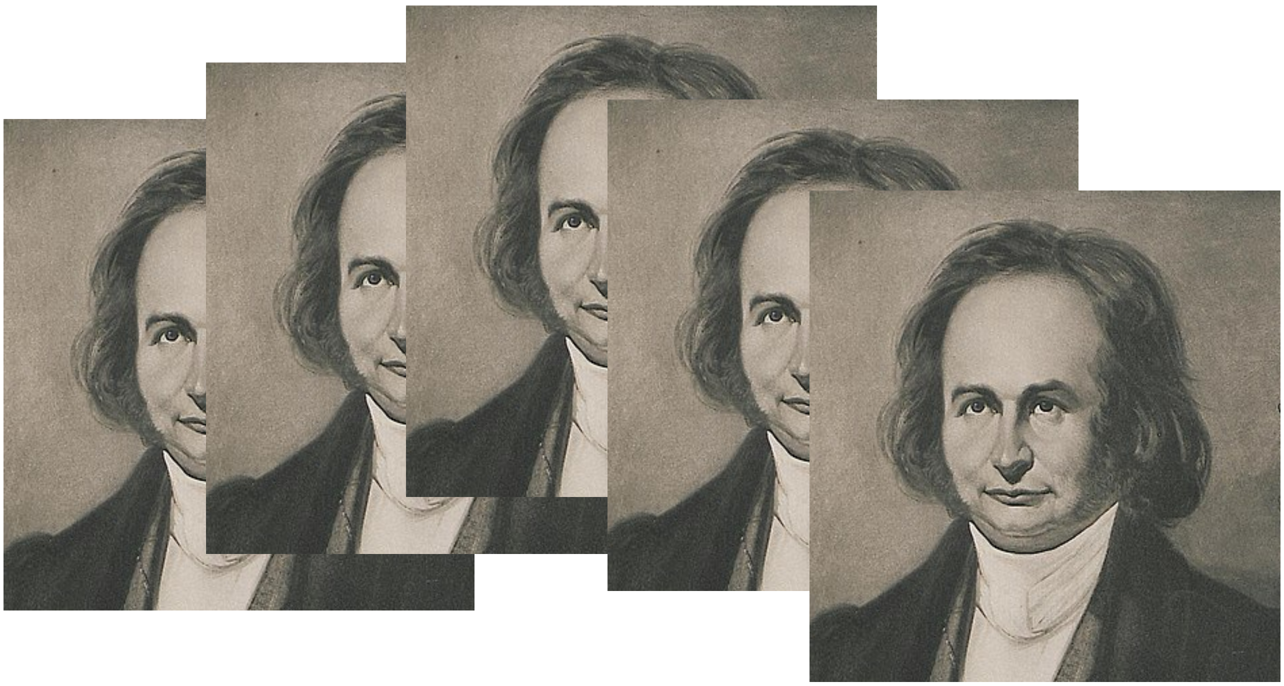
# Geometry of the Universal Jacobian



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ETHZ

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## Log intersections on $\overline{\mathcal{J}ac}^{\,0}_{g,3}(\phi)$

Consider $\overline{\mathcal{J}ac}^{\,0}_{g,3}(\phi)$ for a small and nondegenerate stability condition $\phi$,

$$\dim \overline{\mathcal{J}ac}^{\,0}_{g,3}(\phi) = 4g .$$

Let

$$A_1 = (a_1, b_1, c_1), \qquad a_1 + b_1 + c_1 = 0$$

$$A_2 = (a_2, b_2, c_2), \qquad a_2 + b_2 + c_2 = 0$$

$$A_3 = (a_3, b_3, c_3), \qquad a_3 + b_3 + c_3 = 0$$

$$A_4 = (a_4, b_4, c_4), \qquad a_4 + b_4 + c_4 = 0$$

be 4 vectors with integer coordinates.

As defined in HMPPS, we consider the resolutions of the Abel-Jacobi maps:

$$\overline{\mathcal{M}}^{\phi}_{g,A_i} \xrightarrow{AJ_i} \overline{Jac}^{\,0}_{g,3}(\phi).$$

On the open set

$$\mathcal{M}_{g,3} \subset \overline{\mathcal{M}}^{\phi}_{g,A_i}$$

$$AJ_i(C, p_1, p_2, p_3) = (C, p_1, p_2, p_3, \mathcal{O}_C(a_i p_1 + b_i p_2 + c_i p_3)).$$

We obtain 4 varieties in $CH^g(\overline{Jac}^{\,0}_{g,3}(\phi))$:

Variety dim $3g$

$$V_i = AJ_i(\overline{\mathcal{M}}^{\phi}_{g,A_i}) \subset \overline{Jac}^{\,0}_{g,3}(\phi).$$

Question: Calculate

$$\int_{\overline{\mathcal{J}ac}^{\circ}_{g,3}(\phi)} V_1 \cdot V_2 \cdot V_3 \cdot V_4 .$$

$V_i \in CH^g\left(\overline{\mathcal{J}ac}^{\circ}_{g,3}(\phi)\right)$

associated Chow class

We will solve a related log intersection integral. The idea is to take a limit over all log modifications

$$\widetilde{\mathcal{J}ac}^{\circ}_{g,3}(\phi) \xrightarrow{\alpha} \overline{\mathcal{J}ac}^{\circ}_{g,3}(\phi)$$

Standard boundary log structure

$$\log CH^*\left(\overline{\mathcal{J}ac}^{\circ}_{g,3}(\phi)\right) = \lim_{\alpha} CH^*\left(\widetilde{\mathcal{J}ac}^{\circ}_{g,3}(\phi)\right).$$

The subvarieties $V_i \subset \overline{Jac}^{\circ}_{g,3}(\phi)$

Canonically determine log classes by blowing up until the strict transformations are transverse to the log boundary.

Question$^{\log}$: Calculate

log integration via pushdown

$$\int_{\overline{Jac}^{\circ}_{g,3}(\phi)} V_1^{\log} \cdot V_2^{\log} \cdot V_3^{\log} \cdot V_4^{\log},$$

$$V_i^{\log} \in \log CH^g\left(\overline{Jac}^{\circ}_{g,3}(\phi)\right).$$

For the solution, we will take a special form of log modification.

Let $\widetilde{\mathcal{M}}_{g,3} \xrightarrow{\gamma} \overline{\mathcal{M}}_{g,3}$ be a log modification with respect to the standard boundary.

The log modifications we will consider are:

$$\begin{array}{l} \widetilde{Jac}^{\circ}_{g,3}(\phi) \cong \widetilde{\mathcal{M}}_{g,3} \times_{\overline{\mathcal{M}}_{g,3}} \overline{Jac}^{\circ}_{g,3}(\phi) \\ \quad \Big\downarrow \\ \overline{Jac}^{\circ}_{g,3}(\phi) \end{array}$$

.

There exists a log blow-up $\widehat{\mathcal{M}}_{g,3} \to \overline{\mathcal{M}}_{g,3}$ which resolves all 4 Abel–Jacobi maps $AJ_i$ simultaneously:

$$\begin{array}{ccc} \mathcal{C} & \leftarrow & S_1, S_2, S_3, S_4 \\ \downarrow & & \\ \widehat{\mathcal{M}}_{g,3} & & \end{array}$$

quasi stables must accommodate all 4 line bundles

.

Then we have Abel-Jacobi maps:

$$\widehat{\mathcal{M}}_{g,3} \overset{\xrightarrow{AJ_1}}{\underset{\xrightarrow{AJ_4}}{\overset{\xrightarrow{AJ_2}}{\xrightarrow{AJ_3}}}} \overline{Jac}^{\,0}_{g,3}(\phi)$$

defined by the line bundles $S_1, S_2, S_3, S_4$.

A crucial property: the restriction of $S_i$ to the fibers of $\begin{array}{c}\mathcal{C}\\ \downarrow \\ \widehat{\mathcal{M}}_{g,3}\end{array}$ is $\phi$-stable.

[up to possible extraneous unstable $\mathbb{P}^1$ of degree 0]

We have

$$\widehat{\mathcal{J}ac}^{\,0}_{g,3}(\phi) \cong \widehat{\mathcal{M}}_{g,3} \times_{\overline{\mathcal{M}}_{g,3}} \overline{\mathcal{J}ac}^{\,0}_{g,3}(\phi)$$

and

$$\widehat{\mathcal{M}}_{g,3} \xrightarrow{\widehat{AJ}_1,\ \widehat{AJ}_2,\ \widehat{AJ}_3,\ \widehat{AJ}_4} \widehat{\mathcal{J}ac}^{\,0}_{g,3}(\phi) .$$

Moreover, the strict transforms of $V_i$ are

$$V_i^{\log} = \widehat{AJ}_i\left(\widehat{\mathcal{M}}_{g,3}\right) \subset \widehat{Jac}^{\circ}_{g,3}(\phi)$$

and are transverse. Their classes are

$$V_i^{\log} \in \log CH^g\left(\overline{Jac}^{\circ}_{g,3}(\phi)\right)$$

So we have

$$\int_{\overline{Jac}^{\circ}_{g,3}(\phi)} V_1^{\log}\cdot V_2^{\log}\cdot V_3^{\log}\cdot V_4^{\log} = \int_{\widehat{Jac}^{\circ}_{g,3}(\phi)} V_1^{\log}\cdot V_2^{\log}\cdot V_3^{\log}\cdot V_4^{\log}.$$

Since $\hat{AJ}_{1*}[\hat{\mathcal{M}}_{g,3}] = V_1^{log}$,

$$\int_{\overline{Jac}^{\circ}_{g,3}(\phi)} V_1^{log} \cdot V_2^{log} \cdot V_3^{log} \cdot V_4^{log} = \int_{\hat{\mathcal{M}}_{g,3}} \hat{AJ}_1^{*}\left(V_2^{log} \cdot V_3^{log} \cdot V_4^{log}\right).$$

The first serious geometric issue is to understand the pull-backs

$$\hat{AJ}_1^{*}(V_2^{log}),\quad \hat{AJ}_1^{*}(V_3^{log}),\quad \hat{AJ}_1^{*}(V_4^{log}).$$

**Claim I.** For $i = 1, 2, 3, 4$

$$\hat{AJ}_1^{\,*}\left(V_i^{\log}\right) = \mathrm{UniDR}\left(S_1 - S_i\right).$$

**Proof.** We will use that the restriction of $S_1$ and $S_i$ to the fibers of

$$\begin{array}{c}\mathcal{C}\\ \downarrow \\ \widehat{\mathcal{M}}_{g,3}\end{array}$$

are both $\phi$-stable.

We must show that the map

$$\widehat{\mathcal{M}}_{g,3} \longrightarrow \mathcal{P}ic^{0}_{g,3}$$

Artin stack
Universal
Picard

$$(C, p_1, p_2, p_3) \longmapsto \left(C, p_1, p_2, p_3, S_1 \otimes S_i^{-1}\big|_C\right)$$

restricted to C

only meets the closure of trivial locus

$$\overline{\mathrm{Triv}} \subset \mathrm{Pic}^0_{g,3}$$

Triv is where the line bundle is trivial

when $S_1 \otimes S_i^{-1}\big|_C = \mathcal{O}_C$.

The scheme structure claims then follow easily.

Suppose there is a limit of pointed curves and line bundles

$$\begin{tikzcd} & \mathcal{T} \arrow[dl] \\ \mathcal{C} \arrow[d] & \\ \Delta_0 \arrow[u, bend right, "p_1\ p_2\ p_3"'] & \end{tikzcd}$$

fiberwise trivial away from $0 \in \Delta_0$

disk

where $(\mathcal{C}_0, p_1, p_2, p_3, \mathcal{T}_0)$

$$\cong$$

$$\left(C, p_1, p_2, p_3, S_1 \otimes S_i^{-1}\big|_C\right)$$

We now extend $S_i|_C$ on $\mathcal{C}_0$ arbitrarily

over $\begin{array}{c}\mathcal{C}\\ \downarrow\\ \Delta_0\end{array}$, so we have $\mathcal{S}$ on $\mathcal{C}$

with $\mathcal{S}_0 \cong S_i|_C$. Since $\phi$-stability
$\phi$-stable

is open, we can (after shrinking $\Delta_0$)

assume that the restrictions of $\mathcal{S}$ to

all fibers of $\begin{array}{c}\mathcal{C}\\ \downarrow\\ \Delta_0\end{array}$ are $\phi$-stable.

Consider the family

$$\begin{tikzcd} & \mathcal{T}\otimes\mathcal{S} \arrow[dl] \\ \mathcal{C} \arrow[d] & \\ \Delta_0 \arrow[u, bend right, "p_1, p_2, p_3"'] & \end{tikzcd} .$$

disk

Away from $0 \in \Delta_0$, the restrictions of $\mathcal{T}\otimes\mathcal{S}$ and $\mathcal{S}$ to the fibers are isomorphic. By separatedness of $\phi$-stability, if

$$\begin{aligned} \mathcal{T}\otimes\mathcal{S}\big|_0 &\cong T_0 \otimes S_i\big|_C \\ &\cong S_1 \otimes S_i^{-1}\big|_C \otimes S_i\big|_C \\ &\cong S_1\big|_C \end{aligned}$$

is $\phi$-stable, then $S_1\big|_C \cong S_i\big|_C$. QED

**Claim II.** For $i = 1, 2, 3, 4$

$$\widehat{AJ}_1^{\,*}\left(V_i^{\log}\right) = \log DR\left(S_1 - S_i\right).$$

**Proof.** We will use that the restriction of $S_1$ and $S_i$ to the fibers of $\begin{array}{c}\mathcal{C}\\ \downarrow \\ \widehat{\mathcal{M}}_{g,3}\end{array}$ are both $\phi$-stable.

We will use the Holmes-Schwarz Criterion for when UniDR equals log DR (since we have Claim I.)

Let $\mathcal{U} \subset \widehat{\mathcal{M}}_{g,3}$ be the nonempty open locus where $S_1 \otimes S_i^{-1}$ has multidegree $\underline{0}$ on the fibes of $\begin{array}{c}\mathcal{C}\\ \downarrow \\ \widehat{\mathcal{M}}_{g,3}\end{array}$.

Let $\begin{array}{c}\mathcal{C}\\ \downarrow\\ \Delta_0\end{array}$ be a family of curves obtained

by pulling-back the universal family

$\begin{array}{c}\mathcal{C}\\ \downarrow\\ \widehat{\mathcal{M}}_{g,3}\end{array}$ via a map $\Delta_0 \to \widehat{\mathcal{M}}_{g,3}$

where $\Delta_0^* \to \mathcal{U}$

and $0 \mapsto (C_0, P_1, P_2, P_3) \in \widehat{\mathcal{M}}_{g,3}$

Suppose there exist

a line bundle $\begin{array}{c}\mathcal{C} \xleftarrow{} \Upsilon\\ \downarrow\\ \Delta_0\end{array}$

Such that two conditions hold:

- Away from $o \in \Delta_0$, the restrictions of $\mathcal{T}$ and $S_1 \otimes S_i^{-1}$ to the fibers of $\begin{array}{c}\mathcal{C} \\ \downarrow \\ \Delta_0\end{array}$ are isomorphic.

- $T_0 = T|_{C_0}$ has multidegree $\underline{0}$.

For the Holmes-Schwarz Criterion, we must prove $o \mapsto (C_0, P_1, P_2, P_3) \in \mathcal{U}$.

Consider now the line bundle $S_i$ pulled back to all of $\begin{array}{c}\mathcal{C}\\ \downarrow\\ \Delta_0\end{array}$. Certainly $S_i\big|_{C_0}$ is

$\phi$-stable (and therefore has a $\phi$-stable multidegree).

Via tensor product, we obtain

$$\begin{array}{ccc} & & \mathcal{T}\otimes S_i \\ & \swarrow & \\ \mathcal{C} & & \\ \downarrow & & \\ \Delta_0 & & \end{array}.$$

For $0 \neq t \in \Delta_0$,

$$\mathcal{T}\otimes S_i\big|_{C_t} \cong S_1 \otimes S_i^{-1}\big|_{C_t} \otimes S_i\big|_{C_t}$$

$$\cong S_1\big|_{C_t}.$$

For $o \in \Delta_0$,

$\mathcal{T} \otimes S_i \big|_{C_0}$ has the same multidegrees as $S_i \big|_{C_0}$.

Since $S_i \big|_{C_0}$ is $\phi$-stable, then

$\mathcal{T} \otimes S_i \big|_{C_0}$ must be also $\phi$-stable.

By separatedness of $\phi$-stability,

we conclude $\mathcal{T} \otimes S_i \big|_{C_0} \cong S_1 \big|_{C_0}$

and $o \mapsto (C_0, p_1, p_2, p_3) \in \mathcal{U}$. QED

Returning to the calculation:

$$\int_{\overline{\mathcal{J}ac}^{\circ}_{g,3}(\phi)} V_1^{\log} \cdot V_2^{\log} \cdot V_3^{\log} \cdot V_4^{\log}$$

$$= \int_{\widehat{\mathcal{M}}_{g,3}} \widehat{AJ}_1^{*}\left(V_2^{\log} \cdot V_3^{\log} \cdot V_4^{\log}\right)$$

using Claim II three times

$\Downarrow$

$$= \int_{\overline{\mathcal{M}}_{g,3}} \log DR_{A_1-A_2} \cdot \log DR_{A_1-A_3} \cdot \log DR_{A_1-A_4} .$$

See my Tel Aviv lecture in 2022

I explained in earlier lectures on $\log DR$ how to reduce the last integral to calculation of Buryak-Rossi 19:

use $Sl_3$ invariance

$$\int_{\overline{\mathcal{M}}_{g,3}} \log DR_{A_1-A_2} \cdot \log DR_{A_1-A_3} \cdot \log DR_{A_1-A_4} .$$

$$= \frac{\delta^{2g}}{2^{3g}\, g!\, (2g+1)!!}$$

ANSWER is $\phi$ independent

where $\delta$ is the GCD of all the $2\times 2$ minors of

$$\begin{pmatrix} a_1-a_2 & b_1-b_2 & c_1-c_2 \\ a_1-a_3 & b_1-b_3 & c_1-c_3 \\ a_1-a_4 & b_1-b_4 & c_1-c_4 \end{pmatrix}.$$

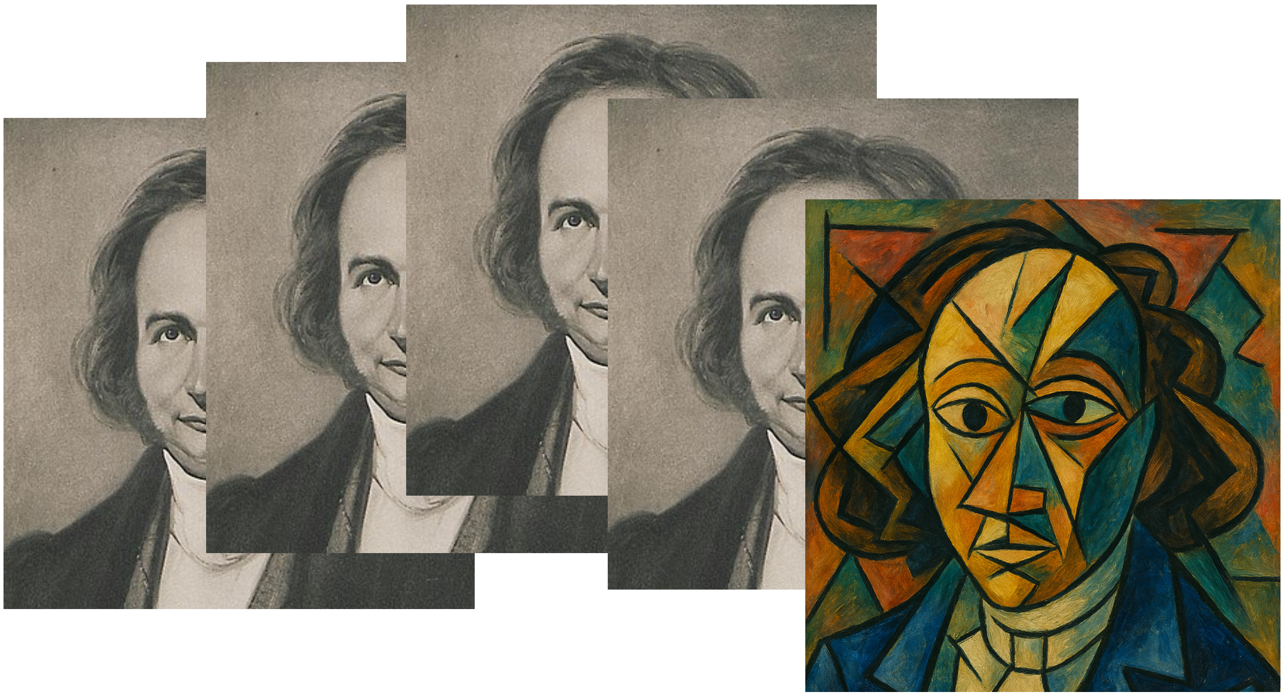
Answer must be symmetric in $A_1, A_2, A_3, A_4$!

Last comment: using Claim II, we obtain the follow statement.

Claim III. Every log intergal of the form

$$\int_{\overline{Jac}^{\circ}_{g,n}(\phi)} \prod_{i=1}^{m} V_i^{\log} \cdot \pi^*(\gamma^{\log}), \qquad \pi : \overline{Jac}^{\circ}_{g,n}(\phi) \to \bar{M}_{g,n},$$

where $V_i^{\log} \in \log CH^g\left(\overline{Jac}^{\circ}_{g,n}(\phi)\right)$ are classes of Abel-Jacobi loci and $\gamma^{\log} \in \log R^*(\bar{M}_{g,n})$, can be effectively computed via log DR theory.



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