is more natural

than $DR_{g,A}$ from several perspectives.

Example: Let $A = (a_1, \dots, a_n)$ $\qquad \sum a_i = \sum b_i = 0$

and $B = (b_1, \dots, b_n)$

given any $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in SL_2(\mathbb{Z})$

We obtain new vectors

$$MA = m_{11} A + m_{21} B$$

$$MB = m_{12} A + m_{22} B$$

SL-invariance also for more vectors

Theorem (Holmes - Pixton - Schmitt 2017)

$$DR^{\log}_{g,A} \cdot DR^{\log}_{g,B} = DR^{\log}_{g,MA} \cdot DR^{\log}_{g,MB}$$

in $\log CH^g(\bar{\mathcal{M}}_{g,n}, \partial \bar{\mathcal{M}}_{g,n})$

Computation (Buryak-Rossi 2019):

$$\int_{\overline{\mathcal{M}}_{g,3}} \pi_* \left(DR^{\log}_{g,A} \cdot DR^{\log}_{g,B} \cdot DR^{\log}_{g,C} \right)$$

$$= \frac{\delta^{2g}}{2^{3g}\, g!\, (2g+1)!!}$$

Later derivations by Bousseau, Ranganathan

by left multiplication

What is δ? Must be an SL_3-invariant

of the 3x3 matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$.

Can't be det (since det = 0).

δ = GCD of all 2x2 minors of

Sign doesn't matter!

A difficulty in studying $DR^{\log}_{g,A}$ is knowing how much to blow-up.

But there is an almost perfect solution for this via stability conditions.

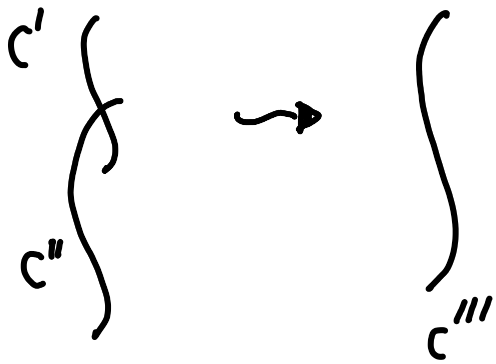
A stability condition θ of type (g,n) is a rule which assigns a rational number to every irreducible component of every stable curve of $\overline{M}_{g,n}$ satisfying

(i) deformation invariance

(ii) compatibility with smoothing of nodes

(iii) $\theta(C) = 0$ for nonsingular $(C, p_1, \dots, p_n)$

Compatibility with smoothing of nodes:



$$\theta(C') + \theta(C'') = \theta(C''')$$

Once we have θ

$\Rightarrow$ moduli stack of degree 0 line bundles* on stable curves

$$\begin{array}{c} Pic^{\theta} \\ \pi \downarrow \\ \overline{M}_{g,n} \end{array}$$

* standard caveat concerning possible singularities at nodes

Studied for over 30 years:

Caporaso, P, Kass-Pagani, Abreu-Pacini, Esteves, Melo, Viviani

follow the conventions here

A review: $\begin{array}{c} L \\ \downarrow \\ C \end{array}$ is θ-stable iff

intersection with complement

$$-\frac{E(\Gamma,\Gamma^c)}{2} + \theta(\Gamma) < \deg L|_\Gamma < \frac{E(\Gamma,\Gamma^c)}{2} + \theta(\Gamma)$$

for all proper subcurves $\Gamma \subset C$

We choose θ to be nondegenerate ← No strictly semistable issues

and small ← trivial bundle is stable

and revisit the Abel-Jacobi diagram:

$$\begin{tikzcd}
 & & \mathrm{Jac}_0 \arrow[d, "\pi"] \arrow[d, bend left=60, "\text{zero section with image } Z"'] \\
\widetilde{\mathcal{M}} \arrow[urr, dashed, "\widetilde{AJ}_A"] \arrow[rr] & & \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left=60, "AJ_A"]
\end{tikzcd}$$

$$\begin{array}{c} \mathrm{Pic}^{\ominus} \\ \downarrow \pi \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

determines a canonical blow-up

$$\begin{array}{ccc} & & \mathcal{L}^{\ominus} \\ & \swarrow & \\ C^{\ominus} & & \\ \downarrow & & \\ \mathrm{Pic}^{\ominus} & & \end{array}$$

universal curve and line bundle

$$\begin{tikzcd} & \mathrm{Pic}^{\ominus} \arrow[d, "\pi"] \\ \overline{\mathcal{M}}^{\ominus}_{g,n} \arrow[ur, "\widetilde{AJ}_A"] \arrow[r] & \overline{\mathcal{M}}_{g,n} \arrow[u, bend left, dashed, "AJ_A"] \arrow[u, bend right, "\text{zero section}"'] \end{tikzcd}$$

zero section with image Z

explicit Combinatorial Subdivision of the Cone complex

$$C(\overline{\mathcal{M}}_{g,n}, \partial \overline{\mathcal{M}}_{g,n})$$

Bae-Holmes-P-Schmitt-Schwarz

Theorem: Universal DR applied to

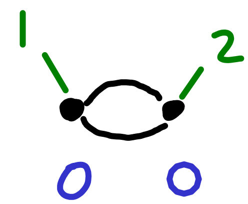
[HMPPS]

$$\begin{array}{ccc} & & \mathcal{L}^{\ominus} \\ & \swarrow & \\ C^{\ominus} & & \\ \downarrow & & \\ \overline{\mathcal{M}}_{g,n} & & \end{array}$$

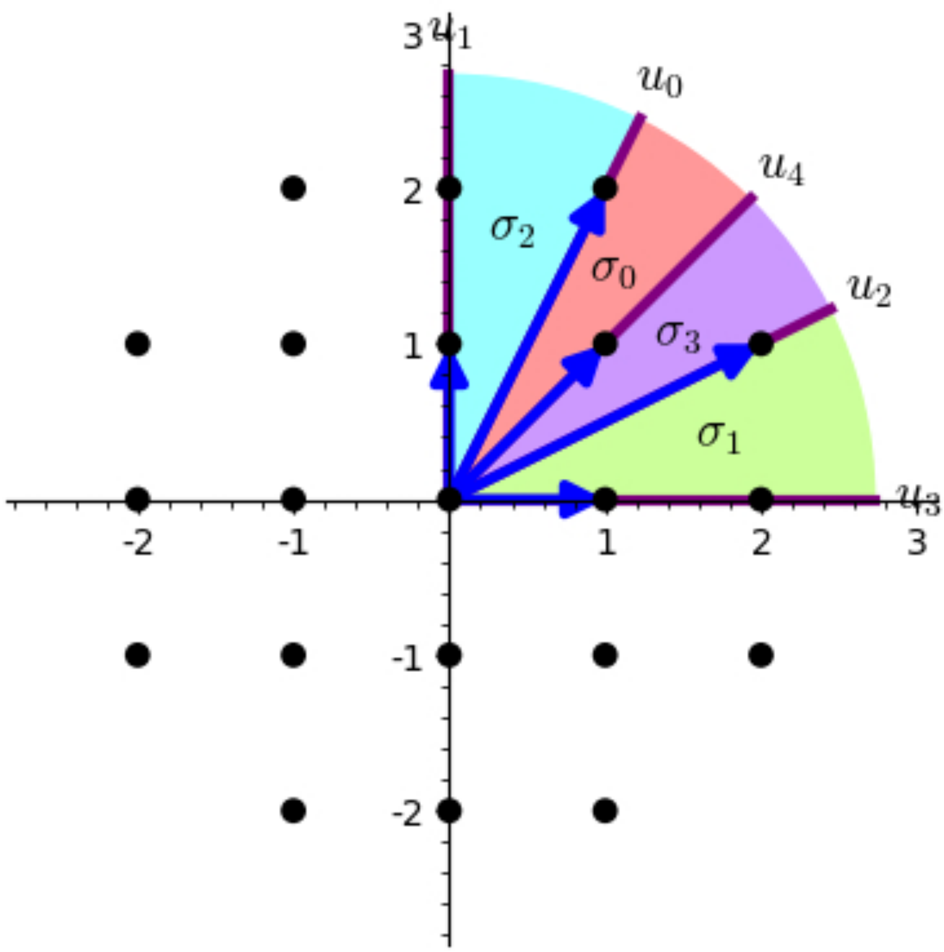
yields $DR^{\log}_{g,A}$

Proof: Uses criterion of Holmes-Schwarz.

Example of shattering the cone complex:

- $\overline{\mathcal{M}}_{1,A}$, $A = (3,-3)$
- Cone $\mathbb{R}^2_{\geq 0}$ corresponding to 

- Stability condition with least shattering



Holmes
Schmitt

<u>Logtaut</u> (Sage package for Admcycles)

Final step in the calculation of $\mathrm{DR}^{\log}_{g,A}$

is to express the output of the

Universal DR formula in $\log R^*_L(\overline{M}_{g,n}, \partial\overline{M}_{g,n})$.

Holmes Schmitt

The answer is explicit (and has even been coded in Sage) but I will explain it

Schematically

$$\log R^*_L(\overline{M}_{g,n}, \partial\overline{M}_{g,n})$$

$$\cup$$

$$\mathrm{DR}^{\log}_{g,A} = \left[\exp\left(-\tfrac{1}{2}\left(\eta + \Phi(f_2)\right)\right) \cdot \Phi(f_1) \right]_g$$

Codim grade

$$\eta = -\sum a_i^2 \psi_i$$

explicit PP on cone complex of $\overline{M}_{g,n}^{\Theta}$

Main Theorem of Holmes-Molcho-P-Pixton-Schmitt from Holmes-Molcho-P-Pixton-Schmitt :

• The definition of f_1 requires a sum over weightings: for a positive integer r, an *admissible weighting mod* r on $\widehat{\Gamma}$ is a flow w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

$$\mathrm{div}(w) = D \in (\mathbb{Z}/r\mathbb{Z})^{V(\widehat{\Gamma})}\,.$$

We define

$$\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)} = \sum_w r^{-h_1(\Gamma)} \prod_{e\in E(\widehat{\Gamma})} \exp\left(\frac{\overline{w}(\vec{e})\cdot\overline{w}(\overleftarrow{e})}{2}\widehat{\ell}_e\right) \in \mathbb{Q}[[\widehat{\ell}_e : e \in E(\widehat{\Gamma})]]\,, \tag{24}$$

where the sum runs over admissible weightings w mod r. Inside the exponential, $\overline{w}(\vec{e})$ and $\overline{w}(\overleftarrow{e})$ denote the unique representative of $w(\vec{e}) \in \mathbb{Z}/r\mathbb{Z}$ and $w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ in $\{0,\ldots,r-1\}$.

As in [25, Appendix], one shows that in each fixed degree in the variables $\widehat{\ell}_e$, the element $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$ is polynomial in r for sufficiently large r. We denote by $\mathrm{Cont}_{(\widehat{\Gamma},D,I)}$ the polynomial in the variables $\widehat{\ell}_e$ obtained by substituting $r=0$ into the polynomial expression for $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$. We define

$$f_1\big|_{\sigma_I} = \mathrm{Cont}_{(\widehat{\Gamma},D,I)}\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \in \mathbb{Q}[[\ell_e : e \in E(\Gamma)]]\,, \tag{25}$$

where we use the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$ associated to σ_I from Claim 2. We claim that these functions fit together to give a well-defined strict piecewise formal power series f_1 on $\widetilde{\Sigma}_\theta$.

• To define f_2 on $\widetilde{\Sigma}_\theta$, we fix a vertex $v_0 \in V(\widehat{\Gamma})$. For every length assignment $\widehat{\ell}$ in the cone τ_I and any vertex $v \in V(\widehat{\Gamma})$, let $\gamma_{v_0\to v}$ be a path from v_0 to v in $\widehat{\Gamma}$. We define

$$\alpha(v) = \sum_{\vec{e}\in\gamma_{v_0\to v}} I(\vec{e})\cdot\widehat{\ell}_e\,, \tag{26}$$

where the sum is over the oriented edges $\vec{e}$ constituting the path $\gamma_{v_0\to v}$. The defining equations of τ_I imply that for $\widehat{\ell} \in \tau_I$ the expression (26) is independent of the chosen path $\gamma_{v_0\to v}$. We define

$$f_2 = \sum_{v\in V(\widehat{\Gamma})} (D + \deg_{k,A})(v)\cdot\alpha(v)\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \in \mathbb{Q}[\ell_e : e \in E(\Gamma)]\,. \tag{27}$$

The substitution of variables $\widehat{\ell} = \widehat{\ell}(\ell)$, which give the inverse of the isomorphism $\tau_I \to \sigma_I$ and thus have image in τ_I, ensure that the expression is independent of the choice of the paths $\gamma_{v_0\to v}$. The expression is independent of the base vertex v_0. since the divisor $D + \deg_{k,A}$ has total degree 0 on $\widehat{\Gamma}$.

Aaron's calculation in $\overline{M}_{g,4}$:

$$\pi_\star \left(DR^{\log}_{g,(-2,2,0,0)} \cdot DR^{\log}_{g,(0,0,-2,2)} \right)$$

$$= DR_{g,(-2,2,0,0)} \cdot DR_{g,(0,0,-2,2)}$$

$$+ \text{Correction}$$

where $\pi_\star : \log CH(\overline{M}_{g,n}) \to CH(\overline{M}_{g,n})$.

The entire theory of the lecture is used to calculate the correction.

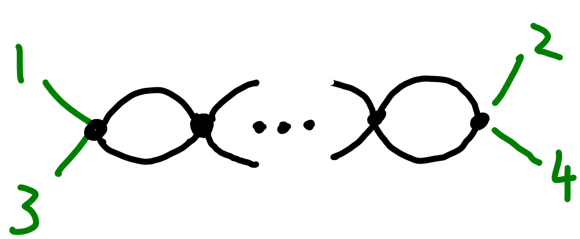
Correction term is :

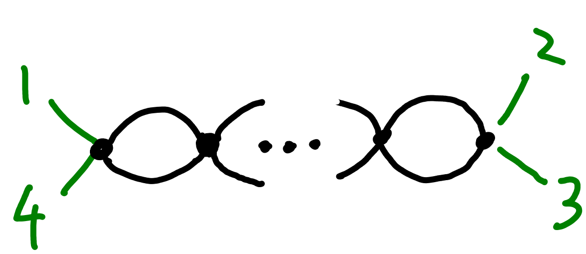
$$\sum_{\substack{1\le m\le g\\ g_0+\cdots+g_m=g-m}} \frac{(-1)^m}{2^m}\left(\xi_* + \hat{\xi}_*\right)\left[\begin{matrix} DR_{g_0}(-2,0,1,1) \\ \cdot \\ DR_{g_0}(0,-2,1,1)\end{matrix} \otimes DR_{g_1}(-1,-1,1,1)^2 \otimes \right.$$

$$\left. DR_{g_2}(-1,-1,1,1)^2 \otimes \cdots \otimes DR_{g_{m-1}}(-1,-1,1,1)^2 \otimes \begin{matrix} DR_{g_m}(-1,-1,2,0)\\ \cdot \\ DR_{g_m}(-1,-1,0,2)\end{matrix}\right]$$

Where $\xi, \hat{\xi} : \prod_{i=0}^{m} \overline{M}_{g_i,4} \to \overline{M}_{g,4}$

via the graphs





Last result of the lecture:

Theorem [HMPPS] Relations!

$$\left[\exp\left(-\frac{1}{2} \left(\eta + \Phi(f_2) \right) \right) \cdot \Phi(f_1) \right]_{>g} = 0$$

in $\log R^*_L(\bar{M}_{g,n}, \partial \bar{M}_{g,n})$.

These are Pixton's log DR relations.

The End