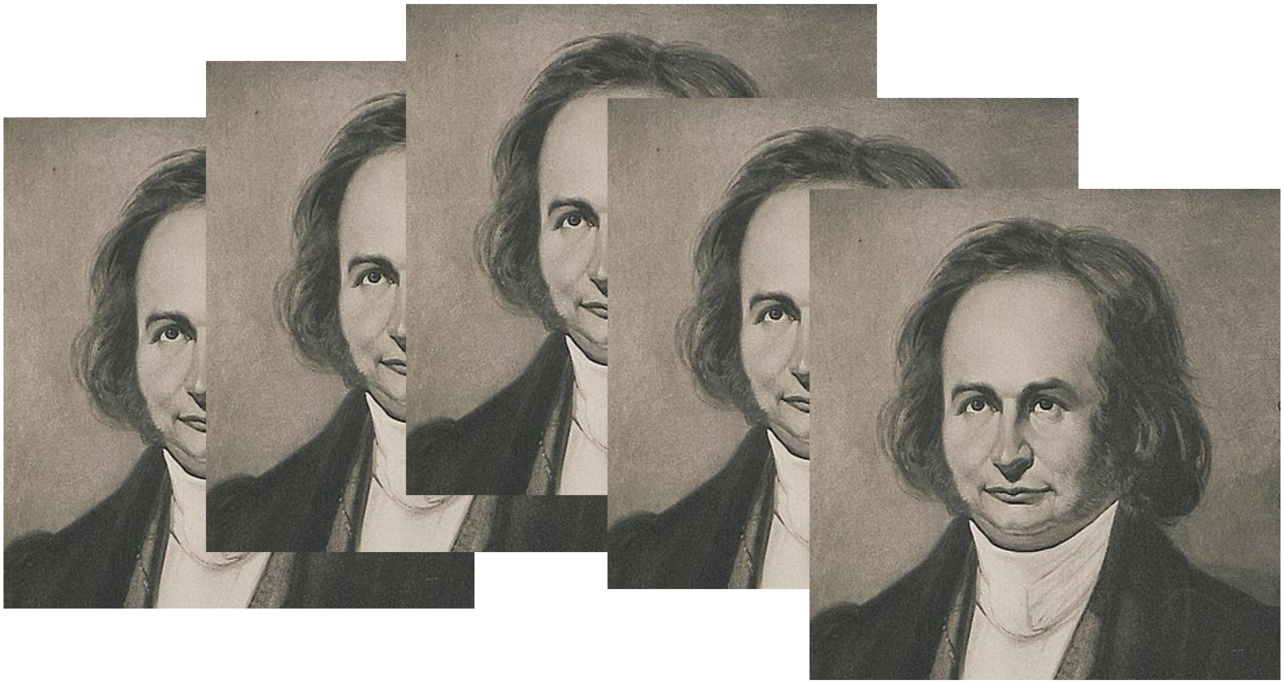
Geometry of the Universal Jacobian



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HU Berlin

Rahul Pandharipande

joint work with

D. Petersen

J. Schmitt

S. Wood

Thanks to
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for improvements

Object of study:

$$\overline{\mathrm{Jac}}^{d}_{g,n}(\phi)$$

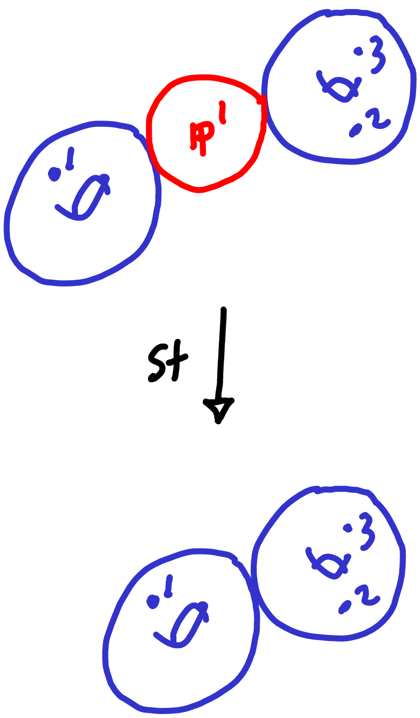
- genus g, n markings with $2g-2+n>0$,
- degree $d \in \mathbb{Z}$,
- ϕ is a non-degenerate Kass-Pagani stability condition of degree d.

There is a morphism

$$\overline{\mathrm{Jac}}^{d}_{g,n}(\phi) \to \overline{\mathcal{M}}_{g,n}$$

$$\overline{\mathrm{Jac}}^{d}_{g,n}(\phi) \ni [C, p_1, \dots, p_n, \mathcal{L}]$$

- $[C, p_1, \dots, p_n]$ is a quasi-stable n pointed curve of genus g



C is nodal with distinct points $p_i \in C^{\text{nonsing}}$, but C is permitted to have disjoint unstable $\mathbb{P}^1$'s over nodes of the stabilization

$$(C, p_1, \dots, p_n) \xrightarrow{st} (C^{st}, p_1, \dots, p_n)$$

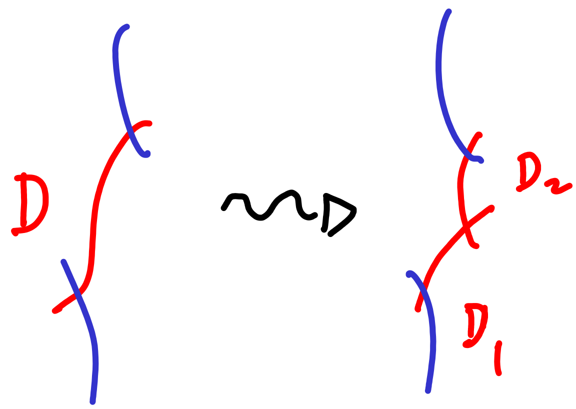
- A Kass-Pagani stability condition ϕ is a rule which assigns a number

$$\phi(D) \in \mathbb{Q}$$

to every irreducible component D of every quasi stable curve C of genus g with n points.

$$\boxed{\text{We also work with the more general Pagani-Tommasi stability conditions, but not in this lecture.}}$$

(i) ϕ is required to satisfy the <u>deformation property</u>



$$\phi(D) = \phi(D_1) + \phi(D_2),$$

(ii) ϕ is required to <u>vanish</u> on unstable $\mathbb{P}^1$'s:

$$\phi(\mathbb{P}^1_{\text{unstable}}) = 0,$$

(iii) The degree of ϕ is

$$\deg(\phi) = \sum_{D \subset C} \phi(D).$$

- $\mathcal{L} \to C$ is a ϕ-stable line bundle of degree d:

  - $\deg \mathcal{L}|_{\mathbb{P}^1_{\text{unstable}}} = 1$

    union of irreducible components of C not in D

  - for all sub curves $D \subset C$ for which neither D nor D^c is a union of unstable $\mathbb{P}^1$'s,

    not just irreducible

    [so $D \neq$ empty set, $D^c \neq$ empty set]

We have the inequality

$$\phi(D) - \frac{E(D,D^c)}{2} < \deg \mathcal{L}|_D < \phi(D) + \frac{E(D,D^c)}{2}.$$

$E(D,D^c)$ equals the number intersections $|D \cap D^c|$

- A Kass-Pagani stability condition ϕ is nondegenerate if equalities never occur in the stability test:

We never have

$$\phi(D) - \frac{E(D, D^c)}{2} = \deg \mathcal{L}|_D \quad \text{or}$$

$$\deg \mathcal{L}|_D = \phi(D) + \frac{E(D, D^c)}{2}$$

for any $\mathcal{L} \to C$ line bundle satisfying $\deg \mathcal{L}|_{P'_{\text{unstable}}} = 1$ on any quasistable curve of genus g with n points.

- Let $2g-2+n>0$,
  let ϕ be a non-degenerate Kass-Pagani stability condition of degree d

Theorem (Kass-Pagani 2019):

$\overline{\mathcal{J}ac}^{d}_{g,n}(\phi)$ is a nonsingular proper Deligne-Mumford stack with a canonical morphism

$$\overline{\mathcal{J}ac}^{d}_{g,n}(\phi) \to \overline{\mathcal{M}}_{g,n}.$$

- Suppose $[C, p_1, \ldots, p_n, \mathcal{L}] \in \overline{Jac}^{d}_{g,n}(\phi)$ for a stability condition ϕ.

Suppose further that $\mathbb{P}^1 \subset C$ is an unstable component with

$$\deg \mathcal{L}|_{\mathbb{P}^1} = 1.$$

The stability condition implies:

$\boxed{C \text{ is not disconnected by the removal of } \mathbb{P}^1 \subset C.}$

Two basic examples of Kass-Pagani stability conditions:

(A) Canonical stability ϕ^{can}

$$\phi^{can}(D) = \deg \omega_C^{log}\big|_D$$

$$\deg(\phi^{can}) = 2g-2+n$$

(B) Point stability ϕ^i for $1 \leq i \leq n$

$$\phi^i(D) = \begin{cases} 0, & p_i \notin D \\ 1, & p_i \in D \end{cases}$$

$$\deg(\phi^i) = 1$$

Varia (see paper of Kass-Pagani 2019):

(i) Linear combination of stability conditions are still stability conditions.

If ϕ_1, ϕ_2 are Kass-Pagani stability conditions of degrees d_1, d_2, then $\alpha\phi_1 + \beta\phi_2$ is a Kass-Pagani stability condition of degree $\alpha d_1 + \beta d_2$. (required to be an integer)

(ii) Kass-Pagani stability conditions of degree 0 for fixed g, n form a finite dimensional $\mathbb{Q}$-vector space.

The neutral element of the $\mathbb{Q}$-vector space is the trivial stability condition ϕ^{tr}

$$\phi^{tr}(D) = 0.$$

The trivial line bundle on a stable curve is always stable for ϕ^{tr}:

$$\overline{Jac}^{0}_{g,n}(\phi^{tr}) \ni [\underbrace{C, p_1, \ldots, p_n}_{\text{stable curve}}, \mathcal{O}_C].$$

dense if nonempty

(iii) Nondegeneracy is an open condition for Kass-Pagani stability conditions of degree d for fixed g, n.

ϕ^{tr} is degenerate for almost all g, n.

(iv) Using linear combinations of ϕ^{can}, ϕ^{i} there always exist nondegenerate degree d Kass-Pagani stability conditions when $n \geq 1$ for any g.

(v) A stability condition ϕ of degree 0 is small if the trivial line bundle is always stable:

- ϕ^{tr} is small,
- smallness is an open condition,
- there exist stability conditions which are small and nondegenerate for $n \geq 1$.

(vi) For $n = 0$, existence is subtler: degree d Kass-Pagani stability conditions if and only if

$$\gcd(d-g+1,\, 2g-2) = 1.$$

Question about cycles and cohomology for $\overline{Jac}^{d}_{g,n}(\phi)$: ($\phi$ always nondegenerate)

(I) Orbifold Euler Characteristic

Theorem (S. Wood 2024):

$$\chi_{orb}\left(\overline{Jac}^{d}_{g,n}(\phi)\right) = \frac{1}{2^g g!}\,\chi_{top}\left(\overline{\mathcal{M}}_{0,n+2g}\right)$$

orbifold

topological

Idea: because of the torus factors, only the most degenerate strata contribute.

A consequence:

$$\chi_{orb}\left(\overline{Jac}^{d}_{g,n}(\phi)\right) = \chi_{orb}\left(\overline{Jac}^{\hat{d}}_{g,n}(\hat{\phi})\right)$$

for different choices of degrees and Kass-Pagani nondegenerate stability conditions

(II) Hodge Numbers

Theorem (P-Petersen-Schmitt-Wood 2025):

$$h^{p,q}\left(\overline{Jac}^{d}_{g,n}(\phi)\right) = h^{p,q}\left(\overline{Jac}^{\hat{d}}_{g,n}(\hat{\phi})\right).$$

- If $n \geq 1$, the equality of Hodge numbers was already known by results of Migliorini - Shende - Viviani 2021
- The $n=0$ case is new.
- The strategy differs from MSV 2021.

We start with a result about the S_n-equivariant Hodge-Deligne polynomial:

Let $\mathrm{Jac}^d_{g,n} \to \mathcal{M}_{g,n}$ be the classical (uncompactified) spaces. The symmetric group S_n acts on both sides.

$$\boxed{\begin{array}{l}\text{Theorem (PPSW 2025):} \\ \text{For fixed } g \text{ and } n, \text{ the} \\ S_n\text{-equivariant Hodge-Deligne} \\ \text{polynomial of } \mathrm{Jac}^d_{g,n} \text{ is} \\ \text{independent of } d.\end{array}}$$

The idea to prove the Hodge number equality in the $d = \hat{d}$ case,

$$h^{p,q}\left(\overline{Jac}^{d}_{g,n}(\phi)\right) = h^{p,q}\left(\overline{Jac}^{d}_{g,n}(\hat{\phi})\right).$$

is to put the strata in bijective correspondence and to use the above on the Hodge-Deligne polynomial to obtain the equality.

If $n \geq 1$, the marking can be used to relate different degrees d and $\hat{d}$.

The $n = 0$ requires special arguments.

Aside (by Qizheng Yin):

$$\boxed{[Jac^{d}_{g}] \text{ and } [Jac^{\hat{d}}_{g}] \text{ are not in general equal in the Grothendieck group } K_0(Var).}$$

Proof: Equality in $K_0(Var)$

$\Rightarrow$ Jac^{d}_{g} and $Jac^{\hat{d}}_{g}$ are stably birational by Larsen-Lunts

$\Rightarrow$ Jac^{d}_{g} and $Jac^{\hat{d}}_{g}$ are birational for $g \gg 0$ since Kodaira dim ≥ 0

$\Rightarrow$ Contradiction by Bini-Fontanari-Viviani [On the birational geometry of the universal Pic].

(III) Tautological classes (simple)

$$\overline{Jac}^{d}_{g,n}(\phi) \xrightarrow{\epsilon} \overline{\mathcal{M}}_{g,n}$$

The simplest tautological classes on $\overline{Jac}^{d}_{g,n}(\phi)$ are defined by

degree decorated stable strata classes

$$[\, J^{\delta}_{\Gamma}(\phi) \,] \in CH^*(\overline{Jac}^{d}_{g,n}(\phi))$$

Γ is a stable graph for $\overline{\mathcal{M}}_{g,n}$ decorated with ψ and κ classes from $\overline{\mathcal{M}}_{g,n}$ with a ϕ-stable degree assignment

$$\delta : \mathrm{Vert}(\Gamma) \to \mathbb{Z}$$

of total degree d.

The simplest tautological algebra is :

$$R^*_s(\overline{\mathcal{J}ac}^d_{g,n}(\phi)) \subset CH^*(\overline{\mathcal{J}ac}^d_{g,n}(\phi))$$

as an algebra

generated by all classes $[J^\delta_\Gamma(\phi)]$.

Not yet written, but proven.

Theorem* (PPSW 2025):

$$R^*_s(\overline{\mathcal{J}ac}^d_{g,n}(\phi)) \cong R^*_s(\overline{\mathcal{J}ac}^d_{g,n}(\hat{\phi}))$$

canonically as $R^*(\overline{\mathcal{M}}_{g,n})$-algebras.

Idea: use a canonical matching of degree decorated stable strata and intersection theory.

The above isomorphism is valid for $n=0$.

If $n \geq 1$, the isomorphisms

$$R^*_s(\overline{\mathcal{J}ac}^{\,d}_{g,n}(\phi)) \cong R^*_s(\overline{\mathcal{J}ac}^{\,\hat{d}}_{g,n}(\hat{\phi}))$$

can also be constructed for $d \neq \hat{d}$ by twisting by markings (after applying the equal degree result).

(IV) Tautological classes (full)

Let $n \geq 1$. Then there exists a universal curve with a universal line bundle

$$\begin{array}{ccc} \mathcal{C} & \leftarrow & \mathcal{L} \\ {\scriptstyle \pi}\downarrow & & \\ \overline{\mathcal{J}ac}^{\,d}_{g,n}(\phi)). & & \end{array}$$

$\mathcal{L}$ is trivialized along the section corresponding to p_1.

A richer tautological algebra,

$$R^*_\theta(\overline{Jac}^d_{g,n}(\phi)) \subset CH^*(\overline{Jac}^d_{g,n}(\phi)),$$

is defined by including more classes as generators: on each degree decorated quasi-stable stratum, include all classes constructed from $\mathcal{L}$,

allow unstable genus 0 vertices of degree 1

ψ_i = cotangent line classes at nodes/markings

$\kappa_{a,b} = \pi_*\left(c_1(\mathcal{L})^a \cdot c_1(\omega_\pi)^b\right)$

$\xi_i = s_i^*\left(c_1(\mathcal{L})\right)$ at nodes/markings.

Certainly we have,

$$R^*_s(\overline{Jac}^d_{g,n}(\phi)) \subset R^*_\theta(\overline{Jac}^d_{g,n}(\phi)).$$

Speculation (PPSW 2025):

$$R^*_\theta(\overline{Jac}^d_{g,n}(\phi)) \cong R^*_\theta(\overline{Jac}^d_{g,n}(\hat{\phi}))$$

canonically as $R^*(\overline{\mathcal{M}}_{g,n})$-algebras.

Many of the steps for the proof have been taken. The main idea is the construction of a Master space where the comparisons can be made.

(Update from Diablerets in April 2025: Sam Molcho explains that the Speculation is likely false.)

(v) Further questions

Question A : Is there an isomorphism

$$\mathcal{H}^*(\overline{Jac}^{\,d}_{g,n}(\phi)) \cong \mathcal{H}^*(\overline{Jac}^{\,d}_{g,n}(\hat{\phi}))$$

as $\mathcal{H}^*(\overline{\mathcal{M}}_{g,n})$-algebras ?

Likely Answer is No.

Question B : Are $\overline{Jac}^{\,d}_{g,n}(\phi)$ and $\overline{Jac}^{\,d}_{g,n}(\hat{\phi})$ derived equivalent?

The answer to Question B should likely be yes following Bini-Fontanari-Viviani 2012 and the work of Arinkin.

Ideas for relations in

$$R^*_s(\overline{Jac}^d_{g,n}(\phi)) \text{ and } R^*_\theta(\overline{Jac}^d_{g,n}(\phi)).$$

(i) Pixton's relations in $R^*(\overline{\mathcal{M}}_{g,n})$ can be pulled-back to both.

(ii) Pixton's DR relations (as developed in the Universal DR context) yield relations in $R^*_\theta(\overline{Jac}^d_{g,n}(\phi))$ as studied by Y. Bae.

Bae
Holmes
P, Schmitt
Schwarz

Question C: Propose a complete set of relations for $R^*_s(\overline{Jac}^d_{g,n}(\phi))$ and $R^*_\theta(\overline{Jac}^d_{g,n}(\phi))$.

A related result by Chiodo-Holmes 24, Bae-Molcho-Pixton 25 using Universal DR theory:

Consider the zero section

$$\overline{\mathcal{M}}_{g,n} \xrightarrow{z} \overline{Jac}^{\circ}_{g,n}(\phi)$$

for a small and nondegenerate stability condition ϕ.

Then $z_*[\overline{\mathcal{M}}_{g,n}] \in R^*_\theta(\overline{Jac}^{\circ}_{g,n}(\phi))$

and is given by the uni DR formula for the universal line bundle:

$$\begin{array}{c} \mathcal{C} \leftarrow \mathcal{L} \\ \pi \downarrow \\ \overline{Jac}^{\circ}_{g,n}(\phi) \end{array}$$

Connections to other work:

(A) Qizheng Yin's paper

Cycles on curves and Jacobians: a tale of two tautological rings. Alg Geom (2016)

Yin studies $R^*_\theta(\mathrm{Jac}^d_{g,1})$ using the Beauville decomposition.

(B) Log DR theory (2022)

Holmes
Molcho
P, Pixton
Schmitt

The universal Jacobians $\overline{\mathrm{Jac}}^d_{g,n}(\phi)$ are used in the proof of the log DR formula.

(C) Pixton's lecture at the Simons Center (2025)

DR cycles, admissible covers, and the top degree part.

A direct approach to DR is presented via $R^*_\theta(\overline{\mathrm{Jac}}^\circ_{g,n}(\phi))$ with Bae and Molcho.

Related to direction (B):

Let $n \geq 1$ and $A = (a_1, \dots, a_n)$ with $\sum a_i = d$.

The resolution of the Abel-Jacobi map is

$$\overline{\mathcal{M}}^{\phi}_{g,A} \xrightarrow{AJ_A} \overline{\mathcal{J}ac}^{d}_{g,n}(\phi)$$

for a nondegenerate stability condition ϕ.

AJ_A defined by $\Theta(\sum a_i p_i)$ on $\mathcal{M}_{g,3}$

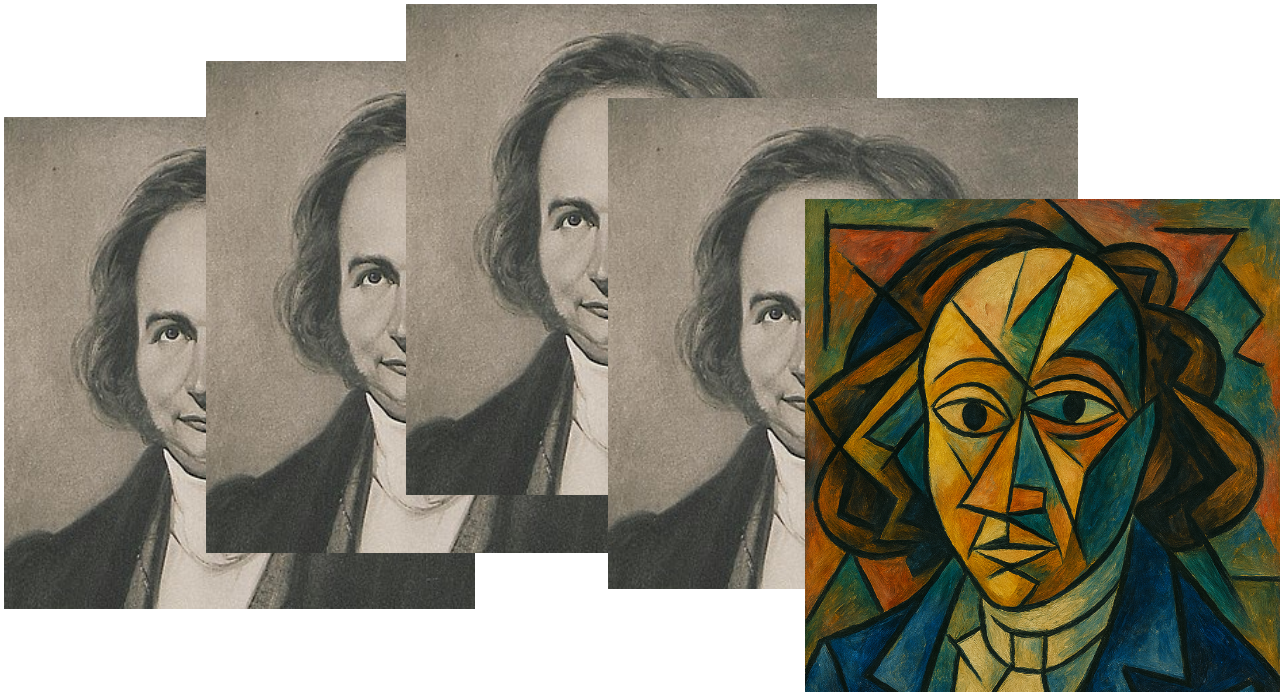
Theorem (Bae-Molcho-Pixton 2025):

$$AJ_{A*}\left[\overline{\mathcal{M}}^{\phi}_{g,A}\right] \in R^*_{\theta}(\overline{\mathcal{J}ac}^{d}_{g,n}(\phi))$$

and is given by the uni DR formula for the line bundle:

$$\begin{array}{c} \mathcal{C} \curvearrowleft \mathcal{L}(-\sum a_i p_i) \\ \downarrow \pi \\ \overline{\mathcal{J}ac}^{d}_{g,n}(\phi)) \end{array}$$

Idea of proof: use the geometric approach to uni DR via intersection theory of the Picard stack.



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