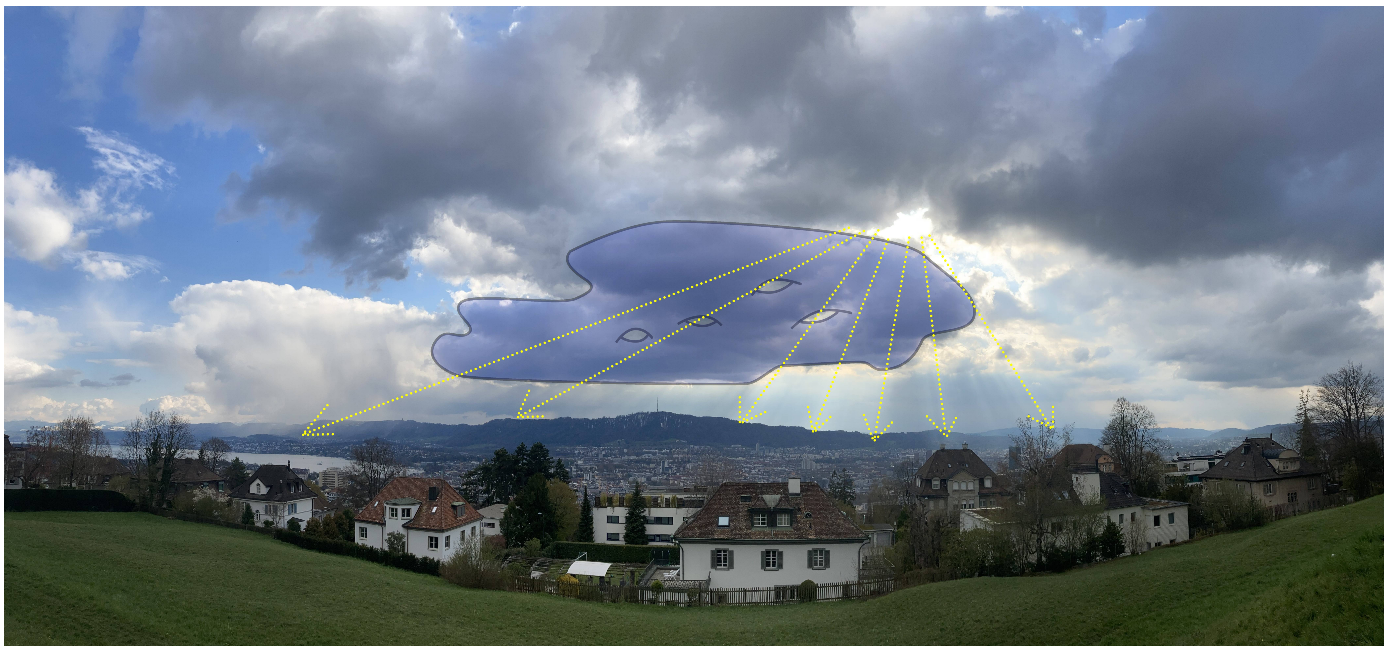
Algebraic Curves, Hurwitz Numbers, and Meromorphic Differentials



Türk Matematik Derneği Seminerleri

13 October 2021

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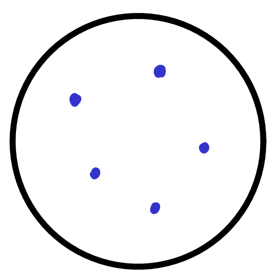
ETHZ

I. Hurwitz Covers

Hurwitz proposed (around 1900) the following problem:

(i) $\mathbb{P}^1 = \bigcirc$ is the Riemann Sphere

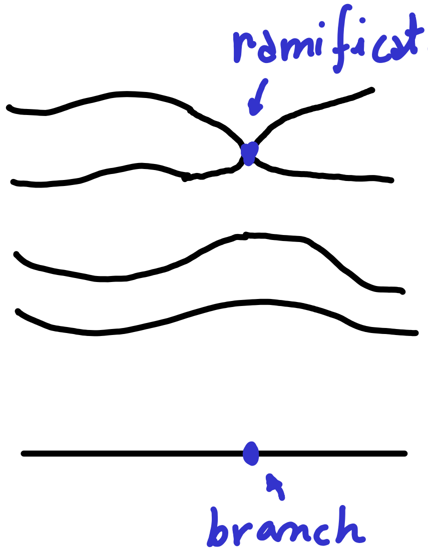
(ii) Choose b points of $\mathbb{P}^1$



(iii) How many Riemann Surfaces appear as degree d covers of $\mathbb{P}^1$ with simple branching over the b points?

$z \to z^2$

d sheets

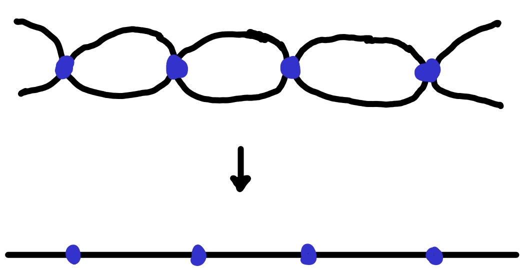


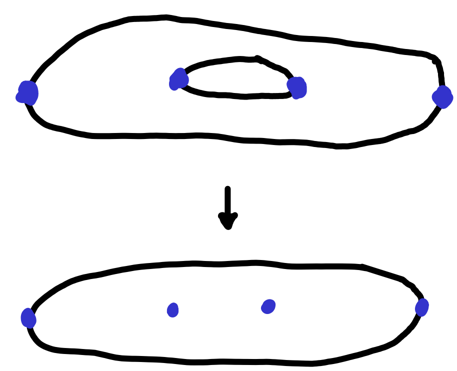
Nonsingular

$$\begin{array}{c} C \\ \big\downarrow \pi \\ \mathbb{P}^1 \end{array} \quad \left\{ \begin{array}{l} \text{algebraic geometry} \\ \text{Complex geometry} \\ \text{Topology} \end{array} \right.$$

Simple branching means 2 sheets come together

Example with $d = 2$, $b = 4$ (genus $= 1$)



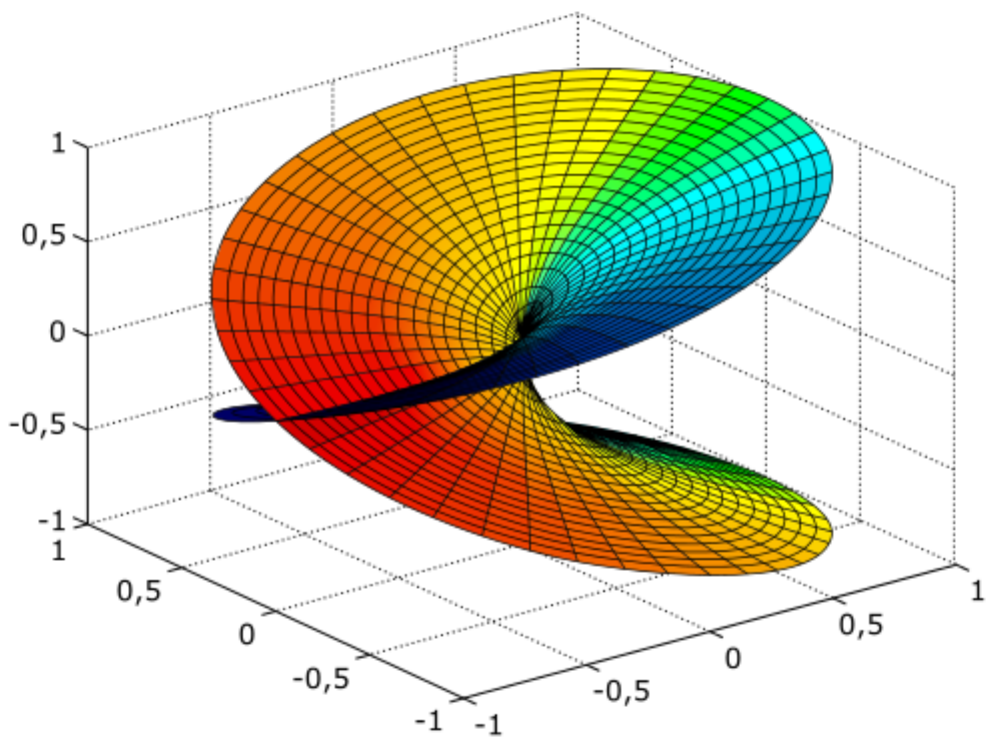


Riemann-Hurwitz Formula :

$$2 \cdot g(C) - 2 = -2d + b$$

Proof by Euler χ

Picture of a ramification point



Graph of $\sqrt{z}$

count by $1/\mathrm{Aut}$

$H_{g,d}$ = Number of covers of $\mathbb{P}^1$ of degree d with $b = 2g+2d-2$ simple branch points

We do not assume cover is connected!

Related by inclusion/exclusion

$H^\circ_{g,d}$ = Count of connected covers

Ueber die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten.

Von

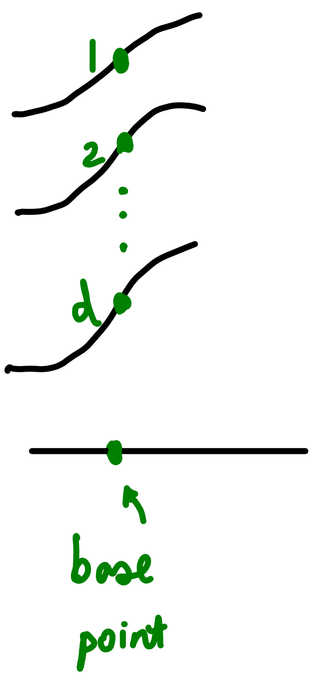
A. Hurwitz in Zürich.

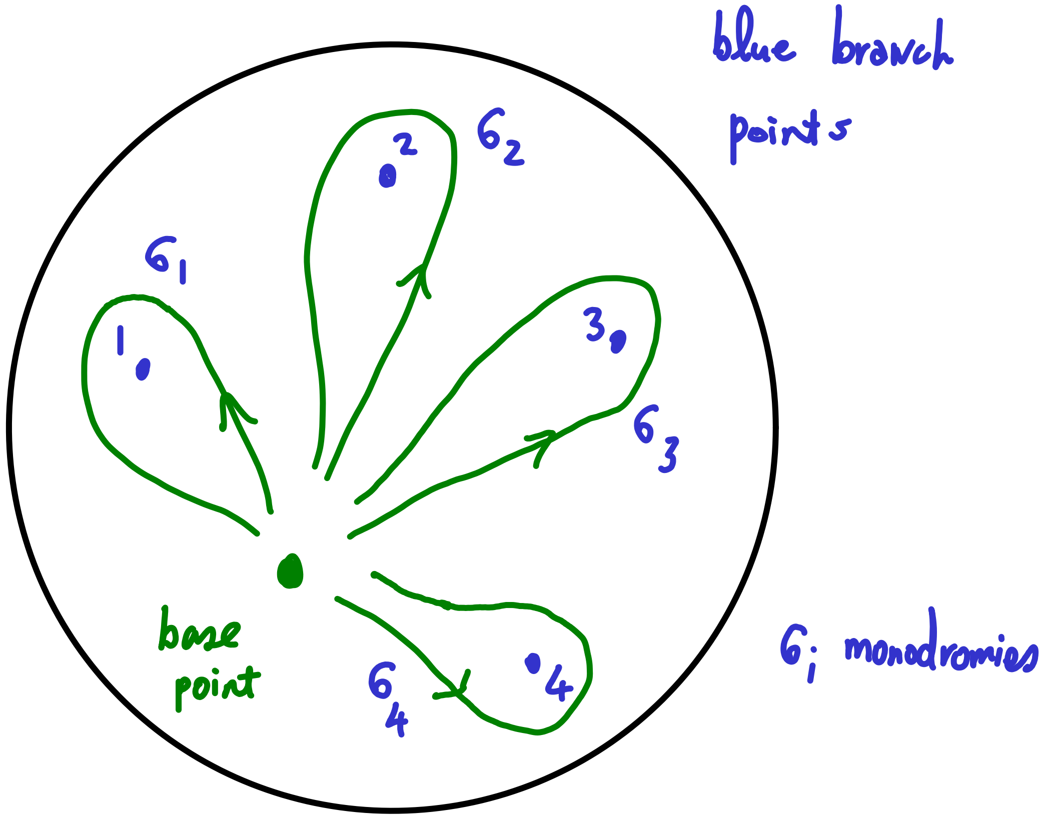
Math. Ann. 1901

Hurwitz presents a solution

Solution via the symmetric group Σ_d

Label preimages of a base point





$\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \Sigma_d$

all are transpositions

Product is trivial: $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4 = \mathrm{Id}$

Theorem (Hurwitz): $\mathcal{H}_{g,d}$ equals

$\frac{1}{d!}$ times the number of solution of

$$\sigma_1 \cdot \sigma_2 \cdots \sigma_b = \mathrm{Id} \in \Sigma_d$$

where $b = 2g+2d-2$ and all σ_i are transpositions

Hurwitz writes

$w = b$

$$(3) \qquad f_S(w) = c_1 f_1^w + c_2 f_2^w + \cdots + c_k f_k^w$$

$\mathcal{H}_{g,d}$

But what are c_i, f_i ?

Als ich im letzten Sommer mit Herrn Dr. E. Lasker, dem bekannten Weltschachmeister und Mathematiker, dieses Resultat besprach, wurde ich durch eine geistvolle Bemerkung des Herrn Lasker zu einer Ueberlegung geführt,

Lasker, the world chess champion, points Hurwitz to the representation theory of Σ_d : Characters, Frobenius

We can compute $H_{g,d}$ using the group algebra and the characters of the symmetric group:

$$H_{g,d} = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 \left(|C_2| \frac{\chi_2^\lambda}{\dim \lambda} \right)^{2g+2d-2}$$

Burnside

λ irrep of Σ_d

χ_2^λ char

C_2 conj class

1913

ETH Archives

II. Toda equations

Gromov-Witten theory concerns

$\int$ ← theory of integration

Topological string

Moduli of maps

$C \to X$

genus g curves

algebraic variety / symplectic manifold

I was working on various aspects of the theory in 1990's.

Eguchi and S-K Yang had conjectured that $GW(\mathbb{P}^1)$ was governed by the Toda equations.

hep-th/9407134

I knew part of $GW(\mathbb{P}^1)$ contained the Hurwitz count.

Connected Covers

degree parameter

$$H(\lambda, y) = \sum_{g \geq 0} \sum_{d > 0} \lambda^{2g-2} e^{dy} \frac{H^{\circ}_{g,d}}{(2g+2d-2)!}$$

genus parameter

branch number

I could write a Toda equation for Hurwitz :

$$\boxed{\exp\Big(H(\lambda, y+\lambda) + H(\lambda, y-\lambda) - 2H\Big) = \lambda^2 e^{-y} \frac{\partial^2 H}{\partial y^2}}$$

which uniquely determines $H^{\circ}_{g,d}$ with no further mathematical input.

Can the Toda equation be proven from the symmetric group formula for the $H_{g,d}$?

Answer : Yes! (Okounkov)

Using the connection found here between the Hurwitz numbers and Toda as a starting point, Okounkov and I were able to solve the entire $GW(\mathbb{P}^1)$.

- P, The Toda equation and GW theory of $\mathbb{P}^1$, Lett. Math. Phys. (2000)
- Okounkov, Toda equation for Hurwitz numbers, Math. Res. Lett. (2000)
- OP, GW theory, Hurwitz Numbers, and Completed cycles, Ann. of Math. (2006)

Brief remarks for the curious:

(i) $$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{Z}\,\underline{k}$$

(ii) Consider $\Lambda^{\frac{\infty}{2}} V$ Fock space $v_\emptyset$, $\langle , \rangle$

(iii) $\tau = \exp(H)$ generating series for $H^\circ_{g,d}$

generating series for $H_{g,d}$

(iv) $$\tau = \langle \Gamma_+ \, e^{yE} e^{\lambda F_2} \, \Gamma_- \, v_\emptyset , v_\emptyset \rangle$$

Energy

2-cycle

Vertex operators

(v) Such matrix products are known to produce τ-functions for Toda

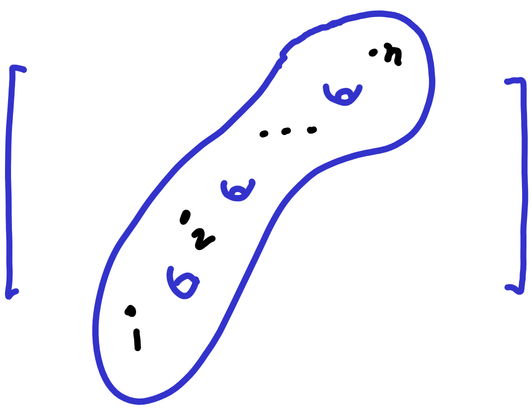
Tohru Eguchi 1948 - 2019

Professor of Physics

University of Tokyo, Kyoto University

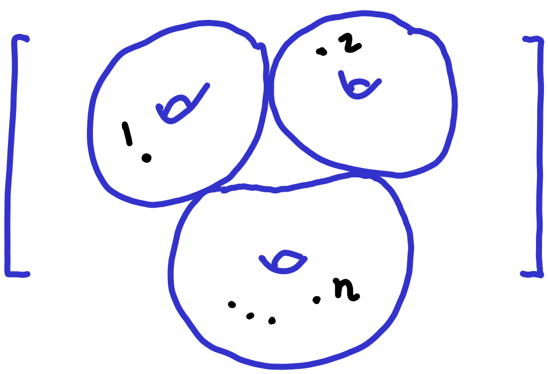
III. KdV

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves.



$\in \mathcal{M}_{g,n}$

$\cap$ Deligne Mumford



$\overline{\mathcal{M}}_{g,n}$

$\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ cotangent line class

$$\begin{array}{ccc} \mathbb{L}_i & \ni & T^*_{C,p_i} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [C, p_1, \dots, p_n] \end{array}$$

$\psi_i = c_1(\mathbb{L}_i)$

Descendent integrals

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \in \mathbb{Q}$$

Define free energy (following Witten 1990)

$$\mathcal{F}(t_0, t_1, t_2, \ldots) = \sum_{g \geq 0} \sum_{n \geq 0} \left\langle \frac{\gamma^n}{n!} \right\rangle_g$$

generating series of descendent integrals

$$\gamma = \sum_{i=0}^{\infty} t_i \tau_i$$

$$\mathcal{U} = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$$

Witten's conjecture: $\mathcal{U}$ satisfies KdV

KdV

$$\frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12} \frac{\partial^3 u}{\partial t_0^3}$$

19th Century, to model shallow water waves

time

spatial coordinate

Proof via Kontsevich's matrix model uses analytic decomposition of moduli space into cells.

Is there any connection to Hurwitz?

Answer: Yes!

Okounkov - P, Gromov-Witten theory, Hurwitz numbers Matrix models (written in 2001)

Connection to Hurwitz starts with ELSV formula:

$$H^0_{g,\mu} = \frac{(2g-2+|\mu|+\ell)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{M}_{g,\ell}} \frac{\sum_{k=0}^{g} (-1)^k \lambda_k}{\prod_{i=1}^{\ell} (1-\mu_i \psi_i)}$$

Hodge classes

μ is a partition of $|\mu|$

ℓ = length of μ

$H^0_{g,\mu}$ is the Hurwitz count of covers

C ← genus g

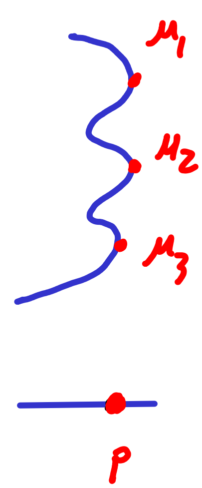
degree $|\mu|$ ⇝ π $\downarrow$

$\mathbb{P}^1$

with $2g-2+|\mu|+\ell$ simple branch points

plus a single point $p \in \mathbb{P}^1$ with

profile $\pi^{-1}(p)$ of shape μ.



Ekedahl
Lando
Shapiro
Vainstein
(2001)

Can also be proven very directly via Relative GW theory

Fantechi – P , Graber-Vakil
(2002) (2005)

The integral on the right side of ELSV:

$\lambda_k \in H^{2k}(\overline{M}_{g,\ell})$

$$\int_{\overline{M}_{g,\ell}} \frac{\sum_{k=0}^{g} (-1)^k \lambda_k}{\prod_{i=1}^{\ell} (1-\mu_i \psi_i)} = \int_{\overline{M}_{g,\ell}} \frac{1}{\prod_{i=1}^{\ell} (1-\mu_i \psi_i)} + \text{lower terms}$$

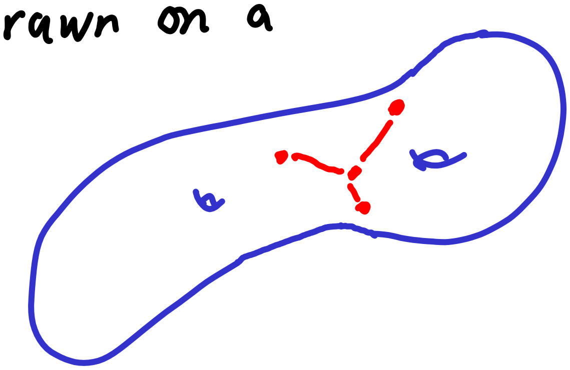
leading term when we send $\mu_i \to \infty$

We see that the asymptotics of $H^{\circ}_{g,\mu}$ carry the full information of descendents.

In the paper with Okounkov, we show how the Hurwitz asymptotics exactly yield Kontsevich's matrix model and therefore prove Witten's Conjecture.

Brief remarks for the curious:

(i) reintepret $H^{\circ}_{g,\mu}$ in terms of branching graphs drawn on a topological surface.



(ii) As $\mu_i \to \infty$, we must study the leading terms in the counts of the branching graphs.

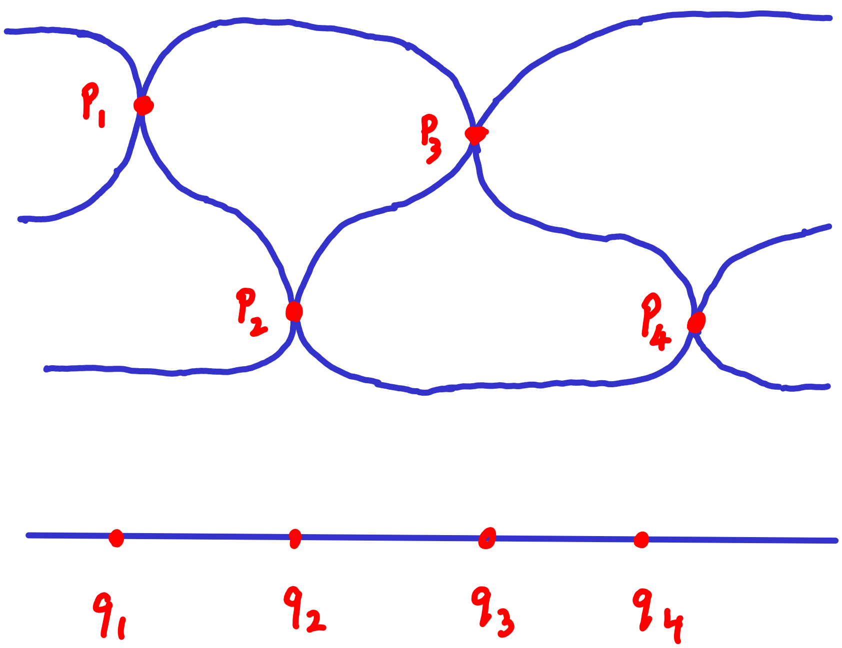
(iii) We exactly match Kontsevich's Comb model

IV. Current directions: Cycles

Main New perspective for Study: $\boxed{\text{Hurwitz covers define a correspondence}}$

A Hurwitz cover has both

a domain curve and a target curve



domain

$[C, p_1, p_2, p_3, p_4] \in \overline{\mathcal{M}}_{g,4}$

target $[\mathbb{P}^1, p_1, p_2, p_3, p_4] \in \overline{\mathcal{M}}_{0,4}$

$$\begin{tikzcd} & \mathcal{H}ur_{g,b} \arrow[dl, "\varepsilon_g"'] \arrow[dr, "\varepsilon_0"] & \\ \mathcal{M}_{g,b} & & \mathcal{M}_{0,b} \end{tikzcd}$$

moduli of such Hurwitz covers

$b = 2g + 2d - 2$

All spaces have natural moduli compactifications

We have a correspondence

$$\begin{tikzcd} & \overline{\mathcal{H}ur}_{g,b} \arrow[dl, "\varepsilon_g"'] \arrow[dr, "\varepsilon_0"] & \\ \overline{\mathcal{M}}_{g,b} & & \overline{\mathcal{M}}_{0,b} \end{tikzcd}$$

Admissible covers
Harris-Mumford
1980s

Deligne-Mumford stable curves
1960s

The fundamental question is to compute the push-forward of the fundamental class

$$(\varepsilon_g \times \varepsilon_0)_* [\overline{\mathcal{H}ur}_{g,b}] \in H^*(\overline{\mathcal{M}}_{g,b} \times \overline{\mathcal{M}}_{0,b})$$

Cycle class calculation

$$\begin{array}{c} \overline{\mathcal{H}ur}_{g,b} \\ \downarrow \varepsilon_0 \\ \overline{\mathcal{M}}_{0,b} \end{array}$$

For the original Hurwitz number:

$$\mathcal{H}^0_{g,d} = \deg(\varepsilon_0)$$

Hurwitz Numbers are a small part of the cohomological information of the correspondence

What form could an answer take?

We would have know how to write

classes in $H^*(\bar{\mathcal{M}}_{g,b})$.

but most of the cohomology

is mysterious

However, there is a subspace which

understand (reasonably) well

$$R^*(\bar{\mathcal{M}}_{g,b}) \subset H^*(\bar{\mathcal{M}}_{g,b})$$

tautological ring, tautological classes

consisting of strata, cotangent lines, kappa's, ...

loci of fixed topological type $\quad \psi_i \quad \kappa_j$

We know three aspects of the answer to the question of calculating the class of the Hurwitz correspondence:

(i) $(\varepsilon_g \times \varepsilon_0)_\star [\overline{Hur}_{g,b}] \in R^\star(\overline{M}_{g,b}) \otimes R^\star(\overline{M}_{0,b})$

Faber-P (2005)

Tautological part

(ii) impractical algorithm to compute

via Relative GW theory

More natural is the double ramification cycle.

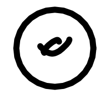
(iii) Pixton's formula for DR cycle

Janda, P, Pixton, Zvonkine (2017)

V Moduli of Meromorphic differentials

By now a large subject, I will just follow a single strand.

Many connections to Hurwitz covers

$\pi : C \to$ 

see Eskin-Okounkov's volume calculations

Let C be a Riemann Surface

We have parallel moduli problems

- Moduli of (C, f) $\Longrightarrow$ Hurwitz Covers

  rational function

- Moduli of (C, ω) $\Longrightarrow$ flat surfaces

  meromorphic differential

$\vdots$

- Moduli of (C, ω^k) $\leftarrow$ meromorphic k-differential

Theorem (Bae-Holmes-P-Schmitt-Schwarz, 2020):

Pixton's formula calculates the push-forwards of fundamental classes of the moduli spaces (C,ω^k) for all $k \geq 1$.

I will end the lecture by explaining (most) of the formula in the $k=1$ case.

Let $\mu = (m_1, \dots, m_n)$ be a vector of integers with $\sum_{i=1}^{n} m_i = 2g-2$

[insist $\exists\, m_i < 0$]

We want to calculate the class $\mathrm{Diff}_{g,\mu}$ of the locus* $(C, p_1, \dots, p_n) \in \overline{M}_{g,n}$ where $\omega_C \cong \mathcal{O}_C(\sum m_i p_i)$ holds.

* Closure or BCGGM space

The Pixton cycle $\mathrm{Pix}^{\mathrm{diff}}_{g,\mu}$ is a sum over stable graphs Γ ← strata of $\overline{M}_{g,n}$

- An admissible weighting of Γ mod r is a function

$$w : H(\Gamma) \to \{0, 1, \dots, r-1\}$$

set of half edges of Γ

Satisfying

half edge corresponding to a marking

(i) $w(j) = m_j + 1 \mod r$

(ii) $w(h) + w(h') = 0 \mod r$

$\bullet\,\underline{\;h\qquad h'\;}\,\bullet$ edge

(iii) $$\sum_{h \mapsto v} w(h) = 2g(v) - 2 + \mathrm{val}(v) \mod r$$

v is a vertex

- $2^g \cdot \mathrm{Pix}_{g,\mu}^{\mathrm{diff}}$ is the r-constant codim g part of

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \frac{1}{r^{h^1(\Gamma)}} \cdot i_{\Gamma *}\left[\prod_{v \in V(\Gamma)} \exp(-k_1(v)) \prod_{i=1}^{n} \exp\left((m_i+1)^2 \psi_i\right) \prod_{e \in E(\Gamma)} \frac{1 - \exp\left(-w(h)w(h')(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]$$

all stable graphs for $\overline{M}_{g,n}$

all admissible weightings mod r

Whole expression is a polynomial in r for $r \gg 0$.

Take the constant term!

$$\mathrm{Pix}^{\mathrm{diff}}_{g,n} \in H^{2g}(\overline{\mathcal{M}}_{g,n}) \leftarrow \text{formula}$$

$$\mathrm{Diff}_{g,n} \in H^{2g}(\overline{\mathcal{M}}_{g,n}) \leftarrow \text{geometric class}$$

Theorem (BHPSS, 2020)

$\mathrm{Pix}^{\mathrm{diff}}_{g,n}$ and $\mathrm{Diff}_{g,n}$ are related by a simple upper triangular transformation.

- Farkas - P, The moduli of twisted canonical divisors, J. Math. Jussieu (2018)
- BHPSS, Pixton's formula and Abel-Jacobi theory on the Picard stack arXiv:2004.08676
- Sage program Admcycles ← Link on Bonn webpage of Johannes Schmitt

The End

ETH archives 1913