The Hilbert scheme of n-dimensional affine space. Zurich, July 1, 2020. ①

Burt Totaro
(joint with M. Hoyois, J. Jelisiejew, D. Nardin, M. Yakerson).

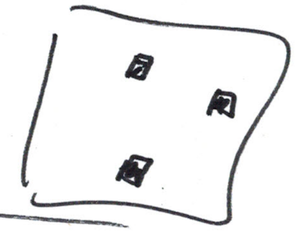
$\mathbb{A}^1$-homotopy type of $\mathrm{Hilb}_d(\mathbb{A}^\infty_k) = \varinjlim_n \mathrm{Hilb}_d \mathbb{A}^n$, say over a field k.

~~This~~. <u>Thm.</u> (HJNTY) $\mathrm{Hilb}_d(\mathbb{A}^\infty) \simeq BGL(d-1) \cong Gr_{d-1}(\mathbb{A}^\infty_k)$.

(1) Review that argument. Prove a stability theorem on $\mathrm{Hilb}_d(\mathbb{A}^n)$ for $n<\infty$.

(2). ~~Bi~~ (T.) $\mathrm{Hilb}_d(\mathbb{A}^n, 0) \hookrightarrow \mathrm{Hilb}_d(\mathbb{A}^n)/_m$ is a homotopy equivalence.

(3). A conjecture on $\mathrm{Hilb}_d(X \times \mathbb{A}^\infty)$.



$\mathbb{A}^n$

$\mathrm{Hilb}_d X :=$ the space of degree-d, 0-dim. closed subschemes X/k

$\mathrm{Hilb}_d \mathbb{A}^n =$ the ~~sp~~space of ideals $I \subset k[x_1, \ldots, x_n]$ such that $k[x_1, \ldots, x_n]/I$ has $\dim_k = d$ as a vector space

$\mathrm{Hilb}_d \mathbb{A}^n$  <u>connected</u> (Hartshorne)

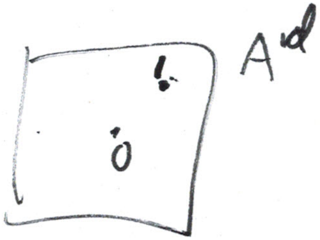
②

Proof of Thm. Define an algebraic stack FFlat_d over $\mathbb{Z}$ (or over k): (considered by B. Poonen)

$$\mathrm{FFlat}_d := \left[\left\{ \begin{array}{l} \text{structures of comm. algebra} \\ \text{on the fixed vector space} \\ \qquad A^d_k \end{array} \right\} \Big/ GL(d) \right] \qquad d \geq 1.$$

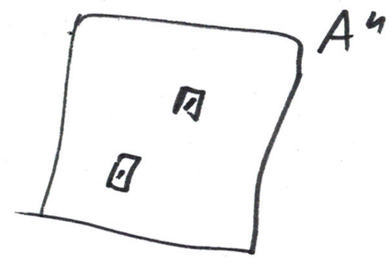
A^d_k $\|$ affine scheme over k.

(quotient stack)



Consider the morphism

$$f : \mathrm{Hilb}_d A^n_k \longrightarrow \mathrm{FFlat}_d.$$



If ~~$d \geq$~~ $n \geq d-1$, then f is smooth and surjective.

Let R be a commutative algebra of degree d over a field k.
The fiber of f over $[R] = [\mathrm{Spec}(R)]$ is

$$\mathrm{Emb}_k(\mathrm{Spec}(R), A^n)$$
$$= \{\text{surjective } k\text{-alg. homs } k[x_1, \ldots, x_n] \twoheadrightarrow R\}.$$
$$\underset{\text{open}}{\subset} \{k\text{-alg. homs } k[x_1, \ldots, x_n] \to R\} \cong R^{\oplus n} \cong k^{nd}.$$

Lemma. The map of lower (1) is $(2n-2d+3)$-connected.

$$f : \mathrm{Hilb}_d \mathbb{A}^n \to \mathrm{FFlat}_d.$$

Pf. The fibers of f are vector spaces minus a closed subset of codimension $\geq n-d+2$. QED.

This uses that if $k[x_1, \ldots, x_n] \to R$ is not surjective, then the k-linear map $\underbrace{k\cdot\{x_1, \ldots, x_n\}}_{\dim. n} \to \underbrace{R/k\cdot 1}_{d-1}$ is not surjective. That space of linear maps has codim. $= n-d+2$. QED

We show that $\boxed{\mathrm{FFlat}_d \simeq_{\mathbb{A}^1} BGL(d-1).}$ Given that, the map

$$\mathrm{Hilb}_d \mathbb{A}^n \to BGL(d-1)$$

is lower (1) $(2n-2d+3)$-connected. So:

$$\underbrace{H^*(BGL(d-1)_{\mathbb{C}}, \mathbb{Z})}_{\mathbb{Z}[c_1, \ldots, c_{d-1}]} \to H^*(\mathrm{Hilb}_d \mathbb{A}^n, \mathbb{Z})$$

is iso in degrees $\leq 2n-2d+2$.

Pf. that $FFlat_d \simeq BGL(d-1) := Vect_{d-1}$ (over $\mathbb{Z}$, or over k).

$\underline{Vect_{d-1} \to FFlat_d}$

$V \mapsto k\cdot 1 \oplus V$, ~~with~~ as a k-algebra, with $V\cdot V = 0$.

dim $d-1$ vector space

$= k[x_1, \dots, x_{d-1}]/(x_i x_j = 0\ \forall i, j)$.

$\underline{FFlat_d \to Vect_{d-1}}$

$A \mapsto A/k\cdot 1$.

dim. d algebra over k

<u>Lemma</u> The composition $FFlat_d \to Vect_{d-1} \to FFlat_d$ is $\mathbb{A}^1$-homotopic to the identity.

<u>Pf.</u> Deform any comm. alg. of degree d, by a <u>canonical</u> $\mathbb{A}^1$-deformation, to the "trivial" algebra assoc. to a vector space of dim. $d-1$.

We use the <u>Rees algebra</u> construction:

<u>Lemma</u>. If R is an k-algebra with an increasing filtration $0 \subset R_0 \subset R_1 \subset R_2 \subset \cdots$ by k-linear subspaces, the Rees algebra

$1 \in R_0$

i:

~~R(A)~~

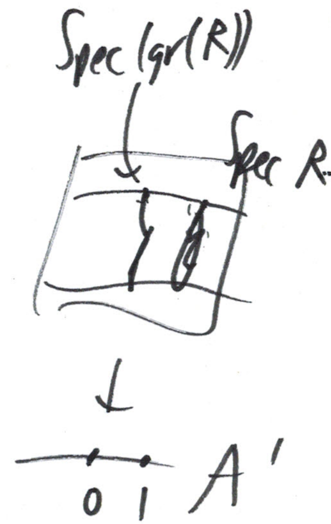
$$\mathrm{Rees}(R) := \bigoplus_{i \geq 0} R_i \cdot t^i$$

This is is a <u>flat</u> $k[t]$-algebra with

$$\mathrm{Rees}(R)/(t) \cong \mathrm{gr}(R)$$

and

$$\mathrm{Rees}(R)/(t-1) \cong R.$$



Apply this to an <u>arbitrary</u> k-alg R of degree d with the filtration $R_0 = k\cdot 1$, $R_1 = R$. Then

$$\mathrm{gr}(R) = \underbrace{k\cdot 1}_{0} \oplus \underbrace{R/\boxed{k\cdot 1}}_{1}$$

is a trivial algebra.

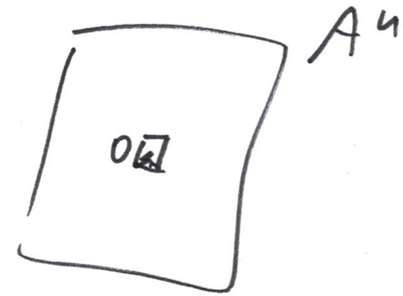
$$R \supsetneq I_1 \supsetneq I_2 \supset$$

ideals.

~~Compare $\mathrm{Hilb}_d(\mathbb{A}^n, 0$~~

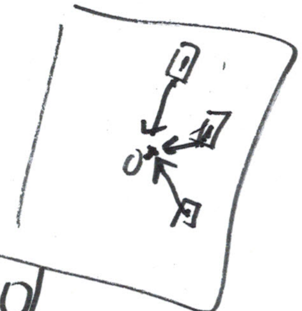
<u>Theorem</u> (T.) $\mathrm{Hilb}_d(\mathbb{A}^n; 0) \hookrightarrow \mathrm{Hilb}_d \mathbb{A}^n$ is a homotopy equivalence over $\mathbb{C}$.

projective



6.

Pf. Consider the action of $\mathbb{G}_m (= \mathbb{C}^*)$ on $\mathbb{A}^n_{\mathbb{C}}$ by scaling, hence an action on $\mathrm{Hilb}_d \mathbb{A}^n$:



$\mathbb{A}^n$:

For any point $S \in \mathrm{Hilb}_d \mathbb{A}^n$,

$\boxed{\lim_{t \to 0} t(S)}$ exists $\in \mathrm{Hilb}_d(\mathbb{A}^n, 0)$.

But: The resulting function

$\mathrm{Hilb}_d(\mathbb{A}^n) \to \mathrm{Hilb}_d(\mathbb{A}^n, 0)$

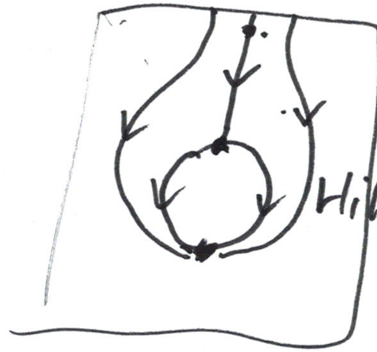
is not continuous.

We have a rational map

$\mathbb{A}^1 \times \mathrm{Hilb}_d(\mathbb{A}^n) \dashrightarrow \mathrm{Hilb}_d(\mathbb{A}^n)$,

but not a morphism,

$\mathrm{Hilb}_d(\mathbb{A}^n)$



$\mathrm{Hilb}_d(\mathbb{A}^n, 0)$

$\mathbb{G}_m$ action

$\mathbb{G}_m$ acting on $\mathbb{P}^1$

0

$x \in \mathbb{P}^1 \mapsto \lim_{t \to 0} tx$ is not continuous

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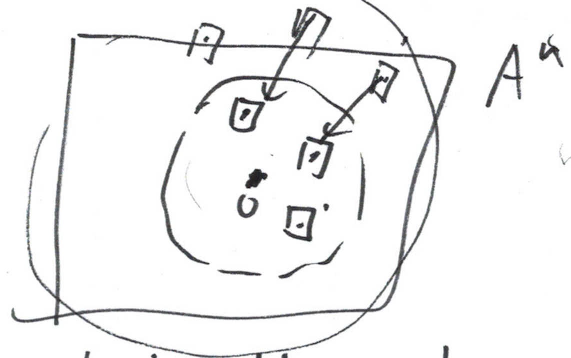
Instead, argue as follows:

Let U_r for $r > 0$ be the open subset of $\mathrm{Hilb}_d(\mathbb{A}^n)$ of subschemes supported in the open ball of radius r in $\mathbb{A}^n_{\mathbb{C}}$.

7.



It's easy that the inclusion $U_r \hookrightarrow U_s$ for $r<s$ is a homotopy, using the $\mathbb{G}_m$-action.

Also $\mathrm{Hilb}_d(\mathbb{A}^n) = \varinjlim_{r\to\infty} U_r$, so ~~Hilb~~

$U_r \hookrightarrow \mathrm{Hilb}_d \mathbb{A}^n$ is a homotopy equiv. for all $r>0$.

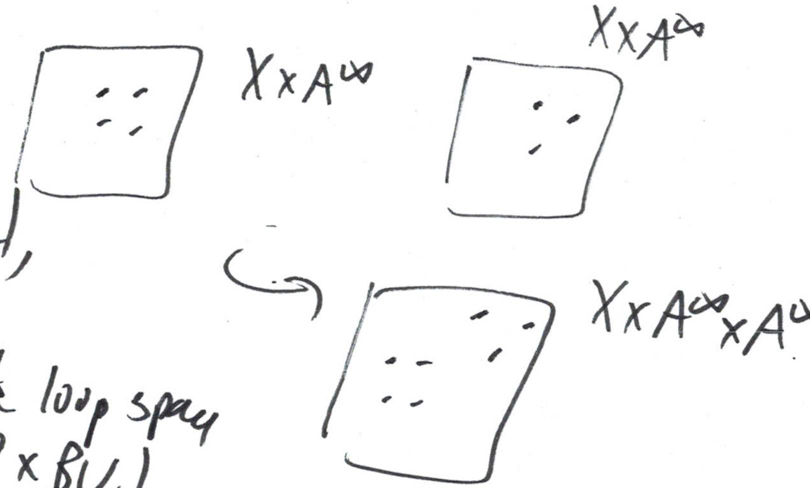
Also, the inclusion $\mathrm{Hilb}_d(\mathbb{A}^n,0) \hookrightarrow U_r$ is a homotopy equivalence for any $r>0$, and that will finish the proof.

By considering PL topology and triangulations, ~~every~~ $\mathrm{Hilb}_d(\mathbb{A}^n,0)$ has arbitrarily small ~~open~~ compact nbds N_i, $i=1,2,\ldots$ with $\mathrm{Hilb}_d(\mathbb{A}^n,0) \hookrightarrow N_i$ a homotopy equivalence.

So $\mathrm{Hilb}_d(\mathbb{A}^n,0)$ has the homotopy type of $\varprojlim^{\mathrm{ho}}$ (all nbds of $\mathrm{Hilb}_d(\mathbb{A}^n,0)$ in $\mathrm{Hilb}_d \mathbb{A}^n$) $\simeq \varprojlim^{\mathrm{ho}} U_r \simeq \mathrm{Hilb}_d(\mathbb{A}^n)$. QED

Conjecture on $\mathrm{Hilb}_d(X \times \mathbb{A}^\infty)$.

Conj. (Nardin) For a smooth variety X over a field k, the group completion of $\coprod_{d \geq 0} \boxed{\mathrm{Hilb}_d(X \times \mathbb{A}^\infty)}$ is $\mathbb{A}^1$-homotopy equivalent to $\Omega^\infty(kgl \wedge X_+)$. $\pi_0 \cong \mathbb{Z}$.



Over $\mathbb{C}$, this space is written as $\Omega^\infty(ku \wedge X_+)$, where ku is the spectrum corresponding to <u>connective</u> top. K-theory. ($\leftrightarrow$ infinite loop space $\mathbb{Z} \times BU$.)

That is,

$$\pi_i\left(\underline{\Omega^\infty(ku \wedge X_+)}\right) \cong ku_i X$$

connective K-homology of X

$$\pi_i KU \cong \begin{cases} \mathbb{Z} & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

$$\pi_i ku = \begin{cases} \mathbb{Z} & \text{if } i \text{ even}, i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$X = pt.$: gp. completion of $\coprod_{d \geq 0} \mathrm{Hilb}_d \mathbb{A}^\infty = \coprod_{d \geq 0} BGL(d-1)_{\mathbb{C}}$

is $\mathbb{Z} \times BU$. $(= \Omega^\infty ku)$.

Graeme Segal, 1970s:

For a compact space X and ~~any~~ any E_∞ top. monoid E, ..

Consider ~~the~~ the space $F(X,E)$ with points:

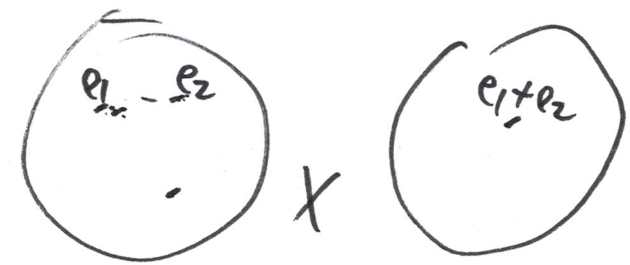
a finite set of points in X

(labeled configuration space)

together with

a point of E

at each of these points in X.



Example. Take $E = \mathbb{N} = \{0, 1, 2, \ldots\}$.

Then

$$F(X, \mathbb{N}) = \cancel{S^\infty X} \coprod_{d \geq 0} S^d X.$$

Segal: $F(X,E)$ is itself an E_∞ top. monoid, and

$$F(X,E)^{gp} \simeq \boxed{\Omega^\infty(e \wedge X_+)}$$

where e is the spectrum corresponding to the ∞-loop space E^{gp}.

Example. Segal $\Rightarrow$ Dold-Thom:

$$S^\infty X = \Omega^\infty(H\mathbb{Z} \wedge X_+)$$

So

$$\pi_i(S^\infty X) = H_i(X, \mathbb{Z}).$$

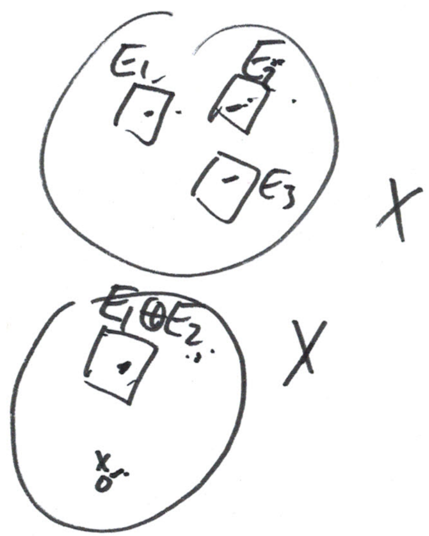
Example. Segal (1977) considered the case of ku.

Let X = compact space.

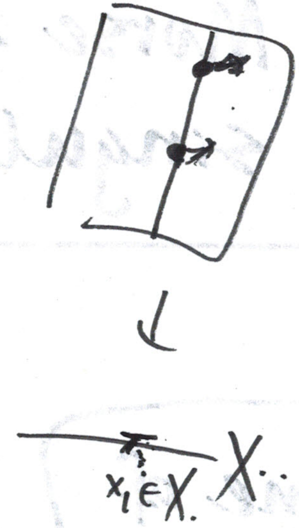
Consider the configuration space of a finite set of points in X, $p_1, \dots, p_r$ with finite-dim. $\mathbb{C}$-linear space $E_1, \dots, E_r \subset \mathbb{C}^\infty$ such that any two of $E_1, \dots, E_r$ are orthogonal in $\mathbb{C}^\infty$.

Then this space $\simeq \Omega^\infty(ku \wedge X_+)$.

$\Omega^\infty(ku \wedge X_+)$

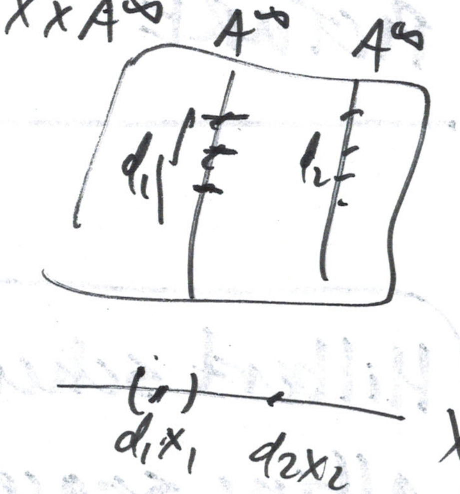


$$\mathrm{Hilb}_d(X\times \mathbb{A}^\infty) \not\to \mathrm{Hilb}_d X,$$



But there is a morphism

$$f: \boxed{\mathrm{Hilb}_d(X\times \mathbb{A}^\infty)} \to S^d X.$$



The fiber of f over a point

$$\sum_{i=1}^r d_i x_i \quad \text{(where } d_i>0,\ x_i\in X(k)\text{).}$$

is

$$\prod_{i=1}^r Y_{d_i},$$

$n=\dim X$.

where Y_d is the space of degree-d subschemes of $\cancel{X}\,\mathbb{A}^n\times\mathbb{A}^\infty$ that are supported on $0\times\mathbb{A}^\infty$.

By our same argument,

$$Y_d \simeq \boxed{BGL(d-1)}.$$

$k[x]/(x^n)$. $\quad k[x_1,\dots,x_n]/(x_ix_j=0)$.