VIRASORO CONSTRAINTS FOR STABLE PAIRS ON TORIC 3-FOLDS

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Abstract. Using new explicit formulas for the stationary GW/PT descendent correspondence for nonsingular projective toric 3-folds, we show that the correspondence intertwines the Virasoro constraints in Gromov-Witten theory for stable maps with the Virasoro constraints for stable pairs proposed in [17]. Since the Virasoro constraints in Gromov-Witten theory are known to hold in the toric case, we establish the stationary Virasoro constraints for the theory of stable pairs on toric 3-folds. As a consequence, new Virasoro constraints for tautological integrals over Hilbert schemes of points on surfaces are also obtained.

Contents

0. Introduction

0.1. **Stable pairs.** Let X be a nonsingular projective 3-fold. A *stable pair* (F, s) on X is a coherent sheaf F on X and a section $s \in H^0(X, F)$ satisfying the following stability conditions:

- F is *pure* of dimension 1,
- the section $s : \mathcal{O}_X \to F$ has cokernel of dimensional 0.

To a stable pair, we associate the Euler characteristic and the class of the support C of the sheaf F,

$$\chi(F) = n \in \mathbb{Z} \quad \text{and} \quad [C] = \beta \in H_2(X, \mathbb{Z}) \,.$$

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For fixed n and β, there is a projective moduli space of stable pairs $P_n(X,\beta)$. Unless β is an effective curve class, the moduli space $P_n(X,\beta)$ is empty. An analysis of the deformation theory and the construction of the virtual cycle $[P_n(X,\beta)]^{vir}$ is given [27]. We refer the reader to [20, 28] for an introduction to the theory of stable pairs.

Tautological descendent classes are defined via universal structures over the moduli space of stable pairs. Let

$$\pi : X \times P_n(X,\beta) \to P_n(X,\beta)$$

be the projection to the second factor, and let

$$\mathcal{O}_{X\times P_n(X,\beta)} \to \mathbb{F}_n$$

be the universal stable pair on $X \times P_n(X,\beta)$. Let[1]

$$\mathsf{ch}_k(\mathbb{F}_n - \mathcal{O}_{X\times P_n(X,\beta)}) \in H^*(X \times P_n(X,\beta))\,.$$

The following *descendent classes* are our main objects of study:

$$\mathsf{ch}_k(\gamma) = \pi_*\left(\mathsf{ch}_k(\mathbb{F}_n - \mathcal{O}_{X\times P_n(X,\beta)})\cdot\gamma\right) \in H^*(P_n(X,\beta))$$

for $k \geqslant 0$ and $\gamma \in H^*(X)$. The summand $-\mathcal{O}_{X\times P_n(X,\beta)}$ only affects ch_0,

$$\mathsf{ch}_0(\gamma) = -\int_X \gamma \in H^0(P_n(X,\beta))\,. \tag{1}$$

Since stable pairs are supported on curves, the vanishing

$$\mathsf{ch}_1(\gamma) = 0$$

always holds.

We will study the following descendent series:

$$\Big\langle \mathsf{ch}_{k_1}(\gamma_1)\cdots\mathsf{ch}_{k_m}(\gamma_m)\Big\rangle_\beta^{X,\mathrm{PT}} = \sum_{n\in\mathbb{Z}} q^n \int_{[P_n(X,\beta)]^{vir}} \prod_{i=1}^m \mathsf{ch}_{k_i}(\gamma_i)\,. \tag{2}$$

For fixed curve class $\beta \in H_2(X,\mathbb{Z})$, the moduli space $P_n(X,\beta)$ is empty for all sufficiently negative n. Therefore, the descendent series (2) has only finitely many polar terms.

Conjecture 1. [27] *The stable pairs descendent series*

$$\Big\langle \mathsf{ch}_{k_1}(\gamma_1)\cdots\mathsf{ch}_{k_m}(\gamma_m)\Big\rangle_\beta^{X,\mathrm{PT}}$$

is the Laurent expansion of a rational function of q for all $\gamma_i \in H^*(X)$ and all $k_i \geqslant 0$.

For Calabi-Yau 3-folds, Conjecture 1 reduces immediately to the rationality of the basic series $\langle 1 \rangle_\beta^{\mathrm{PT}}$ proven via wall-crossing in [2, 31]. In the presence of descendent insertions, Conjecture 1 has been proven for rich class of varieties [22, 23, 24, 25, 26] including all nonsingular projective toric 3-folds.

[1]We will always take singular cohomology with $\mathbb{Q}$-coefficients.

For our study of the GW/PT descendent correspondent and the Virasoro constraints, modified stable pair descendent insertions will be more suitable for us. Let[2]

$$\widetilde{\mathsf{ch}}_k(\alpha) = \mathsf{ch}_k(\alpha) + \frac{1}{24}\mathsf{ch}_{k-2}(\alpha \cdot c_2)\,,$$

where $c_2 = c_2(T_X)$ is the second Chern class of the tangent bundle, and let

$$\Big\langle \widetilde{\mathsf{ch}}_{k_1}(\gamma_1)\cdots\widetilde{\mathsf{ch}}_{k_m}(\gamma_m)\Big\rangle_\beta^{X,\mathrm{PT}} = \sum_{n\in\mathbb{Z}} q^n \int_{[P_n(X,\beta)]^{vir}} \prod_{i=1}^m \widetilde{\mathsf{ch}}_{k_i}(\gamma_i)$$

be the corresponding descendent series.

0.2. **Virasoro constraints for stable pairs.** Let X be a nonsingular projective 3-fold with only (p,p)-cohomology.[3] Let

$$c_i = c_i(T_X) \in H^*(X)\,.$$

The simplest example is $\mathbb{P}^3$ with

$$c_1 = 4\mathsf{H}\,, \quad c_1c_2 = 24\mathsf{p}\,,$$

where H and p are the classes of the hyperplane and the point respectively.

Let $\mathbb{D}_{\mathrm{PT}}^X$ be the commutative $\mathbb{Q}$-algebra with generators

$$\left\{ \mathsf{ch}_i(\gamma) \,\middle|\, i \geqslant 0\,, \gamma \in H^*(X) \right\}$$

subject to the natural relations

$$\begin{aligned} \mathsf{ch}_i(\lambda\cdot\gamma) &= \lambda\,\mathsf{ch}_i(\gamma)\,, \\ \mathsf{ch}_i(\gamma+\widehat{\gamma}) &= \mathsf{ch}_i(\gamma) + \mathsf{ch}_i(\widehat{\gamma}) \end{aligned}$$

for $\lambda \in \mathbb{Q}$ and $\gamma, \widehat{\gamma} \in H^*(X)$.

In order to define the Virasoro constraints for stable pairs, we require three constructions in the algebra $\mathbb{D}_{\mathrm{PT}}^X$:

- Define the derivation R_k on $\mathbb{D}_{\mathrm{PT}}^X$ by fixing the action on the generators:

  $$\mathrm{R}_k(\mathsf{ch}_i(\gamma)) = \left(\prod_{n=0}^{k}(i+d-3+n)\right)\mathsf{ch}_{i+k}(\gamma)\,, \quad \gamma \in H^{2d}(X,\mathbb{Q})$$

  for $k \geqslant -1$. In case $k=-1$, the product is empty and

  $$\mathrm{R}_{-1}(\mathsf{ch}_i(\gamma)) = \mathsf{ch}_{i-1}(\gamma)\,.$$

[2]We set $\mathsf{ch}_\ell(\gamma) = 0$ for $\ell < 0$.

[3]Our results will be about nonsingular projective toric varieties, but the formulas here are all well-defined when there is no odd cohomology and the Hodge classes in the even cohomology are all (p,p). To write the Virasoro constraints for varieties with non-(p,p) cohomology requires the Hodge grading and signs. A treatment is presented in [16] where the Virasoro constraints are checked in several non-(p,p) geometries. The theory leads to surprising predictions for vanishings [16].

- Define the element
$$\mathsf{ch}_a\mathsf{ch}_b(\gamma) = \sum_i \mathsf{ch}_a(\gamma_i^L)\mathsf{ch}_b(\gamma_i^R) \in \mathbb{D}_{\mathrm{PT}}^X$$
where $\sum_i \gamma_i^L \otimes \gamma_i^R$ is the Künneth decomposition of the product,
$$\gamma \cdot \Delta \in H^*(X \times X)\,,$$
with the diagonal Δ. The notation
$$(-1)^{d^L d^R}(a + d^L - 3)!(b + d^R - 3)!\, \mathsf{ch}_a\mathsf{ch}_b(\gamma)$$
will be used as shorthand for the sum
$$\sum_i (-1)^{d(\gamma_i^L)d(\gamma_i^R)}(a + d(\gamma_i^L) - 3)!(b + d(\gamma_i^R) - 3)!\, \mathsf{ch}_a(\gamma_i^L)\mathsf{ch}_b(\gamma_i^R)\,,$$
where $d(\gamma_i^L)$ and $d(\gamma_i^R)$ are the (complex) degrees of the classes. All factorials with negative arguments vanish.
- Define the operator $\mathrm{T}_k : \mathbb{D}_{\mathrm{PT}}^X \to \mathbb{D}_{\mathrm{PT}}^X$ by
$$\mathrm{T}_k = -\frac{1}{2}\sum_{a+b=k+2}(-1)^{d^L d^R}(a + d^L - 3)!(b + d^R - 3)!\, \mathsf{ch}_a\mathsf{ch}_b(c_1) + \frac{1}{24}\sum_{a+b=k} a!b!\, \mathsf{ch}_a\mathsf{ch}_b(c_1c_2)$$
for $k \geqslant -1$. The sum here is over all ordered pairs (a, b) satisfying $a + b = k + 2$ with $a, b \geqslant 0$ (and all factorials with negative arguments vanish). Written in terms of renormalized descendents, the formula simplifies to
$$\mathrm{T}_k = -\frac{1}{2}\sum_{a+b=k+2}(-1)^{d^L d^R}(a + d^L - 3)!(b + d^R - 3)!\, \widetilde{\mathsf{ch}}_a\widetilde{\mathsf{ch}}_b(c_1)\,. \tag{3}$$

Definition 2. *Let* $\mathcal{L}_k^{\mathrm{PT}} : \mathbb{D}_{\mathrm{PT}}^X \to \mathbb{D}_{\mathrm{PT}}^X$ *for* $k \geqslant -1$ *be the operator*
$$\mathcal{L}_k^{\mathrm{PT}} = \mathrm{T}_k + \mathrm{R}_k + (k+1)!\,\mathrm{R}_{-1}\mathsf{ch}_{k+1}(\mathsf{p})\,.$$

Since X is a nonsingular projective 3-fold with only (p, p)-cohomology, Hirzebruch-Riemman-Roch implies
$$\frac{c_1c_2}{24} = \mathsf{p} \in H^6(X)\,,$$
where $\mathsf{p} \in H^6(X)$ in the point class. Hence, for our paper, we can write
$$\mathcal{L}_k^{\mathrm{PT}} = \mathrm{T}_k + \mathrm{R}_k + (k+1)!\,\mathrm{R}_{-1}\mathsf{ch}_{k+1}\Big(\frac{c_1c_2}{24}\Big)\,. \tag{4}$$
The operators for more general varieties X defined in [16] specialize to (4) when all the cohomology is (p, p).

The operators $\mathcal{L}_k^{\mathrm{PT}}$ impose constraints on descendent integrals in the theory of stable pairs which are analogous to the Virasoro constraints of Gromov-Witten theory. We formulate the stable pairs Virasoro constraints as follows.

Conjecture 3. [17] *Let X be a nonsingular projective 3-fold with only (p,p)-cohomology, and let $\beta \in H_2(X,\mathbb{Z})$. For all $k \geqslant -1$ and $D \in \mathbb{D}^X_{\mathrm{PT}}$, we have*

$$\left\langle \mathcal{L}_k^{\mathrm{PT}}(D) \right\rangle_\beta^{X,\mathrm{PT}} = 0\,.$$

Our main result is a statement about *stationary descendents* for nonsingular projective *toric* 3-folds. The subalgebra $\mathbb{D}^{X+}_{\mathrm{PT}} \subset \mathbb{D}^X_{\mathrm{PT}}$ of stationary descendents is generated[4] by

$$\left\{ \mathsf{ch}_i(\gamma) \,\middle|\, i \geqslant 0\,, \gamma \in H^{>0}(X,\mathbb{Q}) \right\}.$$

The operators $\mathcal{L}_k^{\mathrm{PT}}$ are easily seen to preserve $\mathbb{D}^{X+}_{\mathrm{PT}}$. Therefore, the *stationary* Virasoro constraints are well-defined. We prove that the stationary Virasoro constraints hold in the toric case.

Theorem 4. *Let X be a nonsingular projective toric 3-fold, and let $\beta \in H_2(X,\mathbb{Z})$. For all $k \geqslant -1$ and $D \in \mathbb{D}^{X+}_{\mathrm{PT}}$, we have*

$$\left\langle \mathcal{L}_k^{\mathrm{PT}}(D) \right\rangle_\beta^{X,\mathrm{PT}} = 0\,.$$

In the basic case of $\mathbb{P}^3$, Theorem 4 specializes to the Virasoro constraints for stable pairs announced earlier in [20] via (4). A table of data of stable pairs descendent series for $\mathbb{P}^3$ is presented in the Appendix. The Virasoro constraints are seen to provide nontrivial relations.

0.3. **The Virasoro bracket.** For $k \geqslant -1$, we introduce the operators

$$\begin{aligned} \mathrm{L}_k^{\mathrm{PT}} \;=\; & -\frac{1}{2}\sum_{a+b=k+2} (-1)^{d^L d^R}(a+d^L-3)!(b+d^R-3)!\,\mathsf{ch}_a\mathsf{ch}_b(c_1) \\ & +\frac{1}{24}\sum_{a+b=k} a!b!\,\mathsf{ch}_a\mathsf{ch}_b(c_1c_2) \\ & +\mathsf{R}_k\,, \end{aligned}$$

where the sum, as before, is over ordered pairs (a,b) with $a,b \geqslant 0$.

Our conventions with regard to the factorials in the above definition of $\mathrm{L}_k^{\mathrm{PT}}$ differ slightly from those of the definition of $\mathcal{L}_k^{\mathrm{PT}}$. For $\mathrm{L}_k^{\mathrm{PT}}$, *all terms with negative factorial vanish* except *for the term* $(-1)!\,\mathsf{ch}_1(c_1)$. For example, we have

$$\mathrm{L}_{-1}^{\mathrm{PT}} = \mathrm{R}_{-1} + (-1)!\,\mathsf{ch}_1(c_1)\mathsf{ch}_0(\mathsf{p})\,.$$

The new conventions will play a role in the exceptional cases in our analysis. We extend the action of R_k by

$$\mathrm{R}_k((-1)!\,\mathsf{ch}_1(c_1)) = -(k-1)!\,\mathsf{ch}_{k+1}(c_1).$$

We view $(-1)!\mathsf{ch}_1(c_1)$ and

$$\mathrm{R}_{-1}((-1)!\mathsf{ch}_1(c_1)) = -(-2)!\mathsf{ch}_0(c_1)$$

as formal symbols.

[4]Equivalently, $\mathbb{D}^{X+}_{\mathrm{PT}}$ is generated by $\left\{ \widetilde{\mathsf{ch}}_i(\gamma) \,\middle|\, i \geqslant 0\,, \gamma \in H^{>0}(X,\mathbb{Q}) \right\}$.

We define an equivalence relation $\overset{\langle,\rangle}{=}$ for operators $\mathcal{A}, \mathcal{B} : \mathbb{D}_{\mathrm{PT}}^X \to \mathbb{D}_{\mathrm{PT}}^X$ by

$$\mathcal{A} \overset{\langle,\rangle}{=} \mathcal{B} \quad \leftrightarrow \quad \langle \mathcal{A}(D) \rangle_\beta^{X,\mathrm{PT}} = \langle \mathcal{B}(D) \rangle_\beta^{X,\mathrm{PT}} \text{ for all } D \in \mathbb{D}_{\mathrm{PT}}^X \text{ and } \beta \in H_2(X,\mathbb{Z})\,.$$

Inside the bracket, $\mathsf{ch}_0(\mathsf{p})$ acts as -1, and $\mathsf{ch}_1(\gamma)$ acts as 0 for all $\gamma \in H^*(X)$. Moreover, the formal symbols $(-1)!\mathsf{ch}_1(c_1)$ and $(-2)!\mathsf{ch}_0(c_1)$ are *defined* to act as 0 inside the bracket.

Using the equivalence relation $\overset{\langle,\rangle}{=}$, we obtain the Virasoro bracket and the following bracket with $\mathsf{ch}_k(\mathsf{p})$,

$$[\mathrm{L}_n^{\mathrm{PT}}, \mathrm{L}_k^{\mathrm{PT}}] \overset{\langle,\rangle}{=} (k-n)\,\mathrm{L}_{n+k}^{\mathrm{PT}}\,, \qquad [\mathrm{L}_n^{\mathrm{PT}}, (k-1)!\,\mathsf{ch}_k(\mathsf{p})] \overset{\langle,\rangle}{=} (n+k)!\,\mathsf{ch}_{n+k}(\mathsf{p})\,.$$

The operators $\mathcal{L}_k^{\mathrm{PT}}$ are expressed in terms of $\mathrm{L}_k^{\mathrm{PT}}$ by

$$\mathcal{L}_k^{\mathrm{PT}} \overset{\langle,\rangle}{=} \mathrm{L}_k^{\mathrm{PT}} + (k+1)!\,\mathrm{L}_{-1}^{\mathrm{PT}}\mathsf{ch}_{k+1}(\mathsf{p})\,.$$

The occurrences of the negative factorial terms $(-1)!\mathsf{ch}_1(c_1)$ cancel on the right side. The expressions $\mathrm{L}_k^{\mathrm{PT}}$ will play a role in the proof of Theorem 4.

0.4. **Path of the proof.** Our proof of Theorem 4 relies upon two central results. The first is the Virasoro conjecture in Gromov-Witten theory which has been proven for nonsingular projective toric varieties [8]. We refer the reader to the extensive literature on the subject [3, 7, 8, 19, 18, 29]. The second is the stationary GW/PT correspondence of [22, 23, 24] which was cast in terms of vertex operators in [17] and has been proven for nonsingular projective toric 3-folds. We show the stationary GW/PT correspondence intertwines the Virasoro constraints of the two theories. Along the way, we derive a more explicit form for the stationary GW/PT correspondence. Our proof of Theorem 4 yields the following stronger statement.

Theorem 5. *Let X be a nonsingular projective 3-fold with only (p,p)-cohomology for which the following two properties are satisfied:*

(i) *The stationary Virasoro constraints for the Gromov-Witten theory of X hold.*
(ii) *The stationary* GW/ PT *correspondence holds.*

Then, the stationary Virasoro constraints for the stable pairs theory of X hold.

A challenge for the subject is to prove the Virasoro constraints for stable pairs directly using the geometry of the moduli of sheaves. New ideas will almost certainly be required.

0.5. **Gromov-Witten theory.** Let X be a nonsingular projective 3-fold. Gromov-Witten theory is defined via integration over the moduli space of stable maps.

Let C be a possibly disconnected curve with at worst nodal singularities. The genus of C is defined by $1 - \chi(\mathcal{O}_C)$. Let $\overline{M}'_{g,m}(X,\beta)$ denote the moduli space of stable maps with possibly disconnected domain curves C of genus g with *no* collapsed connected components of genus greater or equal to 2. The latter condition[5] requires each non-rational and non-elliptic connected component of C to represent a nonzero class in $H_2(X,\mathbb{Z})$.

[5]The exclusion here of collapsed connected components of genus greater or equal to 2 matches the conventions of [17]. The definition of $\overline{M}'_{g,m}(X,\beta)$ differs slightly from the definitions of [25, 26] where

Let

$$\mathrm{ev}_i : \overline{M}'_{g,m}(X,\beta) \to X\,,$$
$$\mathbb{L}_i \to \overline{M}'_{g,m}(X,\beta)$$

denote the evaluation maps and the cotangent line bundles associated to the marked points. Let $\gamma_1, \ldots, \gamma_m \in H^*(X)$, and let

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}'_{g,m}(X,\beta))\,.$$

The *descendent insertions*, denoted by $\tau_k(\gamma)$ for $k \geqslant 0$, correspond to classes $\psi_i^k \mathrm{ev}_i^*(\gamma)$ on the moduli space of stable maps. Let

$$\Big\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_m}(\gamma_m)\Big\rangle_{g,\beta}^{X,\mathrm{GW}} = \int_{[\overline{M}'_{g,m}(X,\beta)]^{vir}} \prod_{i=1}^m \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i)$$

denote the descendent Gromov-Witten invariants. The associated generating series is defined by

$$\Big\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_m}(\gamma_m)\Big\rangle_{\beta}^{X,\mathrm{GW}} = \sum_{g\in\mathbb{Z}} \Big\langle \prod_{i=1}^m \tau_{k_i}(\gamma_i)\Big\rangle_{g,\beta}^{X,\mathrm{GW}} u^{2g-2}. \tag{5}$$

Since the domain components must map nontrivially, an elementary argument shows the genus g in the sum (5) is bounded from below. Foundational aspects of the theory are treated, for example, in [1, 5, 12].

Using the above definitions, the string equation[6] is easily checked:

$$\Big\langle \tau_0(1)\prod_{i=1}^m \tau_{k_i}(\gamma_i)\Big\rangle_{\beta}^{X,\mathrm{GW}} = \Big\langle \sum_{j=1}^m \prod_{i=1}^m \tau_{k_i-\delta_{i-j}}(\gamma_i)\Big\rangle_{\beta}^{X,\mathrm{GW}} + \text{collapsed contributions.} \tag{6}$$

The Gromov-Witten descendent insertions $\tau_k(\gamma)$ in (5) are defined for $k \geqslant 0$. We include the nonstandard descendent insertions $\tau_{-2}(\gamma)$ and $\tau_{-1}(\gamma)$ by the rule:

$$\Big\langle \tau_k(\gamma)\prod_{i=1}^m \tau_{k_i}(\gamma_i)\Big\rangle_{\beta}^{X,\mathrm{GW}} = \frac{\delta_{k+2}}{u^2}\int_X \gamma \;\cdot\; \Big\langle \prod_{i=1}^m \tau_{k_i}(\gamma_i)\Big\rangle_{\beta}^{X,\mathrm{GW}}\,, \quad \text{for } k<0. \tag{7}$$

We impose Heisenberg relations (100) on the operators $\tau_k(\gamma)$:

$$[\tau_k(\alpha), \tau_l(\beta)] = (-1)^k \frac{\delta_{k+l+1}}{u^2}\int_X \alpha\cdot\beta\,. \tag{8}$$

In particular, the evaluation (7) applies only after commuting the negative descendents to the left.

Assume now that X has only (p,p)-cohomology. Let $\mathbb{D}_{\mathrm{GW}}^X$ be the commutative $\mathbb{Q}$-algebra with generators

$$\big\{\, \tau_i(\gamma)\,\big|\, i \geqslant 0\,, \gamma \in H^*(X)\,\big\}$$

no collapsed connected components are permitted. The difference is minor, see Section 3 of [17] for a discussion.

[6]The standard correction term for the string equation occurs here since we allow collapsed connected components of genus 0 in our definition of the Gromov-Witten descendent series.

subject to the natural relations

$$\tau_i(\lambda \cdot \gamma) = \lambda\, \tau_i(\gamma)\,,$$
$$\tau_i(\gamma + \widehat{\gamma}) = \tau_i(\gamma) + \tau_i(\widehat{\gamma})$$

for $\lambda \in \mathbb{Q}$ and $\gamma, \widehat{\gamma} \in H^*(X)$. The subalgebra $\mathbb{D}^{X+}_{\mathrm{GW}} \subset \mathbb{D}^{X}_{\mathrm{GW}}$ of stationary descendents is generated by

$$\left\{ \tau_i(\gamma) \,\middle|\, i \geqslant 0\,, \gamma \in H^{>0}(X, \mathbb{Q}) \right\}.$$

We will use Getzler's renormalization $\mathfrak{a}_k$ of the Gromov-Witten descendents[7]:

$$\sum_{n=-\infty}^{\infty} z^n \tau_n = \mathsf{Z}^0 + \sum_{n>0} \frac{(\imath u z)^{n-1}}{(1+zc_1)_n} \mathfrak{a}_n + \frac{1}{c_1} \sum_{n<0} \frac{(\imath u z)^{n-1}}{(1+zc_1)_n} \mathfrak{a}_n\,, \tag{9}$$

$$\mathsf{Z}^0 = \frac{z^{-2}u^{-2}}{\mathcal{S}\left(\frac{zu}{\theta}\right)} - z^{-2}u^{-2},$$

where we use standard notation for the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\,.$$

For example[8],

$$\tau_0(\gamma) = \mathfrak{a}_1(\gamma) + \frac{1}{24}\int_X \gamma c_2\,, \tag{10}$$

$$\tau_1(\gamma) = \frac{\imath u}{2}\mathfrak{a}_2(\gamma) - \mathfrak{a}_1(\gamma \cdot c_1)\,. \tag{11}$$

For $k \geqslant 2$ and $\gamma \in H^{>0}(X)$, we have the general formula

$$\begin{aligned} \tau_k(\gamma) = \frac{(\imath u)^k}{(k+1)!}\mathfrak{a}_{k+1}(\gamma) - \frac{(\imath u)^{k-1}}{k!}\left(\sum_{i=1}^{k}\frac{1}{i}\right)\mathfrak{a}_k(\gamma\cdot c_1) \\ + \frac{(\imath u)^{k-2}}{(k-1)!}\left(\sum_{i=1}^{k-1}\frac{1}{i^2} + \sum_{1\leqslant i<j\leqslant k-1}\frac{1}{ij}\right)\mathfrak{a}_{k-1}(\gamma\cdot c_1^2)\,. \end{aligned} \tag{12}$$

0.6. **The** GW/PT **correspondence for essential descendents.** The subalgebra

$$\mathbb{D}^{X\bigstar}_{\mathrm{PT}} \subset \mathbb{D}^{X+}_{\mathrm{PT}}$$

of *essential descendents* is generated by

$$\left\{ \widetilde{\mathsf{ch}}_i(\gamma) \,\middle|\, (i \geqslant 3, \gamma \in H^{>0}(X,\mathbb{Q})) \text{ or } (i = 2, \gamma \in H^{>2}(X,\mathbb{Q})) \right\}.$$

While closed formulas for the full GW/PT descendent transformation of [25] are not known in full generality, the stationary theory is much better understood [17].[9] The transformation takes the simplest form when restricted to essential descendents.

[7] We use $\imath$ for the square root of -1. The genus variable u will usually occur together with $\imath$.

[8] The constant term $\frac{1}{24}\int_X \gamma c_2$ in the formula does not contribute unless $\gamma \in H^2(X)$.

[9] See [13, 14] for an earlier view of descendents and descendent transformations.

The GW/PT transformation restricted to the essential descendents is a linear map

$$\mathfrak{C}^\bullet : \mathbb{D}^{X\star}_{\mathrm{PT}} \to \mathbb{D}^X_{\mathrm{GW}}$$

satisfying

$$\mathfrak{C}^\bullet(1) = 1$$

and is defined on monomials by

$$\mathfrak{C}^\bullet\Big(\widetilde{\mathrm{ch}}_{k_1}(\gamma_1)\dots\widetilde{\mathrm{ch}}_{k_m}(\gamma_m)\Big) = \sum_{P \text{ set partition of } \{1,\dots,m\}} \prod_{S\in P} \mathfrak{C}^\circ\Big(\prod_{i\in S}\widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\Big)\,.$$

The operations $\mathfrak{C}^\circ$ on $\mathbb{D}^{X\star}_{\mathrm{PT}}$ are

$$\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\Big) = \frac{1}{(k_1+1)!}\mathfrak{a}_{k_1+1}(\gamma) + \frac{(\imath u)^{-1}}{k_1!}\sum_{|\mu|=k_1-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1)}{\mathrm{Aut}(\mu)} + \frac{(\imath u)^{-2}}{k_1!}\sum_{|\mu|=k_1-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)} + \frac{(\imath u)^{-2}}{(k_1-1)!}\sum_{|\mu|=k_1-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)}\,, \tag{13}$$

$$\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma')\Big) = -\frac{(\imath u)^{-1}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2}(\gamma\gamma') - \frac{(\imath u)^{-2}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-1}(\gamma\gamma'\cdot c_1) - \frac{(\imath u)^{-2}}{k_1!k_2!}\sum_{|\mu|=k_1+k_2-2}\max(\max(k_1,k_2),\max(\mu_1+1,\mu_2+1))\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\gamma'\cdot c_1)\,, \tag{14}$$

$$\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma')\widetilde{\mathrm{ch}}_{k_3+2}(\gamma'')\Big) = \frac{(\imath u)^{-2}|k|}{k_1!k_2!k_3!}\mathfrak{a}_{|k|-1}(\gamma\gamma'\gamma'')\,, \quad |k| = k_1+k_2+k_3\,. \tag{15}$$

The above sums are over *partitions* of μ of length 2 or 3. The parts of μ are *positive* integers, and we always write

$$\mu = (\mu_1,\mu_2) \quad \text{and} \quad \mu = (\mu_1,\mu_2,\mu_3)$$

with weakly decreasing parts. In equations (13)-(15), we have $k_i \geqslant 0$, and all occurrences of $\mathfrak{a}_0$ and $\mathfrak{a}_{-1}$ are set to 0.

The above formulas for the GW/PT descendent correspondence are proven here from the vertex operator formulas of [17] by a direct evaluation of the leading terms. In the toric case, we have the following explicit correspondence statement.[10]

Theorem 6. *Let X be a nonsingular projective toric 3-fold. Let*

$$\prod_{i=1}^m \widetilde{\mathrm{ch}}_{k_i}(\gamma_i) \in \mathbb{D}^{X\star}_{\mathrm{PT}}\,.$$

[10]A straightforward exercise using our new conventions is to show the abstract correspondence of Theorem 6 is a consequence of [25, Theorem 4]. The novelty of Theorem 6 is the closed formula for the transformation.

Let $\beta \in H_2(X,\mathbb{Z})$ *with* $d_\beta = \int_\beta c_1(X)$. *Then, the* GW/PT *correspondence defined by formulas* (13)-(15) *holds:*

$$(-q)^{-d_\beta/2} \left\langle \prod_{i=1}^{m} \widetilde{\mathsf{ch}}_{k_i}(\gamma_i) \right\rangle_\beta^{X,\mathrm{PT}} = (-\imath u)^{d_\beta} \left\langle \mathfrak{C}^\bullet\Big(\prod_{i=1}^{m} \widetilde{\mathsf{ch}}_{k_i}(\gamma_i) \Big) \right\rangle_\beta^{X,\mathrm{GW}} ,$$

after the change of variables $-q = e^{\imath u}$.

As direct consequence of the formulas (13)-(15), the correspondence taken essential descendents on the stable pairs side to stationary descendents on the stable pairs side.

Proposition 7. *Let* $D \in \mathbb{D}^{X\star}_{\mathrm{PT}}$. *Under the* GW/PT *transformation, we have*

$$\mathfrak{C}^\bullet(D) \in \mathbb{D}^{X+}_{\mathrm{GW}} .$$

0.7. **Plan of the paper.** The key to our proof of Theorem 4 is an intertwining property of $\mathfrak{C}^\bullet$ with respect to Virasoro operators for stable pairs and the Virasoro operators for stable maps. Via the intertwining property, Theorem 4 is a consequence of the stationary GW/PT correspondence of Theorem 6 and the Virasoro constraints for the Gromov-Witten theory of toric 3-folds.

The algebra $\mathbb{D}^X_{\mathrm{PT}}$ carries a *bumping filtration*[11]

$$\mathbb{D}^0_{\mathrm{PT}} \subset \mathbb{D}^1_{\mathrm{PT}} \subset \mathbb{D}^2_{\mathrm{PT}} \subset \mathbb{D}^3_{\mathrm{PT}} \subset \cdots \subset \mathbb{D}^X_{\mathrm{PT}} , \tag{16}$$

where $\mathbb{D}^k_{\mathrm{PT}}$ is spanned by the monomials[12]

$$\prod_{i=1}^{m} \widetilde{\mathsf{ch}}_{k_i}(\gamma_i)$$

for which $\gamma_{s_1} \cdots \gamma_{s_l} = 0$ for all subsets

$$S = \{s_1, \ldots, s_l\} \subset \{1, \ldots, m\} , \quad l > k .$$

In general the filtration (16) has infinite length. But if we restrict the filtration to $\mathbb{D}^{X\star}_{\mathrm{PT}}$, the filtration truncates since

$$\mathbb{D}^3_{\mathrm{PT}} \cap \mathbb{D}^{X\star}_{\mathrm{PT}} = \mathbb{D}^{X\star}_{\mathrm{PT}} .$$

The correspondence

$$\mathfrak{C}^\bullet : \mathbb{D}^{X\star}_{\mathrm{PT}} \to \mathbb{D}^{X+}_{\mathrm{GW}}$$

respects the analogous bumping filtration $\mathbb{D}^k_{\mathrm{GW}} \cap \mathbb{D}^{X+}_{\mathrm{GW}}$ on $\mathbb{D}^{X+}_{\mathrm{GW}}$ with respect to the monomials

$$\prod_{i=1}^{m} \tau_{k_i}(\gamma_i)$$

for which $\gamma_{s_1} \cdots \gamma_{s_l} = 0$ for all subsets

$$S = \{s_1, \ldots, s_l\} \subset \{1, \ldots, m\} , \quad l > k .$$

[11]The bumping filtration is a filtration of vector spaces.

[12]Via the empty monomial ($m = 0$), $\mathbb{D}^0_{\mathrm{PT}}$ is spanned by the unit 1.

Our proof of the intertwining is separated into a calculation for each of the four steps of the restriction of the bumping filtration on $\mathbb{D}^{X\star}_{\mathrm{PT}}$.

We discuss the Virasoro constraints for Gromov-Witten theory in Section 1 and for stable pairs in Section 2. The stationary Virasoro constraints of Theorem 4 are proven in Section 2.4 modulo the intertwining of Theorem 12. The proof of the intertwining property is given in four steps:

(0) We start in Section 3 with the special case where $D \in \mathbb{D}^0_{\mathrm{PT}} \cap \mathbb{D}^{X\star}_{\mathrm{PT}}$ is the trivial monomial 1. The result is Proposition 14 of Section 3.3.
(1) For $D \in \mathbb{D}^1_{\mathrm{PT}} \cap \mathbb{D}^{X\star}_{\mathrm{PT}}$, the required results are proven in Section 4.3.
(2) Proposition 17 and Proposition 18 of Section 5 imply the intertwining property for $D \in \mathbb{D}^2_{\mathrm{PT}} \cap \mathbb{D}^{X\star}_{\mathrm{PT}}$.
(3) We treat $D \in \mathbb{D}^3_{\mathrm{PT}} \cap \mathbb{D}^{X\star}_{\mathrm{PT}} = \mathbb{D}^{X\star}_{\mathrm{PT}}$ in Proposition 19 of Section 5 to complete the proof of Theorem 12.

Let S be a nonsingular projective toric surface. As a consequence of the stationary Virasoro constraints for

$$X = S \times \mathbb{P}^1 \quad \text{and} \quad \beta = n[\mathbb{P}^1]\,,$$

we obtain *new* Virasoro constraints for the integrals of the tautological classes over Hilbert schemes of points $\mathrm{Hilb}^n(S)$ of surfaces S in Section 6. The case of *all* simply connected nonsingular projective surfaces is proven in [16].

After a review of the GW/PT descendent correspondence from the perspective of [17] in Section 7, we complete the proof of Theorem 6 in Section 8. A list of descendent series in degree 1 for $\mathbb{P}^3$ is given in Section 9.

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1. Virasoro constraints for Gromov-Witten theory

1.1. **Overview.** We will discuss here the Virasoro constraints for stable maps. The constraints are equivalent to a procedure for removing the descendents of the canonical class. The procedures may be interpreted as series of the reactions (similar to the reactions discussed in the context of the GW/PT descendent correspondence in [17, Section 3]). Our goal is to write the Virasoro constraints for Gromov-Witten theory in a form which is as close as possible to the Virasoro constraints of Conjecture 3 for stable pairs.

1.2. **Gromov-Witten constraints: original form.** The Virasoro constraints in Gromov-Witten theory were first proposed[13] in [3]. We recall here the original form following [19]. In Section 1.3, a reformulation which is more suitable for the GW/PT correspondence will be presented.

In the discussion below, we fix a basis of $H^*(X)$,

$$\gamma_0, \ldots, \gamma_r\,, \quad \gamma_i \in H^{p_i,q_i}(X)\,, \tag{17}$$

for which $\gamma_0 = 1$, $\gamma_1 = c_1$, and $\gamma_r = [\mathsf{p}]$. We assume[14] $c_1 \neq 0$. We also fix a dual basis

$$\gamma_0^\vee, \ldots, \gamma_r^\vee\,, \quad \int_X \gamma_i \gamma_j^\vee = \delta_{ij}\,.$$

The standard method of describing of the Virasoro constraints uses the generating function for the Gromov-Witten invariants (see [19, section 4]):

$$F^X = \sum_{g\geqslant 0} u^{2g-2} \sum_{\beta\in H_2(X,\mathsf{Z})} q^\beta \sum_{n\geqslant 0} \sum_{\substack{a_1,\ldots,a_n\\ k_1,\ldots,k_n}} t^{a_1}_{k_1}\ldots t^{a_1}_{k_1}\ldots t^{a_n}_{k_n} \left\langle \tau_{k_1}(\gamma_1)\ldots\tau_{k_n}(\gamma_n)\right\rangle^{X,\mathrm{Con}}_{g,\beta}\,,$$

where $\left\langle , \right\rangle^{X,\mathrm{Con}}_{g,\beta}$ is the standard integral over stable maps with connected domains (and stable contracted components of all genera are permitted).

The degree $\beta = 0$ summand F_0^X of F^X does not require knowledge of curves in X. We further split the degree 0 summand into summands of genus $g \leqslant 1$ and genus $g \geqslant 2$:

$$F_0^X = F^X_{0,g\leqslant 1} + F^X_{0,g\geqslant 2}\,.$$

The $g \leqslant 1$ summand takes the form

$$F^X_{0,g\leqslant 1} = u^{-2}\sum_{i,j,k}\left(\frac{t_0^i t_0^j t_0^k}{3!} + \frac{t_0^i t_0^j t_1^k t_0^0}{2!}\right)\int_X \gamma_i\gamma_j\gamma_k - \sum_i \left(\frac{t_0^i}{24} + \frac{t_1^i t_0^0}{24}\right)\int_X \gamma_i c_2 + \ldots\,,$$

where the dots stand for terms divisible by $(t_0^0)^2$. The $g \geqslant 2$ summand $F^X_{0,g\geqslant 2}$ is determined by the string and dilaton equations from the constant maps contributions of [4, Theorem 4].

Let $\widetilde{F}^X$ be the summand of F^X with $\beta \neq 0$. We define

$$Z^X_{0,*} = \exp(F^X_{0,*})\,, \quad \widetilde{Z}^X = \exp(\widetilde{F})\,.$$

[13] The full conjecture also involves ideas of S. Katz.

[14] For Calabi-Yau 3-folds, the Virasoro invariants are a consequence of the string and dilaton equations (and there are no non-trivial stationary invariants).

The Gromov-Witten bracket $\langle , \rangle_{g,\beta}^{X,\mathrm{GW}}$ introduced in Section 0.5 corresponds to the partition function

$$Z^X_{0,g\leqslant 1} \cdot \widetilde{Z}^X = \sum_{g\geqslant \mathbb{Z}} u^{2g-2} \sum_{\beta\in H_2(X,\mathsf{Z})} q^\beta \sum_{n\geqslant 0} \sum_{\substack{a_1,\dots,a_n \\ k_1,\dots,k_n}} t^{a_1}_{k_1} \dots t^{a_1}_{k_1} \dots t^{a_n}_{k_n} \left\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n)\right\rangle^{X,\mathrm{GW}}_{g,\beta} .$$

The full partition function

$$Z^X = \exp(F^X) = Z^X_{0,g\leqslant 1} \cdot Z^X_{0,g\geqslant 2} \cdot \widetilde{Z}^X$$

corresponds to the standard disconnected Gromov-Witten bracket $\langle , \rangle_{g,\beta}^{X,\bullet}$,

$$Z^X = \sum_{g\geqslant 0} u^{2g-2} \sum_{\beta\in H_2(X,\mathsf{Z})} q^\beta \sum_{n\geqslant 0} \sum_{\substack{a_1,\dots,a_n \\ k_1,\dots,k_n}} t^{a_1}_{k_1} \dots t^{a_1}_{k_1} \dots t^{a_n}_{k_n} \left\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n)\right\rangle^{X,\bullet}_{g,\beta} .$$

The Virasoro operators L_k, $k \in \mathbb{Z}_{\geqslant -1}$ are differential operators which satisfy the Virasoro relations,

$$[\mathrm{L}_k, \mathrm{L}_\ell] = (k-\ell)\mathrm{L}_{k+\ell} ,$$

and annihilate the partition function

$$\mathrm{L}_k \, Z^X = 0 . \tag{18}$$

For 3-folds X, the operators are defined by:

$$\begin{aligned}
\mathrm{L}_k = \sum_{m=0}^{\infty} \sum_{i=0}^{k+1} \Bigg(& [p_a + m - 1]^k_i (C^i)^b_a \tilde{t}^a_m \partial_{b,m+k-i} \\
& + \frac{u^2}{2}(-1)^{m+1}[-p_a + 1 - m]^k_i (C^i)^{ab} \partial_{a,m}\partial_{b,k-m-i-1} \Bigg) \\
& + \frac{u^{-2}}{2}(C^{k+1})_{ab} t^a_0 t^b_0 \\
& - \frac{\delta_k}{24} \int_X c_1 c_2 ,
\end{aligned}$$

where the Einstein conventions for summing over repeated indices are followed[15],

$$\tilde{t}^a_m = t^a_m - \delta_{a0}\delta_{m1} , \quad \partial_{a,m} = \partial/\partial t^a_m ,$$

and $[x]^k_j = e_{k+1-j}(x, x+1, \dots, x+k)$. The tensors in the equation are defined in terms of the dual basis:

$$(C^i)^a_b = \int_X \gamma^\vee_a c^i_1 \gamma_b , \quad (C^i)_{ab} = \int_X \gamma_a c^i_1 \gamma_b , \quad (C^i)^{ab} = \int_X \gamma^\vee_a c^i_1 \gamma^\vee_b .$$

[15]Here δ denotes the δ-function: $\delta_k = 0$ unless $k = 0$, $\delta_0 = 1$, and $\delta_{ab} = \delta_{a-b}$.

1.3. **Gromov-Witten constraints: correspondence form.** We rewrite here the Virasoro constraints of Section 1 in the form most natural for the GW/PT descendent correspondence. Since all of our results are for toric varieties, we specialize our discussion here to the case where X is a nonsingular projective 3-fold with only (p,p)-cohomology.

We start by defining derivations R_k^j and quadratic differentials B^k on $\mathbb{D}_{\mathrm{GW}}^X$ by fixing the action on the generators:

- The action of the derivation R_k^j on $\tau_i(\gamma)$ for $k \geqslant -1$, $0 \leqslant j \leqslant 3$, and $\gamma \in H^{2d}(X)$ is
$$\mathrm{R}_k^j(\tau_i(\gamma)) = [i+d-1]_j^k\, \tau_{k+i-j}(\gamma \cdot c_1^j)\,,$$
where $[x]_j^k = e_{k+1-j}(x, x+1, \ldots, x+k)$ and all terms $\tau_{\ell<-2}(\theta)$ are set to zero. As a special case,
$$\mathrm{R}_{-1}^j(\tau_i(\gamma)) = \delta_j\, \tau_{i-1}(\gamma)\,.$$
We will use the notation $\mathrm{R}_k = \sum_{j=0}^3 \mathrm{R}_k^j\,.$
- The action of the quadratic differential B^k on $\tau_0(\gamma)\tau_0(\gamma')$ is
$$\mathrm{B}^k(\tau_0(\gamma)\tau_0(\gamma')) = \int_X \gamma\gamma' c_1^k\,.$$
On all other quadratics terms, B^k acts trivially.

The differential operators $\mathrm{L}_k^{\mathrm{GW}}$, for $k \geqslant -1$, are then defined by the formula:
$$\mathrm{L}_k^{\mathrm{GW}} = \mathrm{R}_k + \frac{u^{-2}}{2}\mathrm{B}^{k+1} + \frac{(\imath u)^2}{2}\mathrm{T}_k - \frac{\delta_k}{24}\int_X c_1 c_2\,,$$
where $\mathrm{T}_k = \sum_{j=0}^3 \mathrm{T}_k^j$ and

$$\mathrm{T}_k^j = \sum_{m=-1}^{k-j+2} (-1)^{m+1} [2-m-d_L]_j^k \; : \tau_{m-1}\tau_{-m+k-j}(c_1^j) : \,, \tag{19}$$

where d_L is the degree of the left term in the co-product[16] (as in Section 0.2). In formula (19), the symbol :: stands for the normal ordering convention: *all negative descendents* $\tau_{<0}(\gamma)$ *are on the left of the positive descendents.*

A calculation then yields the Virasoro bracket and the following bracket with $\tau_k(\mathsf{p})$:

$$[\mathrm{L}_n^{\mathrm{GW}}, \mathrm{L}_k^{\mathrm{GW}}] = (n-k)\,\mathrm{L}_{n+k}^{\mathrm{GW}}\,, \qquad [\mathrm{L}_n^{\mathrm{GW}}, (k+1)!\,\tau_k(\mathsf{p})] = (k+n+2)!\,\tau_{n+k}(\mathsf{p})\,. \tag{20}$$

[16]Define the element
$$\tau_a\tau_b(\gamma) = \sum_i \tau_a(\gamma_i^L)\tau_b(\gamma_i^R) \in \mathbb{D}_{\mathrm{GW}}^X$$
where $\sum_i \gamma_i^L \otimes \gamma_i^R$ is the Künneth decomposition of the product,
$$\gamma \cdot \Delta \in H^*(X \times X)\,,$$
with the diagonal Δ.

Theorem 8. *(Givental [8]) Let X be a nonsingular projective toric 3-fold, and let $\beta \in H_2(X,\mathbb{Z})$. For all $k \geqslant -1$ and $D \in \mathbb{D}^X_{\mathrm{GW}}$, we have*

$$\left\langle \mathrm{L}^{\mathrm{GW}}_k(D) \right\rangle^{X,\bullet}_{\beta} = 0\,.$$

Theorem 8, which is exactly equivalent to constraints (18) for toric 3-folds, was proven by Givental in two steps:

(i) Using the virtual localization formula of [9], the Gromov-Witten theory of X is expressed in terms of graphs sums with descendent integrals over the moduli spaces of curves $\overline{M}_{g,n}$ at the vertices.

(ii) The Virasoro constraints, conjectured by Witten [32] for $\overline{M}_{g,n}$ and proven in [11], are then used to establish the Virasoro constraints for X.

A second proof of Theorem 8, via the Givental-Teleman classification[17] of semisimple CohFTs, was given in [29]. For varieties with non-semisimple Gromov-Witten theory, the Virasoro constraints are known in very few cases.[18]

1.4. **Gromov-Witten constraints: stationary form.** We rewrite the Virasoro constraints in Gromov-Witten theory of Section 1.3 in a form which preserves the algebra of stationary descendents,

$$\mathbb{D}^{X+}_{\mathrm{GW}} \subset \mathbb{D}^X_{\mathrm{GW}}\,.$$

We fix a basis (17) of the cohomology of X which satisfies the following further conventions. Let

$$\gamma_1,\ldots,\gamma_s \in H^2(X)$$

be a basis with $\gamma_1 = c_1$. Let

$$\gamma_{2s},\ldots,\gamma_{s+1} \in H^4(X)$$

be a dual basis with respect to the Poincaré pairing. Let

$$\gamma_0 = 1 \in H^0(X)\,, \quad \gamma_{2s+1} = \mathsf{p} \in H^6(X)$$

span the rest of the cohomology.[19] The Künneth decomposition of the diagonal is

$$\Delta = \sum_{i=0}^{2s+1} \gamma_i \otimes \gamma_{2s+1-i}\,.$$

Consider the term T_k. The only place for descendents of 1 to appear in the operator $\mathrm{L}^{\mathrm{GW}}_k$ is in T^0_k. As most of the terms of T^0_k vanish by definition, we find

$$\frac{1}{2}\mathrm{T}^0_k = (k+1)!\ :\tau_0(1)\tau_{k-1}(\mathsf{p}):\ . \tag{21}$$

We denote the rest of the term by T'_k,

$$\mathrm{T}_k = \mathrm{T}'_k + \mathrm{T}^0_k\,.$$

[17] We refer the reader to [21] for an introduction.

[18] The main known examples are based on the Virasoro constraints for curves proven in [18].

[19] To match with (17), $r = 2s+1$.

Inside the bracket $\langle , \rangle_\beta^{X,\bullet}$, the insertion $\tau_0(1)$ can be removed by the string equation. We are therefore led to define the operator

$$\mathcal{L}_k^{\mathrm{GW}} = \frac{(\imath u)^2}{2}\mathrm{T}'_k + \mathrm{R}_k + \frac{u^{-2}}{2}\mathrm{B}^{k+1} + (\imath u)^2(k+1)!\,\mathrm{R}_{-1}\tau_{k-1}(\mathsf{p})\,, \qquad \mathrm{T}'_k = \sum_{j>0}\mathrm{T}^j_k\,,$$

where $\mathrm{R}_k = \sum_{j=0}^3 \mathrm{R}_k^j$ and R_{-1} is the differentiation defined on the generators by

$$\mathrm{R}_{-1}\tau_k(\gamma) = \tau_{k-1}(\gamma)\,.$$

Inside the bracket $\langle , \rangle_\beta^{X,\bullet}$, we have[20]

$$\mathcal{L}_k^{\mathrm{GW}} \overset{\langle,\rangle}{=} \widetilde{\mathrm{L}}_k^{\mathrm{GW}} + (\imath u)^2(1-\delta_k)(k+1)!\,\widetilde{\mathrm{L}}_{-1}^{\mathrm{GW}}\tau_{k-1}(\mathsf{p})\,, \tag{22}$$

where we have modified the Virasoro operators to exclude the descendents of 1:

$$\widetilde{\mathrm{L}}_k^{\mathrm{GW}} = \mathrm{L}_k^{\mathrm{GW}} - \frac{(\imath u)^2}{2}\mathrm{T}^0_k = \frac{(\imath u)^2}{2}\mathrm{T}'_k + \mathrm{R}_k + \frac{u^{-2}}{2}\mathrm{B}^{k+1} - \frac{\delta_k}{24}\int_X c_1c_2\,. \tag{23}$$

Though the operators $\mathcal{L}_k^{\mathrm{GW}}$ no longer satisfy the Virasoro bracket, the operators $\mathcal{L}_k^{\mathrm{GW}}$ preserve the subalgebra $\mathbb{D}^{X+}_{\mathrm{GW}} \subset \mathbb{D}^X_{\mathrm{GW}}$.

Proposition 9. *Let X be a nonsingular projective toric 3-fold, and let $\beta \in H_2(X,\mathbb{Z})$. For all $k \geqslant -1$ and $D \in \mathbb{D}^{X+}_{\mathrm{GW}^\circ}$, we have*

$$\Big\langle \mathcal{L}_k^{\mathrm{GW}}(D)\Big\rangle_\beta^{X,\bullet} = 0\,.$$

Proof. The case $k=0$ follows because

$$\mathcal{L}_0^{\mathrm{GW}} - \mathrm{L}_0^{\mathrm{GW}} = \mathrm{T}^0_0 = 2 : \tau_0(1)\tau_{-1}(\mathsf{p}) :$$

and $\Big\langle \mathrm{T}^0_0 \ldots \Big\rangle_\beta^{X,\bullet} = 0$. For the other case the argument is below.

Using (22) and (23), we have

$$\Big\langle \mathcal{L}_k^{\mathrm{GW}}(D)\Big\rangle_\beta^{X,\bullet} = \Big\langle \mathrm{L}_k^{\mathrm{GW}}(D) + (\imath u)^2(k+1)!\,\mathrm{L}_{-1}^{\mathrm{GW}}(\tau_{k-1}(\mathsf{p})D)\Big\rangle_\beta^{X,\bullet} - \frac{(\imath u)^2}{2}\Big\langle \mathrm{T}^0_k(D) + (\imath u)^2(k+1)!\,\mathrm{T}^0_{-1}(\tau_{k-1}(\mathsf{p})D)\Big\rangle_\beta^{X,\bullet}. \tag{24}$$

The first bracket on the right side of (24) vanishes by Theorem 8. We can write the second bracket on the right as

$$\frac{(\imath u)^2}{2}\Big\langle \mathrm{T}^0_k(D) + (\imath u)^2(k+1)!\,\mathrm{T}^0_{-1}(\tau_{k-1}(\mathsf{p})D)\Big\rangle_\beta^{X,\bullet} =$$
$$(\imath u)^2\Big\langle (k+1)!\,\tau_0(1)\tau_{k-1}(\mathsf{p})D + (\imath u)^2(k+1)!\,\tau_0(1)\tau_{-2}(\mathsf{p})\tau_{k-1}(\mathsf{p})D\Big\rangle_\beta^{X,\bullet}$$

[20]Note $\mathcal{L}_0^{\mathrm{GW}} = \widetilde{\mathrm{L}}_0^{\mathrm{GW}}$.

using (21). The right side of the above equation, after applying the commutator (7), is

$$(\imath u)^2\Big\langle (k+1)!\,\tau_0(1)\tau_{k-1}(\mathsf{p})D + (\imath u)^2(k+1)!\,\tau_{-2}(\mathsf{p})\tau_0(1)\tau_{k-1}(\mathsf{p})D\Big\rangle_{\beta}^{X,\bullet},$$

which vanishes after applying (8). ☐

In our study of the GW/PT descendent correspondence, we are interested in the Gromov-Witten bracket $\langle , \rangle_{g,\beta}^{X,\mathrm{GW}}$ of Section 0.5 instead of the standard disconnected bracket $\langle , \rangle_{g,\beta}^{X,\bullet}$. Therefore, the following result is important for our study.

Proposition 10. *Let X be a nonsingular projective toric 3-fold, and let $\beta \in H_2(X,\mathbb{Z})$. For all $k \geqslant -1$ and $D \in \mathbb{D}_{\mathrm{GW}^\circ}^{X+}$, we have*

$$\Big\langle \mathcal{L}_k^{\mathrm{GW}}(D)\Big\rangle_{\beta}^{X,\mathrm{GW}} = 0\,.$$

Proof. Since $\mathcal{L}_k^{\mathrm{GW}}$ preserves $\mathbb{D}_{\mathrm{GW}}^{X+}$, we have

$$\mathcal{L}_k^{\mathrm{GW}}(D) \in \mathbb{D}_{\mathrm{GW}}^{X+}\,.$$

Since the Gromov-Witten invariants corresponding to collapsed connected components of genus at least 2 *always* vanish in the presence of stationary descendents,

$$\Big\langle \mathcal{L}_k^{\mathrm{GW}}(D)\Big\rangle_{\beta}^{X,\bullet} = Z_{0,g\geqslant 2}^{X}\Big|_{\{t_k^i=0\}} \cdot \Big\langle \mathcal{L}_k^{\mathrm{GW}}(D)\Big\rangle_{\beta}^{X,\mathrm{GW}}\,.$$

Since $\Big\langle \mathcal{L}_k^{\mathrm{GW}}(D)\Big\rangle_{\beta}^{X,\bullet}$ vanishes by Proposition 9 and

$$Z_{0,g\geqslant 2}^{X}\Big|_{\{t_k^i=0\}} = \exp\left(\sum_{g=2}^{\infty}(-1)^g u^{2g-2}\,\frac{\chi(X)}{2}\int_{\overline{M}_g}\lambda_{g-1}^3\right)$$

is invertible[21], $\Big\langle \mathcal{L}_k^{\mathrm{GW}}(D)\Big\rangle_{\beta}^{X,\mathrm{GW}}$ also vanishes. ☐

2. Theorem 4: Virasoro constraints for stable pairs

2.1. **Intertwining property.** We have already defined the operators $\mathrm{L}_k^{\mathrm{PT}}$ and $\mathcal{L}_k^{\mathrm{PT}}$ on $\mathbb{D}_{\mathrm{PT}}^{X}$ in Sections 0.2 and 0.3:

$$\mathrm{L}_k^{\mathrm{PT}} = \mathrm{T}_k + \mathrm{R}_k\,, \quad \mathcal{L}_k^{\mathrm{PT}} = \mathrm{L}_k^{\mathrm{PT}} + (k+1)!\,\mathrm{L}_{-1}^{\mathrm{PT}}\mathsf{ch}_{k+1}(\mathsf{p})\,,$$

for $k \geqslant -1$. We also have

$$[\mathrm{L}_n^{\mathrm{PT}}, \mathrm{L}_k^{\mathrm{PT}}] \overset{\langle,\rangle}{=} (k-n)\mathrm{L}_{n+k}^{\mathrm{PT}}\,, \quad [\mathrm{L}_n^{\mathrm{PT}}, (k-1)!\,\mathsf{ch}_k(\mathsf{p})] \overset{\langle,\rangle}{=} (n+k)!\,\mathsf{ch}_{n+k}(\mathsf{p})\,. \tag{25}$$

Equations (25) are parallel to equations (20) in Gromov-Witten theory.

[21]See [4, Theorem 4] for the evaluation.

The main computation of the paper is the *intertwining property* which relates the Virasoro operators for the stable pairs and Gromov-Witten theories via the descendent correspondence. We separate the argument into two cases: $k \leqslant 0$ and $k \geqslant 1$. Proposition 11 covers the $k \leqslant 0$ case. The $k \geqslant 1$ case treated in Theorem 12 is harder.

Proposition 11 is proven in Section 2.3 except for steps at the end of the proof which will be completed in the proof of Theorem 12 in Sections 3-5. The argument is an intricate calculation based on a strategy of filtration.

Proposition 11. *For $k = -1, 0$ and $D \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}$, we have*

$$\mathfrak{C}^\bullet \circ \mathrm{L}_k^{\mathrm{PT}}(D) \;=\; (\imath u)^{-k}\, \widetilde{\mathrm{L}}_k^{\mathrm{GW}} \circ \mathfrak{C}^\bullet(D)$$

after the restrictions $\tau_{-2}(\mathsf{p}) = 1$ and $\tau_{-1}(\mathsf{p}) = 0$.

Theorem 12. *For all $k \geqslant 1$ and $D \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}$, we have*

$$\mathfrak{C}^\bullet \circ \mathrm{L}_k^{\mathrm{PT}}(D) = (\imath u)^{-k}\, \widetilde{\mathrm{L}}_k^{\mathrm{GW}} \circ \mathfrak{C}^\bullet(D)$$

after the restrictions $\tau_{-2}(\mathsf{p}) = 1$ and $\tau_{-1}(\gamma) = 0$ for $\gamma \in H^{>2}(X)$.

The evaluations of left sides of the equalities in Proposition 11 and Theorem 12 require a slight generalization of the formulas (13)-(15) which govern the descendent correspondence on $\mathbb{D}_{\mathrm{PT}}^{X\bigstar}$. Additional rules are required for

$$\widetilde{\mathsf{ch}}_0(\gamma), \widetilde{\mathsf{ch}}_1(\gamma) \text{ for } \gamma \in H^{>0}(X) \quad\text{and}\quad \widetilde{\mathsf{ch}}_2(\delta) \text{ for } \delta \in H^2(X). \tag{26}$$

The required rules take a very simple form since $\mathrm{L}_k^{\mathrm{PT}}(D)$ is at most linear[22] in the classes (26) over $\mathbb{D}_{\mathrm{GW}}^{X\bigstar}$:

$$\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_0(\gamma)) = -\int_X \gamma\,, \qquad \mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_0(\gamma)M) = 0\,, \tag{27}$$

$$\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_1(\gamma)) = 0\,, \qquad \mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_1(\gamma)M) = 0\,,$$

where $M \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}$. For $\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_2(\delta)M)$ with $M \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}$, formulas (13)-(15) apply unchanged. The above rules are compatible with the GW/PT descendent correspondence and will be established in Section 8.

The restrictions $\tau_{-2}(\mathsf{p}) = 1$ and $\tau_{-1}(\mathsf{p}) = 0$ in Proposition 11 are well-defined since both $\mathfrak{C}^\bullet \circ \mathrm{L}_k^{\mathrm{PT}}(D)$ and $\widetilde{\mathrm{L}}_k^{\mathrm{GW}} \circ \mathfrak{C}^\bullet(D)$, $k = 0, -1$ will be seen to lie in the commutative algebra generated by $\tau_{-2}(\mathsf{p})$, $\tau_{-1}(\gamma)$, and $\mathbb{D}_{\mathrm{GW}}^{X+}$. The commutation with $\tau_{-2}(p)$ and $\tau_{-1}(\mathsf{p})$ follows from (8).

Similarly, the restrictions $\tau_{-2}(\mathsf{p}) = 1$ and $\tau_{-1}(\gamma) = 0$ for $\gamma \in H^{>2}(X)$ in Theorem 12 are well-defined since both $\mathfrak{C}^\bullet \circ \mathrm{L}_k^{\mathrm{PT}}(D)$ and $\widetilde{\mathrm{L}}_k^{\mathrm{GW}} \circ \mathfrak{C}^\bullet(D)$, $k > 0$ will be seen to lie in the commutative algebra generated by $\tau_{-2}(\mathsf{p})$, $\tau_{-1}(\gamma)$, and $\mathbb{D}_{\mathrm{GW}}^{X\bigstar}$. The algebra $\mathbb{D}_{\mathrm{GW}}^{X\bigstar}$ is generated by the *essential descendents*

$$\left\{\, \tau_i(\gamma) \,|\, (i \geqslant 0, \gamma \in H^{>0}(X, \mathbb{Q})) \text{ or } (i = 0, \gamma \in H^{>2}(X, \mathbb{Q})) \,\right\}.$$

[22] $\mathrm{L}_1^{\mathrm{PT}}(D)$ has a single quadratic term in the classes (26) given by $\widetilde{\mathsf{ch}}_1(\mathsf{p})\widetilde{\mathsf{ch}}_2(c_1)$ which causes no difficulty since $\widetilde{\mathsf{ch}}_1(\mathsf{p})$ does not interact.

Again, commutation follows from (8).

2.2. **Conventions for $(-1)!\mathsf{ch}_1(c_1)$.** In order to complete the definitions of the left sides of Proposition 11 and Theorem 12, we must also include the term $(-1)!\mathsf{ch}_1(c_1)$ in the descendent correspondence $\mathfrak{C}^\bullet$ since such terms occur in $\mathrm{L}_k^{\mathrm{PT}}$.

- The first case is

$$\mathfrak{C}^\circ((-1)!\mathsf{ch}_1(c_1)) = 0\,.$$

- The non-vanishing bumping term is given by

$$\mathfrak{C}^\circ\Big((-1)!\mathsf{ch}_1(c_1)\widetilde{\mathsf{ch}}_{k_1+2}(\gamma)\Big) = -\frac{(\imath u)^{-1}}{k_1!}\Bigg(\mathfrak{a}_{k_1-1}(c_1\gamma) + (\imath u)^{-1}\mathfrak{a}_{k_1-2}(c_1\gamma\cdot c_1) + (\imath u)^{-1}k_1\sum_{|\mu|=k_1-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(c_1\gamma\cdot c_1)\Bigg)\,, \tag{28}$$

where $k_1 \geqslant 2$.

- The higher bumping term is

$$\mathfrak{C}^\circ((-1)!\mathsf{ch}_1(c_1)\widetilde{\mathsf{ch}}_{k_1+2}(\gamma)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma')) = \frac{(\imath u)^{-2}(k_1+k_2-1)}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-2}(c_1\gamma\gamma')\,,$$

$k_1, k_2 \geqslant 0, k_1+k_2 > 1$. There is also an exceptional higher bumping term

$$\mathfrak{C}^\circ((-1)!\mathsf{ch}_1(c_1)\widetilde{\mathsf{ch}}_2(\gamma)\widetilde{\mathsf{ch}}_3(\gamma')) = \tau_{-2}(c_1\gamma\gamma')\,.$$

2.3. **Proof of Proposition 11.** The cases $k=-1,0$ are special in two ways:

(i) We must use the exceptional cases of the operator $\mathfrak{C}^\circ$, in the analysis for $k=-1,0$.
(ii) While the operator $\widetilde{\mathrm{L}}_k^{\mathrm{GW}}$ for $k=-1,0$ has quadratic part $\frac{u^{-2}}{2}B^{k+1}$, $\widetilde{\mathrm{L}}_k^{\mathrm{GW}}$ is a first order operator acting on the stationary sector of descendent algebra for $k>0$.

For these reasons, we treat the $k=-1,0$ cases separately here.

The restrictions in the statement of Proposition 11 allow us freely use

$$\mathsf{ch}_0(\mathsf{p}) = -1\,, \tag{29}$$

which is compatible with $\mathfrak{C}^\bullet$. Similarly, we can use

$$\mathsf{ch}_1(\mathsf{p}) = 0\,. \tag{30}$$

Let us write down the corresponding operators explicitly:

$$\mathrm{L}_{-1}^{\mathrm{PT}} = \mathrm{R}_{-1} - (-1)!\,\mathsf{ch}_1(c_1), \quad \widetilde{\mathrm{L}}_{-1}^{\mathrm{GW}} = \mathrm{R}_{-1} + \frac{u^{-2}}{2}\mathrm{B}^0\,.$$

$$\mathrm{L}_0^{\mathrm{PT}} = \mathrm{R}_0 - \widetilde{\mathsf{ch}}_2(c_1) - \frac{1}{2}\mathsf{ch}_1\mathsf{ch}_1(c_1), \quad \widetilde{\mathrm{L}}_0^{\mathrm{GW}} = \mathrm{R}_0 + \frac{u^{-2}}{2}\mathrm{B}^1 - \tau_0(c_1) - \frac{1}{24}\int_X c_1c_2\,.$$

We have used (29) for $\mathrm{L}_{-1}^{\mathrm{PT}}$. For $\mathrm{L}_0^{\mathrm{PT}}$, only the $d_L = d_R = 2$ summand is nonzero by (30).

Step 1. We check the statement for $D=1$.

The left side of the equality of Proposition 11 for $k = -1$ is

$$\mathfrak{C}^\bullet(\mathrm{L}_{-1}^{\mathrm{PT}}(D)) = -\mathfrak{C}^\bullet((-1)!\mathsf{ch}_1(c_1)) = 0\,.$$

The right side of the equality,

$$\imath u\,\widetilde{\mathrm{L}}_{-1}^{\mathrm{GW}}(\mathfrak{C}^\bullet(1)) = \imath u\,\widetilde{\mathrm{L}}_{-1}^{\mathrm{GW}}(1) = 0\,,$$

matches. For $k = 0$, the left side for $D = 1$ is

$$\mathfrak{C}^\bullet(\mathrm{L}_0^{\mathrm{PT}}(1)) = -\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_2(c_1)) = -\mathfrak{a}_1(c_1) = -\tau_0(c_1) - \frac{1}{24}\int_X c_1c_2\,.$$

The right side,

$$\widetilde{\mathrm{L}}_0^{\mathrm{GW}}(\mathfrak{C}^\bullet(1)) = \widetilde{\mathrm{L}}_0^{\mathrm{GW}}(1) = -\tau_0(c_1) - \frac{1}{24}\int_X c_1c_2\,,$$

matches.

Step 2. We check the statement for $D = \widetilde{\mathsf{ch}}_{k+2}(\gamma)$ with $k \geqslant 0$.

We must expand both sides of the equality of Proposition 11 in terms of τ. The following formula will be used:

$$(31)\quad (\imath u)^k\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k+2}(\gamma)) = \tau_k(\gamma) + \left(\sum_{i=1}^{k}\frac{1}{i}\right)\tau_{k-1}(\gamma\cdot c_1) + \left(\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}\right)\tau_{k-2}(\gamma\cdot c_1^2)$$
$$+\sum_{|\mu|=k-1}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)k!}\left(\tau_{\mu_1-1}\tau_{\mu_2-1}(\gamma\cdot c_1)+(\sum_{i=1}^{\mu_1-1}\frac{1}{i})\tau_{\mu_1-2}(\gamma\cdot c_1^2)\tau_{\mu_2-1}(\mathsf{p})+(\sum_{i=1}^{\mu_2-1}\frac{1}{i})\tau_{\mu_1-1}(\mathsf{p})\tau_{\mu_2-2}(\gamma\cdot c_1^2)\right)$$
$$+\sum_{|\mu|=k-2}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)k!}\tau_{\mu_1-1}\tau_{\mu_2-1}(\gamma\cdot c_1^2) + \sum_{|\mu|=k-3}\frac{\mu_1!\mu_2!\mu_3!}{\mathrm{Aut}(\mu)(k-1)!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{\mu_3-1}(\gamma\cdot c_1^2)\,.$$

We split the analysis of the difference for

$$(32)\qquad \mathfrak{C}^\bullet\circ\mathrm{L}_{-1}^{\mathrm{PT}}(D) - \imath u\,\widetilde{\mathrm{L}}_{-1}^{\mathrm{GW}}\circ\mathfrak{C}^\bullet(D)$$

in stages according to the τ degree of terms. The second term of the difference is simpler since

$$\imath u\,\widetilde{\mathrm{L}}_{-1}^{\mathrm{GW}}\circ\mathfrak{C}^\bullet(D) = \imath u\,\mathrm{R}_{-1}(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_k(\gamma)))$$

and the latter is a easy modification of (31). The first term is more involved since there are two parts: the action of R_{-1} and the interaction with $(-1)!\mathsf{ch}_1(c_1)$.

• We first study the τ linear terms of $(\imath u)^{k-1}\mathfrak{C}^\bullet \circ \mathrm{L}_{-1}^{\mathrm{PT}}(D)$:

$$\Big(\tau_{k-1}(\gamma) + \Big(\sum_{i=1}^{k-1}\frac{1}{i}\Big)\tau_{k-2}(\gamma\cdot c_1) + \Big(\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{ij}\Big)\tau_{k-3}(\gamma\cdot c_1^2)\Big)$$
$$+\Big(\frac{(\imath u)^{k-2}}{k!}\mathfrak{a}_{k-1}(\gamma\cdot c_1) + \frac{(\imath u)^{k-3}}{k!}\mathfrak{a}_{k-2}(\gamma\cdot c_1^2)\Big) =$$
$$\Big(\tau_{k-1}(\gamma) + \Big(\sum_{i=1}^{k-1}\frac{1}{i}\Big)\tau_{k-2}(\gamma\cdot c_1) + \Big(\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{ij}\Big)\tau_{k-3}(\gamma\cdot c_1^2)\Big)$$
$$+\frac{1}{k}\Big(\tau_{k-2}(\gamma\cdot c_1) + \Big(\sum_{i=1}^{k-2}\frac{1}{i}\Big)\tau_{k-3}(\gamma\cdot c_1^2)\Big) + \frac{1}{k(k-1)}\tau_{k-3}(\gamma\cdot c_1^2)\,.$$

We have used here bumping with $(-1)!\mathsf{ch}_1(c_1)$ from (28) to obtain the expression in the second line and an inversion[23] of (12) to justify the second equality. After collecting together the coefficients in front of the τ's in the last expression, we obtain $R_{-1}(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_k(\gamma)))$, exactly as expected.

• We study next the τ-quadratic term of (32). Consider first the terms that have a coproduct $(\gamma\cdot c_1)_i^L\otimes(\gamma\cdot c_1)_i^R$ as argument. Bumping with $(-1)!\mathsf{ch}_1(c_1)$ does not produce such terms – only the terms of the second line of (31) contributes to the terms of (32). These terms cancel exactly.

• The τ-quadratic terms of difference (32) with argument $(\gamma\cdot c_1^2)_i^L\otimes(\gamma\cdot c_1^2)_i^R$ are slightly more involved. The second term of the difference has terms:

$$\sum_{|\mu|=k-2}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)(k-1)!}\Big(\Big(\sum_{i=1}^{\mu_1-1}\frac{1}{i}\Big)\tau_{\mu_1-2}(\gamma\cdot c_1^2)\tau_{\mu_2-1}(\mathsf{p}) + \Big(\sum_{i=1}^{\mu_2-1}\frac{1}{i}\Big)\tau_{\mu_1-1}(\mathsf{p})\tau_{\mu_2-2}(\gamma\cdot c_1^2)\Big)$$
$$+\sum_{|\mu|=k-3}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)(k-1)!}\tau_{\mu_1-1}\tau_{\mu_2-1}(\gamma\cdot c_1^2)\,,$$

where the term on the second line is a result of bumping with $(-1)!\mathsf{ch}_1(c_1)$. After simplifying the last expression, we obtain the corresponding τ-quadratic term of $R_{-1}(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k+2}(\gamma)))$ as expected.

• The last step is to analyze the τ-cubic terms of the difference (32). Since bumping with $\mathsf{ch}_1(c_1)$ is trivial, the terms match exactly.

Similarly, we must analyze the difference

$$\mathfrak{C}^\bullet\circ \mathrm{L}_0^{\mathrm{PT}}(D) - \widetilde{\mathrm{L}}_0^{\mathrm{GW}}\circ\mathfrak{C}^\bullet(D)\,. \tag{33}$$

[23]See (54) for the full formula for the inversion.

Since both R_0 on the stable pairs side and R_0^0 on the Gromov-Witten side scale the descendents by the complex cohomological degree, the difference (33) is equal[24] to

$$-\,\mathfrak{C}^\bullet\left((\widetilde{\mathsf{ch}}_2+\mathsf{ch}_1^2/2)(c_1)\cdot D\right)-\left(\mathrm{R}_0^1+\frac{u^{-2}}{2}\mathrm{B}^1-\tau_0(c_1)-\frac{1}{24}\int_X c_1c_2\right)\circ\mathfrak{C}^\bullet(D)\,. \tag{34}$$

If $D=\widetilde{\mathsf{ch}}_{k+2}(\gamma)$ then $\mathrm{B}^1\circ\mathfrak{C}^\bullet(D)=0$. We have already proved that the difference vanishes for $D=1$. Since

$$\mathfrak{C}^\bullet(\mathsf{ch}_1\mathsf{ch}_1(c_1)\widetilde{\mathsf{ch}}_{k+2}(\gamma))=0\,,$$

the difference (34) is equal to

$$-\,\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_2(c_1)\widetilde{\mathsf{ch}}_{k+2}(\gamma))-\mathrm{R}_0^1(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k+2}(\gamma)))\,. \tag{35}$$

Comparing formulas (13) and (14), we conclude that the latter difference vanishes.

Indeed, let us expand both terms of (35). First,

$$\begin{aligned}\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_2(c_1)\widetilde{\mathsf{ch}}_{k+2}(\gamma))&=-\frac{(\imath u)^{-1}}{k!}\mathfrak{a}_k(\gamma\cdot c_1)-\frac{(\imath u)^{-2}}{k!}\mathfrak{a}_{k-1}(\gamma\cdot c_1^2)-\frac{(\imath u)^{-2}}{(k-1)!}\sum_{|\mu|=k-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\cdot c_1^2)\\&=-(\imath u)^{-k}\left(\tau_{k-1}(\gamma\cdot c_1)+\left(\sum_{i=1}^{k-1}\frac{1}{i}\right)\tau_{k-2}(\gamma\cdot c_1^2)\right)-\frac{(\imath u)^{-k}}{k}\tau_{k-2}(\gamma\cdot c_1^2)\\&\qquad-\frac{(\imath u)^{-k+2}}{(k-1)!}\sum_{|\mu|=k-2}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2-1}(\gamma\cdot c_1^2)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k+2}(\gamma))&=\frac{1}{(k+1)!}\mathfrak{a}_{k+1}(\gamma)+\frac{(\imath u)^{-1}}{k!}\sum_{|\mu|=k-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\cdot c_1)+\ldots\\=(\imath u)^{-k}&\left(\tau_k(\gamma)+\left(\sum_{i=1}^{k}\frac{1}{i}\right)\tau_{k-1}(\gamma\cdot c_1)\right)+\frac{(\imath u)^{-k+2}}{k!}\sum_{|\mu|=k-1}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2-1}(\gamma\cdot c_1)+\ldots\,,\end{aligned}$$

where we have used dots to stand for the terms that are of complex cohomological degree 3. Since

$$\mathrm{R}_0^1(\tau_k(\gamma))=\tau_{k-1}(\gamma\cdot c_1)\,,$$

all the omitted terms are annihilated by R_0^1. The remaining terms of the difference (35) cancel.

Step 3. We check the statement for $D=\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2)$ with $k_i\geqslant 0$.

[24]Note both R_0^2 and R_0^3 are 0.

We start with the difference (32):

$$\mathfrak{C}^\circ(\mathrm{R}_{-1}(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2))) - \mathfrak{C}^\circ((-1)!\widetilde{\mathsf{ch}}_1(c_1)\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2)) \\ - (\imath u)\mathrm{R}_{-1}(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2))) - (\imath u)\frac{u^{-2}}{2}\mathrm{B}^0(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)), \mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2)))) \,. \tag{36}$$

Vanishing of the last expression follows from Proposition 17 and Proposition 18.

The difference (33) as above is equivalent to (34). Since we have already shown the vanishing for $D = 1$ and $D = \widetilde{\mathsf{ch}}_{k+2}(\gamma)$, we need only to check the vanishing of

$$-\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_2(c_1)D) - \frac{1}{2}\mathfrak{C}^\bullet(\mathsf{ch}_1\mathsf{ch}_1(c_1)D) - R_0^1(\mathfrak{C}^\circ(D)) \\ - \frac{u^{-2}}{2}\mathrm{B}^1(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)), \mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2)))\,. \tag{37}$$

The vanishing follows from Propositions 17 and 18.

Step 4. We check the statement for $D = \widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2)\widetilde{\mathsf{ch}}_{k_3+2}(\gamma_3)$ with $k_i \geqslant 0$.

The result follows immediately from the triple bumping relation (94) which holds in complete generality. No special cases require extra attention. ☐

2.4. **Proof of Theorem 4.** The vanishings

$$\langle \mathcal{L}_{-1}^{\mathrm{PT}}(D)\rangle_\beta^{X,\mathrm{PT}} = 0 \quad \text{and} \quad \langle \mathcal{L}_0^{\mathrm{PT}}(D)\rangle_\beta^{X,\mathrm{PT}} = 0 \tag{38}$$

are simple to prove for all $D \in \mathbb{D}_{\mathrm{PT}}^X$. For

$$\mathcal{L}_{-1}^{\mathrm{PT}} = \mathrm{R}_{-1} + \mathrm{R}_{-1}\mathsf{ch}_0(\mathsf{p})\,,$$

the vanishing (38) is immediate from the definition of R_{-1} and (1). For

$$\mathcal{L}_0 = \mathrm{R}_0 - \widetilde{\mathsf{ch}}_2(c_1) - \frac{1}{2}\mathsf{ch}_1\mathsf{ch}_1(c_1) + \mathrm{R}_{-1}\mathsf{ch}_1(\mathsf{p})$$

the vanishing (38) follows from the definition of R_0, the virtual dimension constraints, and the divisor equation:

$$\Big\langle \mathsf{ch}_2(c_1)\,\mathsf{ch}_{k_1}(\gamma_1)\cdots\mathsf{ch}_{k_m}(\gamma_m)\Big\rangle_\beta^{X,\mathrm{PT}} = \int_\beta c_1 \cdot \Big\langle \mathsf{ch}_{k_1}(\gamma_1)\cdots\mathsf{ch}_{k_m}(\gamma_m)\Big\rangle_\beta^{X,\mathrm{PT}}\,.$$

We now assume $k \geqslant 1$. Using the intertwining property of Theorem 12, the stationary GW/PT correspondence of Theorem 6, and the Virasoro constraints in Gromov-Witten theory, we can prove the stationary Virasoro constraints for stable pairs in the toric case.

Let $D \in \mathbb{D}_{\mathrm{PT}}^{X+}$, so D is a monomial in the operators

$$\{\, \widetilde{\mathsf{ch}}_i(\gamma)\,|\, i \geqslant 0,\ \gamma \in H^{>0}(X,\mathbb{Q})\,\}.$$

The first step is to check by hand that the Virasoro constraints

$$\Big\langle \mathcal{L}_k^{\mathrm{PT}}(D)\Big\rangle_\beta^{X,\mathrm{PT}} = 0 \tag{39}$$

of Theorem 4 are compatible all with insertions of the form

$$\widetilde{\mathsf{ch}}_0(\gamma), \widetilde{\mathsf{ch}}_1(\gamma) \text{ for } \gamma \in H^{>0}(X) \text{ and } \widetilde{\mathsf{ch}}_2(\delta) \text{ for } \delta \in H^2(X). \tag{40}$$

If any of the operators (40) appear in D, the Virasoro constraints (39) are true if the Virasoro constraints are true for the monomial obtained by dividing D by the occurring operators (40). We can therefore reduce to the case where D is a monomial in the operators

$$\left\{ \widetilde{\mathsf{ch}}_i(\gamma) \,|\, (i \geqslant 3, \gamma \in H^{>0}(X,\mathbb{Q})) \textit{ or } (i = 2, \gamma \in H^{>2}(X,\mathbb{Q})) \right\}.$$

In other words, $D \in \mathbb{D}_{\mathrm{PT}}^{X\star}$.

The next step is to apply Theorem 6:

$$(-q)^{d_\beta/2} \langle \mathcal{L}_k^{\mathrm{PT}}(D) \rangle_\beta^{X,\mathrm{PT}} = (-\imath u)^{d_\beta} \langle \mathfrak{C}^\bullet(\mathcal{L}_k^{\mathrm{PT}}(D)) \rangle_\beta^{X,\mathrm{GW}} \tag{41}$$

for all $k \geqslant 1$. By the construction of the correspondence [25], the descendents of the point class do not interact with other descendents:

$$\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_{k+2}(\mathsf{p})D) = (\imath u)^{-k} \tau_k(\mathsf{p}) \mathfrak{C}^\bullet(D)\,, \tag{42}$$

for every $D \in \mathbb{D}_{\mathrm{PT}}^{X\star}$.

By combining (41), (42), and the intertwining statement of Theorem 12, we see

$$\begin{aligned}
\langle \mathfrak{C}^\bullet(\mathcal{L}_k^{\mathrm{PT}}(D)) \rangle_\beta^{\mathrm{GW}} &= \langle \mathfrak{C}^\bullet(\mathrm{L}_k^{\mathrm{PT}}(D)) \rangle_\beta^{\mathrm{GW}} + (k+1)!\, \langle \mathfrak{C}^\bullet(\mathrm{L}_{-1}^{\mathrm{PT}}(\mathsf{ch}_{k+1}(\mathsf{p})D)) \rangle_\beta^{\mathrm{GW}} \\
&= (\imath u)^{-k} \langle \widetilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(D)) \rangle_\beta^{\mathrm{GW}} + (\imath u)^{2-k}(k+1)!\, \langle \widetilde{\mathrm{L}}_{-1}^{\mathrm{GW}}(\tau_{k-1}(\mathsf{p})\mathfrak{C}^\bullet(D)) \rangle_\beta^{\mathrm{GW}} \\
&= (\imath u)^{-k} \langle \mathcal{L}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(D)) \rangle_\beta^{\mathrm{GW}} \\
&= 0\,,
\end{aligned}$$

where the last equality is by Proposition 10 which may be applied since

$$\mathfrak{C}^\bullet(D) \in \mathbb{D}_{\mathrm{GW}}^{X+}$$

by Proposition 7. We conclude

$$\langle \mathcal{L}_k^{\mathrm{PT}}(D) \rangle_\beta^{X,\mathrm{PT}} = 0$$

as required. □

We could have also used the intertwining property of Proposition 11 to prove the stable pairs vanishings (38) for $D \in \mathbb{D}_{\mathrm{PT}}^{X+}$, but some additional care must be taken since the insertions $\mathsf{ch}_0(\mathsf{p})$ and $\mathsf{ch}_1(\mathsf{p})$ which occur in the terms

$$(k+1)!\, \langle \mathfrak{C}^\bullet(\mathrm{L}_{-1}^{\mathrm{PT}}(\mathsf{ch}_{k+1}(\mathsf{p})D)) \rangle_\beta^{\mathrm{GW}}$$

for $k = -1$ and 0 are not covered by Proposition 11. We leave the details to the reader.

3. Intertwining I: Basic case

3.1. **Overview.** After an explicit study of various terms of the stationary Gromov-Witten Virasoro constraints in Section 3.2, we prove Theorem 12 in the basic case $D = 1$ in Section 3.3.

3.2. **Leading term.** We analyze here the stationary Virasoro constraints on the Gromov-Witten side defined in Section 1.4.

The leading term T^1_k of T'_k is of the form

$$\frac{1}{2}\mathrm{T}^1_k = \frac{k!}{u^2}\tau_k(c_1) + \frac{1}{2}\sum_{a+b=k-2}(-1)^{d^L-1}(a+d^L-1)!(b+d^R-1)!\tau_a\tau_b(c_1)\,,$$

where $a, b \geqslant 0$ in the sum. By the following result, the term T'_k simplifies if we use the modified descendents $\mathfrak{a}_i$.

Proposition 13. *For all $k \geqslant -1$,*

$$\mathrm{T}'_k = -(\imath u)^{k-2}\sum_{a+b=k+2}(-1)^{d^L d^R}(a+d^L-3)!(b+d^R-3)!\frac{\mathfrak{a}_{a-1}\mathfrak{a}_{b-1}(c_1)}{(a-1)!(b-1)!}\,,$$

where the sum over all $a, b \geqslant 0$ and we use convention $\mathfrak{a}_0 = 0$, $\mathfrak{a}_{-1}/(-1)! = \tau_{-2}$.

Proof. Using formula (12), we expand T'_k in terms of $\mathfrak{a}_i$ to show that the quadratic and cubic in c_1 terms cancel. In the computation, we compare the expressions

$$[-a]^k_2 = (-1)^a a!(k-a)!\left(\sum_{i=1}^{k-a}\frac{1}{i} - \sum_{i=1}^{a}\frac{1}{i}\right), \quad a \geqslant 0\,,\ k \geqslant a\,,$$

$$[-a]^k_3 = (-1)^a a!(k-a)!\left(\sum_{1\leqslant i<j\leqslant k-a}\frac{1}{ij} + \sum_{1\leqslant i<j\leqslant a}\frac{1}{ij} - \left(\sum_{i=1}^{k-a}\frac{1}{i}\right)\left(\sum_{i=1}^{a}\frac{1}{i}\right)\right), \quad a \geqslant 0, k \geqslant a$$

with the coefficients in (12).

The transformation (12) simplifies if we use the following operators and short-hand notations for the sums:

$$\tilde{\mathfrak{a}}_k = \frac{(\imath u)^{k-1}}{k!}\mathfrak{a}_k\,, \quad \chi^k_l = \sum_{j=1}^{k}\frac{1}{j^l}\,, \quad \chi^k_{1,1} = \sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}\,.$$

In the formulas below, all operators $\tilde{\mathfrak{a}}_0$ are set to be zero. We apply transformation to T^1_k to obtain:

$$\sum_{m=-1}^{k+1}(-1)^{d_L-1}(m+d_L-2)!(k-m-d_L+2)!\times \Bigg(\tilde{\mathfrak{a}}_m\tilde{\mathfrak{a}}_{-m+k}(c_1) - \left(\chi^m_1 - \chi^{k-m-1}_1\right)\tilde{\mathfrak{a}}_m\tilde{\mathfrak{a}}_{-m+k-1}(c_1^2) + \left(\chi^m_1\chi^{k-m-2}_1 + \chi^m_2 + \chi^m_{1,1} + \chi^{-m+k-2}_2 + \chi^{-m+k-2}_{1,1}\right)\tilde{\mathfrak{a}}_m\tilde{\mathfrak{a}}_{-m+k-2}(c_1^3)\Bigg)\,. \tag{43}$$

To write the transformation of T_k^2, we split the sum for T_k^2 into two subsums, the first with $d_L = 2$ and the second with $d_L = 3$:

$$\sum_{m=-1}^{k}(-1)(m)!(k-m)!(\chi_1^{k-m}-\chi_1^m)\Big(\tilde{\mathfrak{a}}_m(c_1^2)\tilde{\mathfrak{a}}_{-m+k-1}(\mathsf{p})-\chi_1^{m-1}\tilde{\mathfrak{a}}_{m-1}\tilde{\mathfrak{a}}_{-m-k-1}(c_1^3)\Big)+$$
$$(m+1)!(k-m-1)!(\chi_1^{k-m-1}-\chi_1^{m-1})\Big(\tilde{\mathfrak{a}}_m(\mathsf{p})\tilde{\mathfrak{a}}_{-m+k-1}(c_1^2)-\chi_1^{k-m-2}\tilde{\mathfrak{a}}_m\tilde{\mathfrak{a}}_{-m-k-2}(c_1^2)\Big).$$

Finally, the transformation of T_k^3 to $\mathfrak{a}$ variables is

$$\sum_{m=-1}^{k}(m+1)!(k-m-1)!\Big(\chi_{1,1}^{m-1}+\chi_{1,1}^{k-m-1}-\chi_1^{m-1}\chi_1^{k-m-1}\Big)\tilde{\mathfrak{a}}_m\tilde{\mathfrak{a}}_{-m+k-2}(c_1^3).$$

After summing the terms T_k^j for $j = 1, 2, 3$, we find that only the first term in (43) does not cancel. □

3.3. **Intertwining for $D = 1$.** For the most of computations in Section 3, we will require the simplest case of the stationary GW/PT transformation $\mathfrak{C}^\bullet$ of Section 0.6,

$$\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k+2}(\gamma)) = \frac{1}{(k+1)!}\mathfrak{a}_{k+1}(\gamma)+\frac{(\imath u)^{-1}}{k!}\sum_{|\mu|=k-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1)}{\mathrm{Aut}(\mu)}+$$
$$\frac{(\imath u)^{-2}}{k!}\sum_{|\mu|=k-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)}+\frac{(\imath u)^{-2}}{(k-1)!}\sum_{|\mu|=k-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)}\,. \tag{44}$$

Our first result is the simplest case of Theorem 12.

Proposition 14. *For all $k \geqslant 1$, we have*

$$\mathfrak{C}^\bullet(\mathrm{L}_k^{\mathrm{PT}}(1)) = (\imath u)^{-k}\,\widetilde{\mathrm{L}}_k^{\mathrm{GW}}(1)\,.$$

Proof. Since the operators R_k annihilate 1, we must prove

$$\mathfrak{C}^\bullet(\mathrm{T}_k) = (\imath u)^{-k}\left(\frac{(\imath u)^2}{2}\mathrm{T}_k'\right)\,. \tag{45}$$

From Section 0.2, we have the following formula on the stable pairs side:

$$\mathrm{T}_k = -\frac{1}{2}\sum_{a+b=k+2}(-1)^{d^Ld^R}(a+d^L-3)!(b+d^R-3)!\,\widetilde{\mathsf{ch}}_a\widetilde{\mathsf{ch}}_b(c_1)\,.$$

On the Gromov-Witten side, we have

$$\mathrm{T}_k' = -(\imath u)^{k-2}\sum_{a+b=k+2}(-1)^{d^Ld^R}(a+d^L-3)!(b+d^R-3)!\frac{\mathfrak{a}_{a-1}\mathfrak{a}_{b-1}(c_1)}{(a-1)!(b-1)!}$$

by Proposition 13. Using (44), the quadratic term in the $\mathfrak{a}$-insertions of $\mathfrak{C}^\bullet(\mathrm{T}_k)$ exactly matches the full right side of (45). We will prove the other terms of $\mathfrak{C}^\bullet(\mathrm{T}_k)$ all vanish.

The stable pairs term T_k is the sum of three subsums:

$$\frac{1}{2}\sum_{a+b=k+2}\Bigg((a-2)!b!\,\widetilde{\mathsf{ch}}_a(c_1)\widetilde{\mathsf{ch}}_b(\mathsf{p})+a!(b-2)!\,\widetilde{\mathsf{ch}}_a(\mathsf{p})\widetilde{\mathsf{ch}}_b(c_1) - (a-1)!(b-1)!\sum_{s+1\leqslant\bullet,\star\leqslant 2s}\alpha_{\bullet\star}\,\widetilde{\mathsf{ch}}_a(\gamma_\bullet)\widetilde{\mathsf{ch}}_b(\gamma_\star)\Bigg)\,, \tag{46}$$

where last term uses[25]

$$c_1\cdot\gamma_{2s+1-\bullet}=\sum_\star \alpha_{\bullet\star}\gamma_\star\,.$$

After applying $\mathfrak{C}^\bullet$ to (46) we obtain quadratic, cubic, and quartic monomials in $\mathfrak{a}$. We will show the cubic and quartic terms vanish.

We start with the analysis of the quartic term of $\mathfrak{C}^\bullet(\mathrm{T}_k)$. The first term (46) yields the quartic part:

$$\frac{1}{2}\int_X c_1^3\cdot\sum_{a+b=k+2}\Bigg((a-2)!b!\cdot\frac{\mathfrak{a}_{b-1}(\mathsf{p})}{(b-1)!}\cdot\frac{(\imath u)^{-2}}{(a-3)!}\sum_{|\mu|=a-5}\frac{\mathfrak{a}_{\mu_1}(\mathsf{p})\mathfrak{a}_{\mu_2}(\mathsf{p})\mathfrak{a}_{\mu_3}(\mathsf{p})}{\mathrm{Aut}(\mu)} +(b-2)!a!\cdot\frac{\mathfrak{a}_{a-1}(\mathsf{p})}{(a-1)!}\cdot\frac{(\imath u)^{-2}}{(b-3)!}\sum_{|\mu|=b-5}\frac{\mathfrak{a}_{\mu_1}(\mathsf{p})\mathfrak{a}_{\mu_2}(\mathsf{p})\mathfrak{a}_{\mu_3}(\mathsf{p})}{\mathrm{Aut}(\mu)}\Bigg)\,.$$

The last term of (46) yields the following quartic part (with the sum over the same range of a and b as above):

$$-\frac{1}{2}\int_X c_1^3\,\cdot\,(a-1)!(b-1)!\cdot\frac{(\imath u)^{-1}}{(a-2)!}\sum_{|\mu'|=a-3}\frac{\mathfrak{a}_{\mu_1'}(\mathsf{p})\mathfrak{a}_{\mu_2'}(\mathsf{p})}{\mathrm{Aut}(\mu')}\cdot\frac{(\imath u)^{-1}}{(b-2)!}\sum_{|\mu''|=b-3}\frac{\mathfrak{a}_{\mu_1''}(\mathsf{p})\mathfrak{a}_{\mu_2''}(\mathsf{p})}{\mathrm{Aut}(\mu'')}\,,$$

where, in both formulas, we have used convention $|\mu|=\sum_i\mu_i$.

These two quartic parts cancel each other. Indeed, let us analyze the factor in front of

$$\frac{1}{2(\imath u)^2}\int_X c_1^3\cdot\mathfrak{a}_{\lambda_1}(\mathsf{p})\mathfrak{a}_{\lambda_2}(\mathsf{p})\mathfrak{a}_{\lambda_3}(\mathsf{p})\mathfrak{a}_{\lambda_4}(\mathsf{p})$$

in both expressions. For simplicity, let us assume $|\mathrm{Aut}(\lambda)|=1$. Then, the factor in the first quartic part is a sum with four terms:

$$\sum_{i=1}^4(\lambda_i+1)\left(\sum_{j\neq i}(\lambda_j+1)\right)\,. \tag{47}$$

The factor in the second formula is a sum with three terms:

$$-\sum(\lambda_{i_1}+\lambda_{i_2}+2)(\lambda_{j_1}+\lambda_{j_2}+2)\,, \tag{48}$$

where the sum is over all splittings

$$\{1,2,3,4\}=\{i_1,i_2\}\cup\{j_1,j_2\}\,.$$

[25] We use the subscripts $\bullet$ and $\star$ in order to avoid i,j,a,b which are already taken.

The factors (47) and (48) are sums of twelve monomials of $\lambda_i + 1$ and are opposites of each other. The case when $|\operatorname{Aut}(\lambda)| > 1$ is analogous.

Finally, we analyze the cubic terms. Let us first analyze the cubic terms of the form $\mathfrak{a}_i(\mathsf{p})\mathfrak{a}_j(\mathsf{p})\mathfrak{a}_l(\mathsf{p})$. Since

$$\mathsf{ch}_{k+2}(c_1)\mathsf{ch}_0(\mathsf{p}) = (-1)\mathsf{ch}_{k+2}(c_1)\,,$$

the cubic part of the first term of (46) with $b = 0$ is:

$$-k\int_X \frac{c_1^3}{2(\imath u)^2}\sum_{|\mu|=k-1}\frac{\mathfrak{a}_{\mu_1}(\mathsf{p})\mathfrak{a}_{\mu_2}(\mathsf{p})\mathfrak{a}_{\mu_3}(\mathsf{p})}{\operatorname{Aut}(\mu)}\,. \tag{49}$$

A similar cubic part is produced by the second term of (46) with $a = 0$.

The other cubic parts of the first term of (46) are:

$$\int_X c_1^3 \sum_{a+b=k+2}\frac{b}{2(\imath u)^2}\mathfrak{a}_{b-1}(\mathsf{p})\sum_{|\mu|=a-4}\frac{\mathfrak{a}_{\mu_1}(\mathsf{p})\mathfrak{a}_{\mu_2}(\mathsf{p})}{\operatorname{Aut}(\mu)} + \frac{b}{2\imath u}\mathfrak{a}_{b-1}(\mathsf{p})\sum_{|\mu|=a-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(c_1^2)}{\operatorname{Aut}(\mu)}\,. \tag{50}$$

Similar term is yielded by the second term of (46).

If $\operatorname{Aut}(\mu) = 1$, then the factor in front of monomial

$$\frac{1}{2(\imath u)^2}\mathfrak{a}_{\lambda_1}(\mathsf{p})\mathfrak{a}_{\lambda_2}(\mathsf{p})\mathfrak{a}_{\lambda_3}(\mathsf{p})$$

of (50) is the sum of three terms

$$(\lambda_1 + 1) + (\lambda_2 + 1) + (\lambda_3 + 1)$$

and, hence, cancels with corresponding monomial from (49).

The cubic part of the last term of (46) is

$$-\frac{(a-1)}{2\imath u}\sum_{\bullet,\star}\alpha_{\bullet\star}\,\mathfrak{a}_{b-1}(\gamma_\star)\sum_{|\mu|=a-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(c_1\cdot\gamma_\bullet)}{\operatorname{Aut}(\mu)}$$
$$-\frac{(b-1)}{2\imath u}\sum_{\bullet,\star}\alpha_{\bullet\star}\,\mathfrak{a}_{a-1}(\gamma_\star)\sum_{|\mu|=b-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(c_1\cdot\gamma_\bullet)}{\operatorname{Aut}(\mu)}\,,$$

over all $a, b \geqslant 0$ satisfying $a + b = k + 2$. The sum cancels with the last term of (50). ☐

4. Intertwining II: Non-interacting insertions

4.1. Overview. The main result of Section 4 is a proof of Theorem 12 for

$$D \in \mathbb{D}^1_{\mathrm{PT}} \cap \mathbb{D}^{X\bigstar}_{\mathrm{PT}\circ}\,, \tag{51}$$

where D is a product of $\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)$ satisfying

$$\gamma_i \cdot \gamma_j = 0 \quad \text{for} \quad i \neq j\,.$$

We treat the singleton $D = \widetilde{\mathsf{ch}}_k(\mathsf{p})$ in Proposition 15. An intricate computation is required for Proposition 16 which settles the cases $D = \widetilde{\mathsf{ch}}_k(\gamma)$ where

$$\gamma \in H^i(X) \quad \text{for } i = 2 \text{ and } 4.$$

Finally, in Section 4.3, the general case (51) is formally deduced from the singletons.

4.2. **Intertwining shift operators.** We first relate the operators $\mathrm{R_k}$ appearing in the Virasoro constraints on the stable pairs and Gromov-Witten sides. Recall,

$$\widetilde{\mathsf{ch}}_k(\alpha) = \mathsf{ch}_k(\alpha) + \frac{1}{24}\mathsf{ch}_{k-2}(\alpha \cdot c_2)\,, \tag{52}$$

so $\widetilde{\mathsf{ch}}_k(\mathsf{p}) = \mathsf{ch}_k(\mathsf{p})$.

Proposition 15. *For all $k \geqslant 1$ and all $i \geqslant 2$, we have*

$$\mathfrak{C}^\bullet(\mathrm{R}_k(\mathsf{ch}_i(\mathsf{p}))) = (\imath u)^{-k}\,\mathrm{R}_k(\mathfrak{C}^\bullet(\mathsf{ch}_i(\mathsf{p})))\,.$$

Proof. The left side of the equation is

$$\mathfrak{C}^\bullet(\mathrm{R}_k(\mathsf{ch}_i(\mathsf{p}))) = \mathfrak{C}^\bullet\left(\frac{(i+k)!}{(i-1)!}\mathsf{ch}_{i+k}(\mathsf{p})\right) = \frac{(i+k)!}{(i-1)!}\frac{\mathfrak{a}_{i+k-1}(\mathsf{p})}{(i+k-1)!} = \frac{(i+k)}{(i-1)!}\,\mathfrak{a}_{i+k-1}(\mathsf{p})\,,$$

where we have used the definition of R_k for stable pairs and equation (13) for the correspondence.

The right side of the equation is

$$\begin{aligned}
\mathrm{R}_k(\mathfrak{C}^\bullet(\mathsf{ch}_i(\mathsf{p}))) &= \mathrm{R}_k\left(\frac{\mathfrak{a}_{i-1}(\mathsf{p})}{(i-1)!}\right) \\
&= \mathrm{R}_k\left(\frac{\tau_{i-2}(\mathsf{p})}{(\imath u)^{i-2}}\right) \\
&= \frac{(i+k)!}{(i-1)!}\frac{\tau_{i+k-2}(\mathsf{p})}{(\imath u)^{i-2}} \\
&= \frac{(i+k)}{(i-1)!}(\imath u)^k\,\mathfrak{a}_{i+k-1}(\mathsf{p})\,,
\end{aligned}$$

where we have used (13) for the correspondence, equation (12), and the definition of R_k for Gromov-Witten theory. The two sides match. □

Proposition 16. *For all $k \geqslant 1$, $\widetilde{\mathsf{ch}}_i(\gamma) \in \mathbb{D}_{\mathrm{PT}}^{\star X}$, $\gamma \in H^{\geqslant 2}(X)$ we have*

$$\mathfrak{C}^\bullet(\mathrm{L}_k^{\mathrm{PT}}(\widetilde{\mathsf{ch}}_i(\gamma))) = (\imath u)^{-k}\,\widetilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_i(\gamma)))\,.$$

Proof. We start with the easiest case and proceed to the hardest case.

Case $\gamma \in H^6(X)$. The case $\gamma = \mathsf{p}$ follows immediately from the previous results:

$$\begin{aligned}
\mathfrak{C}^\bullet(\mathrm{L}_k^{\mathrm{PT}}(\mathsf{ch}_i(\mathsf{p}))) &= \mathfrak{C}^\bullet(\mathrm{T}_k\,\mathsf{ch}_i(\mathsf{p}) + \mathrm{R}_k(\mathsf{ch}_i(\mathsf{p}))) \\
&= \mathfrak{C}^\bullet(\mathrm{T}_k)\mathfrak{C}^\bullet(\mathsf{ch}_i(\mathsf{p})) + (\imath u)^{-k}\mathrm{R}_k(\mathfrak{C}^\bullet(\mathsf{ch}_i(\mathsf{p})) \\
&= (\imath u)^{-k}\tilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(\mathsf{ch}_i(\mathsf{p})))\,.
\end{aligned}$$

The second equality follows from Proposition 15 and (42). The third equality uses (45).

Case $\gamma \in H^4(X)$. We compute the difference

$$(\imath u)^k \mathfrak{C}^\bullet(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma))) - \mathrm{R}_k(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_i(\gamma)))\,. \tag{53}$$

Since $\gamma \cdot c_2 = 0$, we have $\widetilde{\mathsf{ch}}_k(\gamma) = \mathsf{ch}_k(\gamma)$ by (52).

We start by expanding the first term of the difference:

$$\begin{aligned}\mathfrak{C}^\circ(\mathrm{R}_k(\mathsf{ch}_i(\gamma))) &= \mathfrak{C}^\circ\left(\frac{(i+k-1)!}{(i-2)!}\mathsf{ch}_{k+i}(\gamma)\right)\\ &= \frac{(i+k-1)!}{(i-2)!}\left(\frac{\mathfrak{a}_{i+k-1}(\gamma)}{(i+k-1)!} + \frac{(\imath u)^{-1}}{(i+k-2)!}\sum_{\mu_1+\mu_2=i+k-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{2}(\gamma\cdot c_1)\right).\end{aligned}$$

To proceed, we invert the correspondence (12):

$$\frac{(\imath u)^k \mathfrak{a}_{k+1}}{(k+1)!}(\gamma) = \tau_k(\gamma) + \left(\sum_{i=1}^{k}\frac{1}{i}\right)\tau_{k-1}(\gamma\cdot c_1) + \left(\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}\right)\tau_{k-2}(\gamma\cdot c_1^2)\,. \tag{54}$$

We then obtain

$$\begin{aligned}(\imath u)^k\mathfrak{C}^\circ(\mathrm{R}_k(\mathsf{ch}_i(\gamma))) = \frac{(i+k-1)!}{(i-2)!}&\left(\frac{\tau_{i+k-2}(\gamma)}{(\imath u)^{i-2}} + \left(\sum_{j=1}^{i+k-2}\frac{1}{j}\right)\frac{\tau_{i+k-3}(\gamma\cdot c_1)}{(\imath u)^{i-2}}\right.\\ &\left.+\frac{(\imath u)^{-i+4}}{(i+k-2)!}\sum_{\mu_1+\mu_2=i+k-3}\mu_1!\mu_2!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{2}(\gamma\cdot c_1)\right).\end{aligned} \tag{55}$$

We write the second term of the difference as

$$\mathrm{R}_k(\mathfrak{C}^\circ(\mathsf{ch}_i(\gamma))) = \mathrm{R}_k\left(\frac{\mathfrak{a}_{i-1}(\gamma)}{(i-1)!} + \frac{(\imath u)^{-1}}{(i-2)!}\sum_{\mu_1+\mu_2=i-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{2}(\gamma\cdot c_1)\right). \tag{56}$$

After applying the inversion (54), we have

$$\mathrm{R}_k\left(\frac{\tau_{i-2}(\gamma)}{(\imath u)^{i-2}} + \left(\sum_{j=1}^{i-2}\frac{1}{j}\right)\frac{\tau_{i-3}(\gamma\cdot c_1)}{(\imath u)^{i-2}} + \frac{(\imath u)^{4-i}}{(i-2)!}\sum_{\mu_1+\mu_2=i-3}\mu_1!\mu_2!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{2}(\gamma\cdot c_1)\right).$$

We expand the above expression fully to obtain

$$\begin{aligned}&\frac{(i+k-1)!\tau_{i+k-2}(\gamma)}{(\imath u)^{i-2}(i-2)!}\\ &+\frac{(i+k-1)!}{(\imath u)^{i-2}(i-2)!}\left(\sum_{j=i-1}^{k+i-1}\frac{1}{j}\right)\tau_{i+k-3}(\gamma\cdot c_1) + \frac{(i+k-1)!}{(\imath u)^{i-2}(i-2)!}\left(\sum_{j=1}^{i-2}\frac{1}{j}\right)\tau_{i-k+3}(\gamma\cdot c_1)\\ &+\frac{(\imath u)^{-i+4}}{(i-2)!}\sum_{\mu_1+\mu_2=i-3}\Big((\mu_1+k+1)!\mu_2!\frac{\tau_{\mu_1+k-1}\tau_{\mu_2-1}}{2}(\gamma\cdot c_1) + \mu_1!(\mu_2+k+1)!\frac{\tau_{\mu_1-1}\tau_{\mu_2+k-1}}{2}(\gamma\cdot c_1)\Big),\end{aligned} \tag{57}$$

where we have used formula

$$[i]_1^k = \frac{(i+k)!}{(i-1)!}\sum_{j=i}^{i+k}\frac{1}{j}$$

in the expansion of $\mathrm{R}_k(\tau_{i-2}(\gamma))$.

To complete our computation of the difference (53), we observe several cancellations. The first term of (55) cancels with first term of (57). The second term of (55) almost cancels with the sum of the second and third terms of (57), the only terms that does not cancel is

$$-\frac{(i+k-2)!}{(\imath u)^{i-2}(i-2)!}\tau_{i+k-3}(\gamma\cdot c_1) \tag{58}$$

Finally, we rewrite the last term of (55) as

$$\frac{(\imath u)^{-i+4}}{(i-2)!}\sum_{\mu_1+\mu_2=i+k-3}(\mu_1+1)!\mu_2!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{2}(\gamma\cdot c_1)+\mu_1!(\mu_2+1)!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{2}(\gamma\cdot c_1)\,.$$

Then, we see that the last term of (56) cancels with the last term of (55) if $\mu_1 \geqslant k+1$ and $\mu_2 \geqslant k+1$. Thus the difference (53) equals

$$\frac{(\imath u)^{-i+4}}{(i-2)!}\left(\sum_{\mu_1+\mu_2=i+k-3,\,\mu_1\leqslant k}(\mu_1+1)!\mu_2!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{2}(\gamma\cdot c_1)\right.$$
$$\left.+\sum_{\mu_1+\mu_2=i+k-3,\,\mu_2\leqslant k}\mu_1!(\mu_2+1)!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{2}(\gamma\cdot c_1)\right). \tag{59}$$

We now include the T_k and T'_k terms in the difference. We have

$$(\imath u)^k\mathfrak{C}^\bullet(\mathrm{L}_k^{\mathrm{PT}}(\mathsf{ch}_i(\gamma)))-\widetilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(\mathsf{ch}_i(\gamma)))=$$
$$(\imath u)^k\mathfrak{C}^\bullet(\mathrm{R}_k(\mathsf{ch}_i(\gamma)))-\mathrm{R}_k(\mathfrak{C}^\bullet(\mathsf{ch}_i(\gamma)))+(\imath u)^k\mathfrak{C}^\bullet(\mathrm{T}_k(\mathsf{ch}_i(\gamma)))-\frac{(\imath u)^2}{2}\mathrm{T}'_k(\mathfrak{C}^\bullet(\mathsf{ch}_i(\gamma))) \tag{60}$$

Using (45), the T_k and T'_k terms in (60) simplify to

$$\frac{(\imath u)^k}{2}\sum_{a+b=k+2}(a-2)!b!\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_a(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{b-2}}\right)\tau_{b-2}(\mathsf{p})$$
$$\frac{(\imath u)^k}{2}\sum_{a+b=k+2}a!(b-2)!\tau_{a-2}(\mathsf{p})\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_b(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{a-2}}\right). \tag{61}$$

To complete our proof, we require the bumping formula (14):

$$\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k_1+2}(c_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma))=-\frac{1}{k_1!k_2!}(\imath u)^{-1}\mathfrak{a}_{k_1+k_2}(c_1\gamma)\,. \tag{62}$$

Since $\gamma \in H^4(X)$, all the other terms of (14) vanish. We apply the bumping formula (61). In particular, the first term of (61),

$$(a-2)!b!\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_a(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{b-2}}\right)\tau_{b-2}(\mathsf{p}) = -(\imath u)^{-a-b-i+6}\frac{(a+i-2)!b!}{(i-2)!}\tau_{a+i-3}(\gamma\cdot c_1)\tau_{b-2}(\mathsf{p})$$

cancels with the first term of (59). Similarly, the second term of (61) cancels with the second term of (59).

Let us observe that the term of last expression with $a=1$ by the exceptional bumping (28) turns into the terms of (59) with $\mu_1 = k$ or $\mu_2 = k$. Similarly, the term with $b=0$ cancels out with the term (58).

Also the assumption $\widetilde{\mathsf{ch}}_i(\gamma) \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}$ implies that $i \geqslant 2$ thus no negative factorials appear in the above computations.

Case $\gamma \in H^2(X)$.

If $\gamma \in H^2(X)$, the T_k and T'_k terms of the formula (60) acquires extra summands:

$$\begin{aligned}(63)\quad (\imath u)^k\mathfrak{C}^\bullet(\mathrm{L}_k^{\mathrm{PT}}(\widetilde{\mathsf{ch}}_i(\gamma))) - \widetilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_i(\gamma))) &= \\ (\imath u)^k\mathfrak{C}^\bullet(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma))) - \mathrm{R}_k(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_i(\gamma))) & \\ +\frac{(\imath u)^k}{2}\Bigg[\sum_{a+b=k+2}(a-2)!b!\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_a(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{b-2}}\right)\tau_{b-2}(\mathsf{p}) + a!(b-2)!\tau_{a-2}(\mathsf{p})\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_b(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{a-2}}\right) & \\ -\sum_{a+b=k+2}(a-1)!(b-1)!\sum_{0<\bullet,\star<2s+1}\alpha_{\bullet\star}\Big(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_a(\gamma_\bullet)\cdot\mathsf{ch}_i(\gamma))\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_b(\gamma_\star)) & \\ +\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_a(\gamma_\bullet))\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_b(\gamma_\star)\cdot\mathsf{ch}_i(\gamma))\Big)\Bigg], &\end{aligned}$$

where we have used[26] $c_1 \cdot \gamma_{2s+1-\bullet} = \sum_\star \alpha_{\bullet\star}\gamma_\star$. Nevertheless, the strategy used in the previous case can be pursued also for $\gamma \in H^2(X)$. The computation, which is carried out below, is of course more complicated.

We will study the difference

$$(64)\qquad (\imath u)^k\mathfrak{C}^\bullet(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma))) - \mathrm{R}_k(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_i(\gamma)))$$

[26] In (63), the elements $\gamma_\bullet, \gamma_\star$ are of complex cohomological degree 2.

with $\gamma \in H^2(X)$. The expansion of the first term is:

$$(65)\quad (\imath u)^k \mathfrak{C}^\circ(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma))) = (\imath u)^k \frac{(k+i-2)!}{(i-3)!}\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{i+k}(\gamma))$$
$$= (\imath u)^k \frac{(k+i-2)!}{(i-3)!}\left(\frac{\mathfrak{a}_{i+k-1}(\gamma)}{(i+k-1)!} + \frac{(\imath u)^{-1}}{(i+k-2)!}\sum_{|\mu|=i+k-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\cdot c_1)\right.$$
$$+\frac{(\imath u)^{-2}}{(i+k-2)!}\sum_{|\mu|=i+k-4}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\cdot c_1^2)$$
$$\left.+\frac{(\imath u)^{-2}}{(i+k-3)!}\sum_{|\mu|=i+k-5}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}}{\mathrm{Aut}(\mu)}(\gamma\cdot c_1^2)\right).$$

The second term of the difference (64) is more involved since we must transform the descendents $\mathfrak{a}$ to the standard descendents τ before applying the shift operator R_k:

$$(66)\quad \mathrm{R}_k(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_i(\gamma))) =$$
$$(\imath u)^{-(i-2)}\mathrm{R}_k\left(\tau_{i-2}(\gamma) + \left(\sum_{j=1}^{i-2}\frac{1}{j}\right)\tau_{i-3}(\gamma\cdot c_1) + \left(\sum_{1\leqslant j<l\leqslant i-2}\frac{1}{jl}\right)\tau_{i-4}(\gamma\cdot c_1^2)\right)$$
$$+(\imath u)^{-(i-5)}\mathrm{R}_k\left(\frac{(\imath u)^{-1}}{(i-2)!}\left(\sum_{|\mu|=i-3}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\left(\tau_{\mu_1-1}\tau_{\mu_2-1}(\gamma\cdot c_1) + \left[\left(\sum_{j=1}^{\mu_1}\frac{1}{j}\right)\tau_{\mu_1-2}\tau_{\mu_2-1}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\left(\sum_{j=1}^{\mu_2-1}\frac{1}{j}\right)\tau_{\mu_1-1}\tau_{\mu_2-2}\right](\gamma\cdot c_1^2)\right)\right) + \frac{1}{(i-3)!}\sum_{|\mu|=i-5}\frac{\mu_1!\mu_2!\mu_3!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{\mu_3-1}(\gamma\cdot c_1^2)\right).$$

Notice the upper limits in the first harmonic sum is μ_1, the terms with $j=\mu_1$ correspond to the third term of (13).

We will study the right hand side of (63) using (65) and (66) in three steps corresponding to the τ-degree.

- Consider first the τ-linear terms. The τ-linear terms of (65) are

$$(67)\quad (\imath u)^k\frac{(i+k-2)!}{(i-3)!}\left(\frac{1}{(\imath u)^{i+k-2}}\left(\tau_{i+k-2}(\gamma) + \left(\sum_{j=1}^{i+k-2}\frac{1}{j}\right)\tau_{i+k-3}(c_1\cdot\gamma)\right.\right.$$
$$\left.\left.+\left(\sum_{1\leqslant j<l\leqslant i+k-2}\frac{1}{jl}\right)\tau_{i+k-4}(c_1^2\cdot\gamma)\right)\right).$$

The τ-linear terms of (66) is more complicated:

$$(\imath u)^{-i+2}\frac{(i+k-2)!}{(i-3)!}\left(\tau_{i+k-2}(\gamma)+\left(\sum_{j=i-2}^{i+k-2}\frac{1}{j}\right)\tau_{i+k-3}(\gamma\cdot c_1)\right. \tag{68}$$

$$+\left(\sum_{i-2\leqslant j<l\leqslant i+k-2}\frac{1}{jl}\right)\tau_{i+k-4}(\gamma\cdot c_1^2)+\left(\sum_{j=1}^{i-2}\frac{1}{j}\right)\left[\tau_{i+k-3}(\gamma\cdot c_1)+\left(\sum_{j=i-2}^{i+k-2}\frac{1}{j}\right)\tau_{i+k-4}(\gamma\cdot c_1^2)\right]$$

$$\left.+\left(\sum_{1\leqslant j<l\leqslant i-2}\frac{1}{jl}\right)\tau_{i+k-4}(\gamma\cdot c_1^2)\right).$$

The $\tau_{i+k-2}(\gamma)$ terms of (67) and (68) match, so cancel in the difference (64). The $\tau_{i+k-3}(\gamma\cdot c_1)$ terms in (67) and (68) almost cancel: the difference is

$$(\imath u)^{-i+2}\frac{(i+k-2)!}{(i-2)!}\tau_{i+k-3}(\gamma\cdot c_1)\,. \tag{69}$$

For the $\tau_{i+k-4}(\gamma\cdot c_1^2)$ terms, we split the prefactor in (68) as

$$\sum_{j=1}^{i-2}\frac{1}{j}=\frac{1}{i-2}+\sum_{j=1}^{i-3}\frac{1}{j}$$

and the last coefficient of (68) as

$$\sum_{1\leqslant j<l\leqslant i-2}\frac{1}{jl}=\sum_{1\leqslant j<l\leqslant i-3}\frac{1}{jl}+\frac{1}{i-2}\sum_{1\leqslant j\leqslant i-3}\frac{1}{j}\,.$$

Then, we see the difference of the $\tau_{i+k-4}(\gamma\cdot c_1^2)$ terms in (67) and (68) is

$$(\imath u)^{-i+2}\frac{(i+k-2)!}{(i-2)!}\left(\sum_{j=1}^{i+k-2}\frac{1}{j}\right)\tau_{i+k-4}(c_1^2\cdot\gamma)\,. \tag{70}$$

On the right hand side of equation (63), the τ-linear terms (69) and (70) of the difference (64) are canceled with

$$\frac{(\imath u)^k}{2}\left[k!0!\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_{k+2}(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{-2}}\right)\tau_{-2}(\mathsf{p})+0!k!\tau_{-2}(\mathsf{p})\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_{k+2}(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{-2}}\right)\right]$$

using $\tau_{-2}(\mathsf{p})=1$. In fact, after applying (14), we find

$$(\imath u)^k\frac{k!}{(\imath u)^{-2}}\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{k+2}(c_1)\widetilde{\mathsf{ch}}_i(\gamma))$$

$$=-\frac{(\imath u)^{k+1}}{(i-2)!}\left(\mathfrak{a}_{k+i-2}(c_1\gamma)+(\imath u)^{-1}\mathfrak{a}(c_1^2\gamma)\right)+\dots$$

$$=-\frac{(\imath u)^{2-i}}{(i-2)!}\left((k+i-2)!\left(\tau_{k+i-3}(\gamma\cdot c_1)+(\sum_{i=1}^{k+i-3}\frac{1}{i})\tau_{k+i-4}(\gamma\cdot c_1^2)\right)+(k+i-3)!\tau_{k+i-4}(\gamma\cdot c_1^2)\right)+\dots\,.$$

where the dots stand for the τ-quadratic terms. The second equality follows from the formula (54).

• Consider next the τ-quadratic terms. We start with the quadratic terms of complex cohomological degree 2. The corresponding terms from (65) are:

$$(\imath u)^k \frac{(k+i-2)!}{(i-3)!} \sum_{|\mu|=i+k-3} \frac{(\imath u)^{-\mu_1-\mu_2+2}\mu_1!\mu_2!}{(\imath u)(i+k-2)!\,\mathrm{Aut}(\mu)} \tau_{\mu_1-1}\tau_{\mu_2-1}(\gamma\cdot c_1)\,. \tag{71}$$

The computation of the corresponding terms in (66) are more involved since the action of the shift operator R_k depends on the complex cohomological degree of the descendent:

$$(\imath u)^{-i+4}\frac{1}{(i-2)!}\sum_{|\mu|=i-3}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\left(\frac{(\mu_1+k)!}{(\mu_1-1)!}\tau_{\mu_1+k-1}(\gamma\cdot c_1)\tau_{\mu_2-1}(\mathsf{p})+\right.$$
$$\left.\frac{(\mu_2+k+1)!}{(\mu_2)!}\tau_{\mu_1-1}(\gamma\cdot c_1)\tau_{\mu_2-1+k}(\mathsf{p})\right)\,. \tag{72}$$

The linear combination of the first term of (72) with $\mu_1+k-1=a$ and second term with $\mu_1-1=a$ is equal to the corresponding term of (71) with $\mu_1-1=a$. Hence, these cancel in the difference. Similar cancellations happen with rest of the terms. The resulting difference of (71) and (72) is

$$\frac{(\imath u)^{-i+4}}{(i-3)!}\left(\sum_{|\mu|=i+k-3,\,\mu_1\leqslant k}\mu_1!\mu_2!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{\mathrm{Aut}(\mu)}(\gamma\cdot c_1)\right.$$
$$\left.+\sum_{|\mu|=i+k-3,\,\mu_2\leqslant k}\mu_1!\mu_2!\frac{\tau_{\mu_1-1}\tau_{\mu_2-1}}{\mathrm{Aut}(\mu)}(\gamma\cdot c_1)\right) \tag{73}$$

We will cancel (73) with the τ-quadratic terms of complex cohomological degree 2 in the sum

(74)

$$\frac{(\imath u)^k}{2}\Bigg[\sum_{a+b=k+2}(a-2)!b!\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_a(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{b-2}}\right)\tau_{b-2}(\mathsf{p})+a!(b-2)!\tau_{a-2}(\mathsf{p})\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_b(c_1)\mathsf{ch}_i(\gamma)}{(\imath u)^{a-2}}\right)$$
$$-\sum_{a+b=k+2}(a-1)!(b-1)!\sum_{\bullet,\star}\alpha_{\bullet\star}\Big(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_a(\gamma_\bullet)\cdot\mathsf{ch}_i(\gamma))\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_b(\gamma_\star))$$
$$+\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_a(\gamma_\bullet))\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_b(\gamma_\star)\cdot\mathsf{ch}_i(\gamma))\Big)\Bigg]\,.$$

More precisely, the first and second terms of the last expression yield

$$\frac{(\imath u)^{-i+4}}{2}\left[b\frac{(a+i-4)!(b-1)!}{(i-2)!}\tau_{a+i-5}(\gamma\cdot c_1)\tau_{b-2}(\mathsf{p})+a\frac{(a-1)!(b+i-4)!}{(i-2)!}\tau_{a-2}(\mathsf{p})\tau_{b+i-5}(\gamma\cdot c_1)\right],$$

and the last two terms yield[27]

$$-\frac{(\imath u)^{-i-4}}{2}\Bigg[(a-1)\frac{(a+i-4)!(b-1)!}{(i-2)!}\tau_{a+i-5}(\gamma_\bullet\cdot\gamma)\tau_{b-2}(\gamma_\star)+ (b-1)\frac{(a-1)!(b+i-4)!}{(i-2)!}\tau_{a-2}(\gamma_\bullet\cdot\gamma)\tau_{b+i-5}(\gamma_\star)\Bigg].$$

The cancellation then follows from

$$\sum_{\bullet,\star}\alpha_{\bullet\star}\,(\gamma_\bullet\cdot\gamma)\otimes\gamma_\star=\mathsf{p}\otimes(\gamma\cdot c_1)\quad\text{and}\quad\sum_{\bullet,\star}\alpha_{\bullet\star}\,\gamma_\bullet\otimes(\gamma_\star\cdot\gamma)=(\gamma\cdot c_1)\otimes\mathsf{p}\ .$$

We have cancelled all τ-quadratic terms of complex cohomological degree 2 in (63).

Let us also observe that the terms of (74) with $a=1$ and with $b=1$ cancel out by exceptional bumping with (28) with the term of (73) with $\mu_1=k$ or $\mu_2=k$.

A longer computation is needed to deal with τ-quadratic terms of complex cohomological degree 3. Since all such terms have $\gamma\cdot c_1^2$ as an argument, we drop the cohomology insertion from the notation. The corresponding terms from (65) are:

$$(75)\quad (\imath u)^k\frac{(k+i-2)!}{(i-3)!}\sum_{|\mu|=i+k-4}\frac{(\imath u)^{-\mu_1-\mu_2}}{\mathrm{Aut}(\mu)(i+k-2)!}\left(\mu_1!\mu_2!+(\mu_1+1)!\mu_2!\left(\sum_{j=1}^{\mu_1}\frac{1}{j}\right)+\mu_1!(\mu_2+1)!\left(\sum_{j=1}^{\mu_2}\frac{1}{j}\right)\right)\tau_{\mu_1-1}\tau_{\mu_2-1}\,.$$

The corresponding terms from (66) are:

$$(76)\quad \frac{(\imath u)^{-i+4}}{(i-2)!}\sum_{|\mu|=i-3}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\Bigg[\frac{(\mu_1+k)!}{(\mu_1-1)!}\left(\sum_{j=\mu_1}^{\mu_1+k}\frac{1}{j}\right)\tau_{\mu_1+k-2}\tau_{\mu_2-1} +\frac{(\mu_2+k)!}{(\mu_2-1)!}\left(\sum_{j=\mu_2}^{\mu_2+k}\frac{1}{j}\right)\tau_{\mu_1-1}\tau_{\mu_2+k-2}+\left(\sum_{j=1}^{\mu_1}\frac{1}{j}\right)\left(\frac{(\mu_1+k)!}{(\mu_1-1)!}\tau_{\mu_1+k-2}\tau_{\mu_2-1} +\frac{(\mu_2+k+1)!}{\mu_2!}\tau_{\mu_1-2}\tau_{\mu_2+k-1}\right)+\left(\sum_{j=1}^{\mu_2-1}\frac{1}{j}\right)\left(\frac{(\mu_2+k)!}{(\mu_2-1)!}\tau_{\mu_1-1}\tau_{\mu_2+k-2} +\frac{(\mu_1+k+1)!}{\mu_1!}\tau_{\mu_1+k-1}\tau_{\mu_2-2}\right)\Bigg]\,.$$

The expression (76) is simplified by the following strategy. We number the six τ-quadratic terms by their order of occurrence in (76). The first term of (76) combines with the third term. The second term combines with the fifth term. We also split off the

[27]The sum over $\bullet,\star$ with coefficient $\alpha_{\bullet\star}$ is implicit.

summands with $j = \mu_1 + k$ and $j = \mu_2 + k$ from the first and second terms respectively, as well as the summand with $j = \mu_1$ from the third term. Then, (76) equals

$$(77)\quad \frac{(\imath u)^{-i+4}}{(i-2)!}\Bigg(\sum \mu_1 \frac{(\mu_1+k-1)!\mu_2!}{\mathrm{Aut}(\mu)}\tau_{\mu_1+k-2}\tau_{\mu_2-1} + \mu_2 \frac{\mu_1!(\mu_2+k-1)!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2+k-2}$$
$$+\mu_1 \frac{(\mu_1+k)!\mu_2!}{\mathrm{Aut}(\mu)}\left(\sum_{j=1}^{\mu_1+k-1}\frac{1}{j}\right)\tau_{\mu_1+k-2}\tau_{\mu_2-1} + \mu_2\frac{\mu_1!(\mu_2+k)!}{\mathrm{Aut}(\mu)}\left(\sum_{j=1}^{\mu_2+k-1}\frac{1}{j}\right)\tau_{\mu_1-1}\tau_{\mu_2+k-2}$$
$$+(\mu_2+k+1)\frac{\mu_1!(\mu_2+k)!}{\mathrm{Aut}(\mu)}\left(\sum_{j=1}^{\mu_1-1}\frac{1}{j}\right)\tau_{\mu_1-2}\tau_{\mu_2+k-1}$$
$$+(\mu_1+k+1)\frac{(\mu_1+k)!\mu_2!}{\mathrm{Aut}(\mu)}\left(\sum_{j=1}^{\mu_2-1}\frac{1}{j}\right)\tau_{\mu_1+k-1}\tau_{\mu_2-2}$$
$$+\frac{(\mu_1+k)!\mu_2!}{\mathrm{Aut}(\mu)}\tau_{\mu_1+k-2}\tau_{\mu_2-1} + \frac{(\mu_1-1)!(\mu_2+k+1)!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-2}\tau_{\mu_2+k-1}\Bigg),$$

where the sum is over $\mu_1 \geqslant \mu_2$, $|\mu| = i-3$.

Let us fix an integer a satisfying $a > k-2$. We observe that the sum of the first term from the first line (77) with $\mu_1 = a+2-k$ and the second term in the last line with $\mu_2 = a+1-k$ will cancel with the first term of (75) with $\mu_1 = a+1$. Also, the sum of the second term from the first line with $\mu_2 = a+2-k$ and the first term of the last line with $\mu_1 = a+1-k$ will cancel with the first term of (75) with $\mu_2 = a+1$.

Similarly, the sum of the first term for the second line of (77) with $\mu_1 = a+2-k$ and the first term from the third line of (77) with $\mu_1 = a+2$ cancels with the second term of (75) with $\mu_1 = a+1$. Finally, the sum of the second term from the second line of (77) with $\mu_1 = a+1$ and the last term from the last line of (77) with $\mu_1 = a+1-k$ cancels with the last term of (75) with $\mu_1 = a+1$.

After all of these cancellations, we are left with

$$(78)\quad \sum \frac{(\imath u)^{-i+4}\mu_1!\mu_2!}{\mathrm{Aut}(\mu)(i-3)!}\left(1+(\mu_1+1)\left(\sum_{j=1}^{\mu_1}\frac{1}{j}\right)+(\mu_2+1)\left(\sum_{j=1}^{\mu_2}\frac{1}{j}\right)\right)\tau_{\mu_1-1}\tau_{\mu_2-1}\,,$$

where $\sum$ is the sum of two sub sums: the first is over $\mu_1+\mu_2 = i+k-4$, $\mu_1 \leqslant k$ and the second is over $\mu_1+\mu_2 = i+k-4$, $\mu_2 \leqslant k$.

In the difference (63), the expression (78) is canceled by the corresponding τ-quadratic terms of complex cohomological degree 3 of (74). More precisely, the first and second terms of (74) yield

$$\frac{(\imath u)^{-i+4}}{2}\left[b\frac{(a+i-4)!(b-1)!}{(i-2)!}\tau_{a+i-6}(\gamma\cdot c_1^2)\tau_{b-2}(\mathsf{p})+a\frac{(a-1)!(b+i-4)!}{(i-2)!}\tau_{a-2}(\mathsf{p})\tau_{b+i-6}(\gamma\cdot c_1^2)\right],$$

after we apply (14) to these terms and drop τ-cubic terms and the terms of cohomological degree other than 3. In particular, the factors in first and second terms are produced by the $\mathfrak{a}$-linear term of (14) proportional to c_1.

The last two terms of (74) yield[28]

$$-\frac{(\imath u)^{-i+4}}{2}\Bigg[(a-1)\frac{(a+i-4)!(b-1)!}{(i-2)!}\tau_{a+i-6}(\gamma_\bullet\cdot\gamma)\tau_{b-2}(\gamma_\star\cdot c_1)+ (b-1)\frac{(a-1)!(b+i-4)!}{(i-2)!}\tau_{a-2}(\gamma_\bullet\cdot\gamma)\tau_{b+i-6}(\gamma_\star\cdot c_1)\Bigg],$$

after we apply only the parts of (13) and (14) that are not c_1-proportional, then we use the $\mathfrak{a}$ to τ the transition formula (54) and drop the τ cubic terms and the terms of homological degree other than 3.

Together these two sums combine and cancel the first term of (78). To cancel the last two terms of (78), we follow the same pattern. We first apply c_1^0-part of (14) to the first and second terms of (74) and then apply c_1-part of the $\mathfrak{a}$ to τ transition formula (54). Next, we apply the c_1^0-parts of (13) and the (14) and the c_1^1-part of (54) to the last two terms of (74). After dropping the τ-cubic terms and the terms of complex cohomological degree other than 3, we exactly cancel the remaining terms of (78).

• Consider finally the τ-cubic terms. The cohomological arguments of these terms are $c_1^2\cdot\gamma$, so as in the previous computation, we drop the cohomology insertion from the notation.

After expanding the corresponding terms of (65), we obtain:

$$\frac{(\imath u)^{-i}(i+k-2)}{(i-3)!}\sum_{|\mu|=i+k-5}\frac{\mu_1!\mu_2!\mu_3!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{\mu_3-1}\,. \tag{79}$$

On the other hand, the corresponding terms from (66) are more complicated:

$$\frac{(\imath u)^{-i}}{(i-3)!}\sum_{|\mu|=i-5}\frac{\mu_1!\mu_2!\mu_3!}{\mathrm{Aut}(\mu)}\Bigg(\frac{(\mu_1+k+1)!}{\mu_1!}\tau_{\mu_1+k-1}\tau_{\mu_2-1}\tau_{\mu_3-1} +\frac{(\mu_2+k+1)!}{\mu_2!}\tau_{\mu_1-1}\tau_{\mu_2+k-1}\tau_{\mu_3-1}+\frac{(\mu_3+k+1)!}{\mu_3!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{\mu_3+k-1}\Bigg),$$

In (79), we have $i+k-2=\sum_{j=1}^3(\mu_j+1)$. Therefore, the difference between the last two expressions is the sum of the monomials

$$\Bigg(\sum_{j,\mu_j\leqslant k-2}(\mu_j+2)\Bigg)\frac{(\imath u)^{-i}\mu_1!\mu_2!\mu_3!}{(i-3)!\,\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{\mu_3-1}\,. \tag{80}$$

Let us restrict our attention to the case when i is bigger than k, the other cases are analogous. After applying the reaction from the last line of (14), we obtain a formula for

[28]The sum over $\bullet,\star$ with coefficient $\alpha_{\bullet\star}$ is implicit.

the expressions in the second line of (63):

$$(81)\quad (a-2)!b!\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_a(c_1)\widetilde{\mathsf{ch}}_i(\gamma)}{(\imath u)^{b-2}}\right)\tau_{b-2}(\mathsf{p}) = \tau\text{-quadratic terms } + \frac{(\imath u)^{-k-i}b!}{(i-2)!}\left(\sum_{|\mu|=a+i-6}\max(\max(\mu_1+1,\mu_2+1),i-2)\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2-1}\right)\tau_{b-2}\,,$$

$$(82)\quad a!(b-2)!\tau_{a-2}(\mathsf{p})\mathfrak{C}^\circ\left(\frac{\widetilde{\mathsf{ch}}_b(c_1)\widetilde{\mathsf{ch}}_i(\gamma)}{(\imath u)^{a-2}}\right) = \tau\text{-quadratic terms } + \frac{(\imath u)^{-k-i}a!}{(i-2)!}\left(\sum_{|\mu|=b+i-6}\max(\max(\mu_1+1,\mu_2+1),i-2)\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\tau_{\mu_1-1}\tau_{\mu_2-1}\right)\tau_{a-2}\,.$$

The terms of (81) and (82) with $\max(\mu_1+1,\mu_2+1)\leqslant i-2$ contribute the monomials:

$$(83)\qquad (\imath u)^{-i}b\cdot\frac{\mu_1!\mu_2!(b-1)!}{(i-3)!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{b-2}\,,\qquad (\imath u)^{-i}a\cdot\frac{\mu_1!\mu_2!(a-1)!}{(i-3)!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{a-2}\,.$$

Note $a+b=k+2$ in (63). Since $\max(\mu_1+1,\mu_2+1)\leqslant i-2$ and $|\mu|=a+i-2$ or $|\mu|=b+i-2$ we imply that $\mu_1+1,\mu_2+1\geqslant k-1$. Thus the corresponding terms of (81) and (82) cancel with the monomials (80) such that there is only one j with $\mu_j\leqslant k-2$.

The terms in (81) and (82) with $\max(\mu_1+1,\mu_2+1)>i-2$ yield terms:

$$(\imath u)^{-i}b(\mu'+1)\cdot\frac{\mu_1!\mu_2!(b-1)!}{(i-2)!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{b-2}\,,\qquad (\imath u)^{-i}a(\mu'+1)\cdot\frac{\mu_1!\mu_2!(a-1)!}{(i-2)!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{a-2}\,,$$

where $\mu'=\max(\mu_1,\mu_2)$. Both of these terms are of the form:

$$(84)\qquad (\imath u)^{-i}(\mu_1+1)(\mu_2+1)\cdot\frac{\mu_1!\mu_2!\mu_3!}{(i-2)!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{\mu_3-1}\,,$$

with $\mu_1+1>i-2$ and $|\mu|=i+k-4$. Since we assumed that $i>k$, we have $\mu_1+\mu_2<k-4$ in (84). The discussed terms therefore combine to yield the sum of monomials:

$$(85)\qquad (\imath u)^{-i}(\mu_1+\mu_2+2)(\mu_3+1)\cdot\frac{\mu_1!\mu_2!\mu_3!}{(i-2)!}\tau_{\mu_1-1}\tau_{\mu_2-1}\tau_{\mu_3-1}\,,$$

where $\mu_3+1>i-2$ and $\mu_1,\mu_2\leqslant k-2$.

The terms (85) combine with the terms from the expansion of the last two lines of (63). Indeed, since $\gamma_\bullet$, $\gamma_\star$ in the last two lines of (63) are of complex cohomological degree 2, the τ-terms result from use of the c_1^1-part of (13) and of the c_1^0-part of (14). The expansion of these terms is a sum of monomials

$$(86)\qquad -(\imath u)^{-i}(b-1)(a-1)\frac{(a+i-4)!\mu_1!\mu_2!}{(i-2)!}\tau_{a+i-5}\tau_{\mu_1-1}\tau_{\mu_2-1}\,,$$

where $|\mu|=b-3$.

The combination of (86) with $a = \mu_3 - i + 4$, $b = \mu_1 + \mu_2 + 3$ and (85) matches (80), since, in (86), we have

$$(b-1)(a-1) = (\mu_1+\mu_2+2)(\mu_3-i+3) = (\mu_1+\mu_2+2)(\mu_3+1) - (\mu_1+\mu_2+2)(i-2)\,.$$

We have cancelled all τ-cubic terms.

The assumption $\widetilde{\mathsf{ch}}_i(\gamma) \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}$ implies $i \geqslant 3$. Therefore, in the above computations, we do not see negative factorials in denominators. □

4.3. **Proof of Theorem 12 for $\mathbb{D}^1_{\mathrm{PT}} \cap \mathbb{D}^{X\bigstar}_{\mathrm{PT}}$.** Theorem 12, for all $D \in \mathbb{D}^1_{\mathrm{PT}} \cap \mathbb{D}^{X\bigstar}_{\mathrm{PT}}$, is an immediate consequence of Proposition 16 for singletons by the following simple argument. Let

$$D = \prod_{i=1}^m \widetilde{\mathsf{ch}}_{k_i}(\gamma_i) \in \mathbb{D}^1_{\mathrm{PT}} \cap \mathbb{D}^{X\bigstar}_{\mathrm{PT}}\,,$$

where $\gamma_i\gamma_j = 0 \in H^*(X)$ for all $i \neq j$.

By definition, for $k \geqslant 1$,

$$\begin{aligned}
\mathfrak{C}^\bullet\left(\mathrm{L}_k^{\mathrm{PT}}(D)\right) &= \mathfrak{C}^\bullet\Big(\mathrm{L}_k^{\mathrm{PT}}\big(\prod_{i=1}^m \widetilde{\mathsf{ch}}_{k_i}(\gamma_i)\big)\Big)\\
&= \mathfrak{C}^\bullet\left(\mathrm{T}_k\prod_{i=1}^m \widetilde{\mathsf{ch}}_{k_i}(\gamma_i) + \sum_{j=1}^m \mathrm{R}_k(\widetilde{\mathsf{ch}}_{k_j}(\gamma_j))\prod_{i\neq j}\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)\right).
\end{aligned}$$

Since $\gamma_i\gamma_j = 0$ for $i \neq j$,

$$\mathfrak{C}^\bullet\Big(\mathrm{T}_k\prod_{i=1}^m \widetilde{\mathsf{ch}}_{k_i}(\gamma_i)\Big) = (-m+1)\mathfrak{C}^\bullet(\mathrm{T}_k)\prod_{i=1}^m \mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)) + \sum_{j=1}^m \mathfrak{C}^\bullet(\mathrm{T}_k\widetilde{\mathsf{ch}}_{k_j}(\gamma_j))\prod_{i\neq j}\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_{k_i}(\gamma_i))\,.$$

By Proposition 16,

$$\begin{aligned}
(\imath u)^{-k}\,\widetilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_i(\gamma_i))) &= \mathfrak{C}^\bullet\left(\mathrm{L}_k^{\mathrm{PT}}(\widetilde{\mathsf{ch}}_i(\gamma_i))\right)\\
&= \mathfrak{C}^\bullet(\mathrm{T}_k)\,\mathfrak{C}^\bullet\big(\widetilde{\mathsf{ch}}_i(\gamma_i)\big) + \mathfrak{C}^\bullet\big(\mathrm{T}_k\widetilde{\mathsf{ch}}_i(\gamma_i)\big) + \mathfrak{C}^\bullet\Big(\mathrm{R}_k\big(\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)\big)\Big)\,.
\end{aligned}$$

We conclude

$$\mathfrak{C}^\bullet\left(\mathrm{L}_k^{\mathrm{PT}}(D)\right) = \sum_{j=1}^m (\imath u)^{-k}\,\widetilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_j(\gamma_j)))\prod_{i\neq j}\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)) - (m-1)\mathfrak{C}^\bullet(\mathrm{T}_k)\prod_{i=1}^m \mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_{k_i}(\gamma_i))\,.$$

On the other hand,

$$(\imath u)^{-k}\,\widetilde{\mathrm{L}}_k^{\mathrm{GW}}\left(\mathfrak{C}^\bullet(D)\right) = \sum_{j=1}^m (\imath u)^{-k}\,\widetilde{\mathrm{L}}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_j(\gamma_j)))\prod_{i\neq j}\mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)) - (m-1)(\imath u)^{-k}\left(\frac{(\imath u)^2}{2}\right)\mathrm{T}_k\prod_{i=1}^m \mathfrak{C}^\bullet(\widetilde{\mathsf{ch}}_{k_i}(\gamma_i))\,.$$

The proof is completed by applying (45). □

5. Intertwining III: Interacting insertions

5.1. **Overview.** We complete here the proof of Theorem 12. Since non-interacting insertions have already been treated in Section 4, we must address the interacting cases. In the desired equation,

$$\mathfrak{C}^\bullet \circ \mathrm{L}_k^{\mathrm{PT}}(D) = (\imath u)^{-k}\, \widetilde{\mathrm{L}}_k^{\mathrm{GW}} \circ \mathfrak{C}^\bullet(D)\,, \tag{87}$$

the stable pairs descendent insertions of $D \in \mathbb{D}_{\mathrm{PT}}^{X\star}$ can interact with each other via the GW/PT descendent correspondence on both sides of (87). In addition, the stable pairs descendents can also interact with constant term of the Virasoro constraints on the left side. We must control all of these interactions.

5.2. **Interactions among two insertions.** We start with results which control the interactions of two descendent insertions.

Proposition 17. *Let $\gamma' \in H^2(X)$, $\gamma'' \in H^4(X)$, and let $i \geqslant 3$, $j \geqslant 2$. Then, for $k \geqslant -1$, we have*

$$(\imath u)^k\, \mathfrak{C}^\circ(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma''))) = \mathrm{R}_k(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma'')))\,.$$

Proof. We first compute the left side of the equation. After applying the shifts, we obtain

$$\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma'')) = \frac{(i+k-2)!}{(i-3)!}\widetilde{\mathsf{ch}}_{i+k}(\gamma')\widetilde{\mathsf{ch}}_j(\gamma'') + \frac{(j+k-1)!}{(j-2)!}\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_{j+k}(\gamma'')\,.$$

We apply the correspondence to the both terms:

$$\begin{aligned}
\mathfrak{C}^\circ(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma''))) &= (\imath u)^{-1}\left(\frac{1}{(i-3)!(j-2)!} + \frac{(j+k-1)}{(i-2)!(j-2)!}\right)\mathfrak{a}_{i+j+k-4}(\gamma'\gamma'') \\
&= (\imath u)^{-i-j-k+4}\frac{(i+j+k-3)!}{(i-2)!(j-2)!}\tau_{i+j+k-5}(\gamma'\gamma'')\,.
\end{aligned}$$

The right side of the equation is

$$\begin{aligned}
\mathrm{R}_k(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma''))) &= \mathrm{R}_k\left(\frac{(\imath u)^{-1}}{(i-2)!(j-2)!}\mathfrak{a}_{i+j-4}(\gamma'\gamma'')\right) \\
&= (\imath u)^{-i-j+4}\frac{(i+j-4)!}{(i-2)!(j-2)!)}\mathrm{R}_k(\tau_{i+j-5}(\gamma'\gamma'')) \\
&= (\imath u)^{-i-j+4}\frac{(i+j+k-3)!}{(i-2)!(j-2)!}\tau_{i+j+k-5}(\gamma'\gamma'')\,,
\end{aligned}$$

which matches the left side. □

Proposition 18. *Let $\gamma', \gamma'' \in H^2(X)$, and let $i, j \geqslant 3$. Then, for $k \geqslant -1$, we have*

$$(88)\quad (\imath u)^k\, \mathfrak{C}^\circ(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma''))) - \mathrm{R}_k(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_i(\gamma')\mathsf{ch}_j(\gamma''))) = \sum_{a+b=k+2} (a-2)!b!\, \mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_i(\gamma')\cdot \widetilde{\mathsf{ch}}_j(\gamma'')\cdot \widetilde{\mathsf{ch}}_a(c_1))\, \mathfrak{C}^\circ(\mathsf{ch}_b(\mathsf{p})) + a!(b-2)!\, \mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_i(\gamma')\cdot \widetilde{\mathsf{ch}}_j(\gamma'')\cdot \widetilde{\mathsf{ch}}_b(c_1))\, \mathfrak{C}^\circ(\mathsf{ch}_a(\mathsf{p})) - \sum_{a+b=k+2}(a-1)!(b-1)!\sum_{\bullet,\star}\alpha_{\bullet\star}\Big(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_a(\gamma_\bullet)\cdot \widetilde{\mathsf{ch}}_i(\gamma'))\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_b(\gamma_\star)\widetilde{\mathsf{ch}}_j(\gamma'')) + \mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_a(\gamma_\bullet)\widetilde{\mathsf{ch}}_j(\gamma''))\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_b(\gamma_\star)\cdot \widetilde{\mathsf{ch}}_i(\gamma'))\Big)\,.$$

Proof. We follow the same strategy as in the proof of Proposition 16. We first compute

$$\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma'')) = \frac{(k+i-2)!}{(i-3)!}\widetilde{\mathsf{ch}}_{i+k}(\gamma')\widetilde{\mathsf{ch}}_j(\gamma'') + \frac{(k+j-2)!}{(j-3)!}\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_{j+k}(\gamma'')\,.$$

After applying the correspondence, we obtain

$$(89)\quad \mathfrak{C}^\circ(\mathrm{R}_k(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma''))) = -\frac{1}{(i-3)!(j-2)!}\left[\frac{\mathfrak{a}_{i+j+k-4}(\gamma'\gamma'')}{\imath u} + \frac{\mathfrak{a}_{i+j+k-5}(\gamma'\gamma''\cdot c_1)}{(\imath u)^2} + (\imath u)^{-2}\sum_{|\mu|=i+j+k-6}\frac{f(i+k,j;\mu_1,\mu_2)}{\mathrm{Aut}(\mu)}\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma'\gamma''\cdot c_1)\right] - \frac{1}{(i-2)!(j-3)!}\left[\frac{\mathfrak{a}_{i+j+k-4}(\gamma'\gamma'')}{\imath u} + \frac{\mathfrak{a}_{i+j+k-5}(\gamma'\gamma''\cdot c_1)}{(\imath u)^2} + (\imath u)^{-2}\sum_{|\mu|=i+j+k-6}\frac{f(i,j+k;\mu_1,\mu_2)}{\mathrm{Aut}(\mu)}\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma'\gamma''\cdot c_1)\right],$$

where $f(i,j;\mu_1,\mu_2) = \max(\max(i-2,j-2),\max(\mu_1+1,\mu_2+1))$.

The second term of the difference is easier:

$$(90)\quad \mathrm{R}_k(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma''))) = -\frac{(\imath u)^{-i-j+4}}{(i-2)!(j-2)!}\mathrm{R}_k\left((i+j-4)!\left(\tau_{i+j-5}(\gamma'\gamma'') + \left(\sum_{s=1}^{i+j-4}\frac{1}{s}\right)\tau_{i+j-6}(\gamma'\gamma''\cdot c_1)\right) + (\imath u)^{-2}\sum_{|\mu|=i+j-6}\frac{f(i,j;\mu_1,\mu_2)}{\mathrm{Aut}(\mu)}\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma'\gamma''\cdot c_1)\right).$$

We now analyze the difference. The τ-linear terms of complex cohomological degree 2 in $(\imath u)^k$ times (89) and (90) are matching sums of the monomials:

$$(\imath u)^{-i-j+4}\frac{(i+j+k-4)!}{(i-2)!(j-2)!}(i+j-4)\tau_{i+j+k-5}(\gamma'\gamma'')\,.$$

The τ-linear terms of cohomological degree 3 almost match. To be precise, the corresponding terms in (89) are sums the monomials:

$$(\imath u)^{-i-j+4}\frac{(i+j+k-4)!}{(i-2)!(j-2)!}(i+j-4)\left(\sum_{s=1}^{i+j+k-4}\frac{1}{s}\right)\tau_{i+j+k-6}(\gamma'\gamma''\cdot c_1)\,.$$

Respectively, the corresponding terms in (90) are sums of the same monomials plus an extra term

$$(\imath u)^{-i-j+4}\frac{(i+j+k-4)!}{(i-2)!(j-2)!}\tau_{i+j+k-6}(\gamma'\gamma''\cdot c_1)\,.$$

This extra term gets canceled by the term from the second line of (88) with $b=0$ because of (29).

Therefore, the difference of $(\imath u)^k$ times (89) and (90) consists only of the τ-quadratic terms of complex cohomological degree 3. We omit cohomological classes since all the cohomological arguments are $\gamma'\gamma''\cdot c_1$. The corresponding part of (89) is

(91)
$$\frac{(\imath u)^{-i-j+2}}{(i-2)!(j-2)!}\sum_{|\mu|=i+j+k-6}\frac{\mu_1!\mu_2!}{\mathrm{Aut}(\mu)}\left[(i-2)f(i+k,j;\mu)+(j-2)f(i,j+k;\mu)\right]\tau_{\mu_1-1}\tau_{\mu_2-1}\,,$$

where we assume that f vanishes whenever one of the argument is negative.

We must compare (91) with the expansion of the last four lines of (88). The first two of the last four lines of (88) expand to

$$-\frac{(\imath u)^{-i-j+2}}{(i-2)!(j-2)!}\sum_{a+b=k+2}(i+j+a-6)b\frac{(i+j+a-7)!(a-1)!}{2}\tau_{i+j+a-8}\tau_{b-2}$$
$$+(i+j+b-6)a\frac{(i+j+b-7)!(a-1)!}{2}\tau_{i+j+b-8}\tau_{a-2}\,.$$

The last two lines of the last four lines of (88) expand to

$$\frac{(\imath u)^{-i-j+2}}{(i-2)!(j-2)!}\sum_{a+b=k+2}(a-1)(b-1)\Big((a+i-4)!(b+j-4)!\tau_{a+i-5}\tau_{b+j-5}+$$
$$(a+j-4)!(b+i-4)!\tau_{a+j-5}\tau_{b+i-5}\Big)\,.$$

These last two expressions are the τ-cubic contribution to the (88) which result from the bumping of $\widetilde{\mathsf{ch}}_i(\gamma')\widetilde{\mathsf{ch}}_j(\gamma'')$ with the constant term T_k. The corresponding coefficient in front of τ-cubic monomial is given by the formula (93) below.

To complete the proof, we must match the coefficients in front of the terms in sums above. That is we need to compare two expressions below for all μ satisfying $|\mu|=i+j+k-6$:

(92)
$$(i-2)f(i+k,j;\mu)+(j-2)f(i,j+k;\mu)-(\mu_1+1)f(i,j;\mu_1-k,\mu_2)$$
$$-(\mu_2+1)f(i,j;\mu_1,\mu_2-k)\,,$$

$$[\mu_1+1]_{\leqslant k}(\mu_2+1)+(\mu_1+1)[\mu_2+1]_{\leqslant k}-[\mu_1-i+3]_{\geqslant 0}[\mu_2-j+3]_{\geqslant 0} - [\mu_1-j+3]_{\geqslant 0}[\mu_2-i+3]_{\geqslant 0}\,, \tag{93}$$

where $[a]_{\leqslant b}$ and $[a]_{\geqslant b}$ are cut off functions which equal a if a satisfies inequalities $a \geqslant b$ and $a \leqslant b$ respectively (and are zero otherwise). The matching now is a long and routine check. We give some details.

We can always assume $\mu_1 \geqslant \mu_2$ and $i \geqslant j$. Let us further assume k is small and $\mu_1 \geqslant i+k$. If $\mu_2 \geqslant k$, then the function (92) equals

$$(i+j-4)(\mu_1+1)-(\mu_1+1)(\mu_1-k+1)-(\mu_2-1)(\mu_1+1)=0\,.$$

The assumed inequalities force all terms in (93) to vanish.

Next, we assume all but last inequality are true, that is $\mu_2 < k$. Then the expression (92) becomes

$$(i+j-4)(\mu_1+1)-(\mu_1+1)(\mu_1-k+1)=(\mu_1+1)(\mu_2+1)\,.$$

On the other hand, in (93), only the second expression does not vanish – the second expression matches (92). Rest of the case can be treated analogously. ☐

5.3. **Interactions among three insertions.** The last interaction to consider is among three descendent insertions. Because of the stationary assumption, there is only one case to control.

Proposition 19. *Let $\gamma',\gamma'',\gamma''' \in H^2(X)$, and let $i_1,i_2,i_3 \geqslant 3$, Then, for $k \geqslant -1$, we have*

$$(\imath u)^k\,\mathfrak{C}^\circ\left(\mathrm{R}_k(\widetilde{\mathsf{ch}}_{i_1}(\gamma')\widetilde{\mathsf{ch}}_{i_2}(\gamma'')\widetilde{\mathsf{ch}}_{i_3}(\gamma'''))\right)-\mathrm{R}_k\left(\mathfrak{C}^\circ\left(\widetilde{\mathsf{ch}}_{i_1}(\gamma')\widetilde{\mathsf{ch}}_{i_2}(\gamma'')\widetilde{\mathsf{ch}}_{i_3}(\gamma''')\right)\right)=0\,.$$

For the proof, we will use the explicit correspondence formula (15) for the triple interaction:

$$\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{i_1}\widetilde{\mathsf{ch}}_{i_2}\widetilde{\mathsf{ch}}_{i_3})(\gamma)=\frac{(|i|-6)(\imath u)^{-2}}{(i_1-2)!(i_2-2)!(i_3-2)!}\mathfrak{a}_{|i|-7}(\gamma) \tag{94}$$

where $|i|=i_1+i_2+i_3$.

Proof of Proposition 19. We first compute the left side of the equation. To start,

$$\begin{aligned}\mathrm{R}_k(\widetilde{\mathsf{ch}}_{i_1}(\gamma')\widetilde{\mathsf{ch}}_{i_2}(\gamma'')\widetilde{\mathsf{ch}}_{i_3}(\gamma''')) \quad = \quad & \frac{(i_1+k-2)!}{(i_1-3)!}\widetilde{\mathsf{ch}}_{i_1+k}(\gamma')\widetilde{\mathsf{ch}}_{i_2}(\gamma'')\widetilde{\mathsf{ch}}_{i_3}(\gamma''') \\ & +\frac{(i_2+k-2)!}{(i_2-3)!}\widetilde{\mathsf{ch}}_{i_1}(\gamma')\widetilde{\mathsf{ch}}_{i_2+k}(\gamma'')\widetilde{\mathsf{ch}}_{i_3}(\gamma''') \\ & +\frac{(i_3+k-2)!}{(i_3-3)!}\widetilde{\mathsf{ch}}_{i_1}(\gamma')\widetilde{\mathsf{ch}}_{i_2}(\gamma'')\widetilde{\mathsf{ch}}_{i_3+k}(\gamma''')\,.\end{aligned}$$

After applying the triple bumping and the transition from $\mathfrak{a}$ descendents to τ descendents, we obtain:

$$\mathfrak{C}^\circ(\mathrm{R}_k(\widetilde{\mathsf{ch}}_{i_1}(\gamma')\widetilde{\mathsf{ch}}_{i_2}(\gamma'')\widetilde{\mathsf{ch}}_{i_3}(\gamma'''))) = (|i|+k-6)(\imath u)^{-2}\left(\frac{1}{(i_1-3)!(i_2-2)!(i_3-2)!} + \frac{1}{(i_1-2)!(i_2-3)!(i_3-2)!} + \frac{1}{(i_1-2)!(i_2-2)!(i_3-3)!}\right)\mathfrak{a}_{|i|+k-7}(\gamma'\gamma''\gamma''') = (\imath u)^{-|i|-k+6}(|i|-6)\frac{(|i|+k-6)!}{(i_1-2)!(i_2-2)!(i_3-2)!}\tau_{|i|+k-8}(\gamma'\gamma''\gamma''')\,. \tag{95}$$

On the other hand, the right side of the equation equals

$$\mathrm{R}_k(\mathfrak{C}^\circ(\widetilde{\mathsf{ch}}_{i_1}(\gamma')\widetilde{\mathsf{ch}}_{i_2}(\gamma'')\widetilde{\mathsf{ch}}_{i_3}(\gamma'''))) = \frac{(\imath u)^{-2}(|i|-6)}{(i_1-2)!(i_2-2)!(i_3-2)!}\mathrm{R}_k(\mathfrak{a}_{|i|-7}(\gamma'\gamma''\gamma''')) = (\imath u)^{-|i|+6}(|i|-6)\frac{(|i|+k-6)!}{(i_1-2)!(i_2-2)!(i_3-2)!}\tau_{|i|+k-8}(\gamma'\gamma''\gamma''')\,,$$

which matches $(\imath u)^k$ times (95). □

5.4. **Proof of Theorem 12.** Let $k \geqslant 1$, and let $D \in \mathbb{D}^{X\star}_{\mathrm{PT}}$. To prove the equality

$$\mathfrak{C}^\bullet \circ \mathrm{L}^{\mathrm{PT}}_k(D) = (\imath u)^{-k}\,\widetilde{\mathrm{L}}^{\mathrm{GW}}_k \circ \mathfrak{C}^\bullet(D)\,,$$

after the restrictions $\tau_{-2}(\mathsf{p}) = 1$ and $\tau_{-1}(\gamma) = 0$ for $\gamma \in H^{>2}(X)$, we will expand both sides. The non-interacting case was already proven in Section 4.3. Equality in the general case will use Propositions 14, 15, 16, 17, 18, and 19.

In the formulas below, we will use short-hand notation for the constant term of $\mathrm{L}^{\mathrm{PT}}_k$:

$$\mathrm{T}_k = \sum_j \mathrm{T}^L_{k,j}\mathrm{T}^R_{k,j}\,,$$

where L and R denote the left and right sides in (3).

For $D = \prod_{i=1}^{\ell} D_i \in \mathbb{D}^{X\star}_{\mathrm{PT}}$, we have

$$\mathfrak{C}^\bullet(\mathrm{L}^{\mathrm{PT}}_k(D)) = \mathfrak{C}^\bullet(\mathrm{T}_k D + \mathrm{R}_k(D)) = \sum_{P'}\sum_j\prod_{S\in P'}\mathfrak{C}^\circ(\mathrm{T}^S_{k,j}D^S) + \sum_{P''}\sum_{t=1}^{\ell(P'')}\mathfrak{C}^\circ(\mathrm{R}_k(D^{S_t}))\prod_{S\in P'',\,S\neq S_t}\mathfrak{C}^\circ(D^S)\,. \tag{96}$$

The first sum is over partitions P' of $\{1,\ldots,\ell,L,R\}$ and

$$D^S = \prod_{i\in S\cap\{1,\ldots,\ell\}} D_i\,,\quad \mathrm{T}^S_{k,j} = \prod_{\gamma\in S\cap\{L,R\}}\mathrm{T}^\gamma_{k,j}\,.$$

The second sum is over partitions P'' of $\{1,\ldots,\ell\}$.

We must compare the (96) with $(\imath u)^{-k}$ times

$$\mathrm{L}_k^{\mathrm{GW}}(\mathfrak{C}^\bullet(D)) = \mathrm{L}_k^{\mathrm{GW}}(\sum_P \prod_{S\in P} \mathfrak{C}^\circ(D^S)) \tag{97}$$

$$= \sum_{P'} \mathrm{T}_k \prod_{S\in P'} \mathfrak{C}^\circ(D^S) + \sum_{P''} \sum_{t=1}^{\ell(P'')} \mathrm{R}_k(\mathfrak{C}^\circ(D^{S_t})) \prod_{S\in P'',\, S\neq S_t} \mathfrak{C}^\circ(D^S)\,.$$

where both sums run over partitions P', P'' of $\{1,\ldots,\ell\}$.

Since we only work with the stationary descendents, we can assume that the parts of partitions in the formulas have at most three elements. We will match the terms of (96) and $(\imath u)^{-k}$ times (97) depending on the size of S_t.

• If $|S_t| = 3$, then the terms in (96) and (97) with $P'' = \tilde{P} \sqcup S_t$ are matched by Proposition 19.

• If $|S_t| = 2$ with $S_t = \{p, q\}$, then we use Propositions 17 and 18 to match the terms of (96) with $P'' = \tilde{P} \sqcup S_t$ and with P' equal to

$$\tilde{P} \sqcup \{S_t, L\} \sqcup \{R\}\,, \quad \tilde{P} \sqcup \{S_t, R\} \sqcup \{L\}\,, \quad \tilde{P} \sqcup \{p, R\} \sqcup \{q, L\}\,, \quad \tilde{P} \sqcup \{p, L\} \sqcup \{q, R\}\,,$$

with the terms of (97) with $P'' = \tilde{P} \sqcup S_t$.

• If $|S_t| = 1$ with $S_t = \{p\}$, then we use Proposition 15 and Proposition 16 to identify the terms of (96) with $P'' = \tilde{P} \sqcup S_t$ and with P' equal to

$$\tilde{P} \sqcup \{p, L\} \sqcup \{R\}\,, \quad \tilde{P} \sqcup \{p, R\} \sqcup \{L\}$$

with the terms of (97) with $P'' = \tilde{P} \sqcup S_t$.

• The terms of (96) with $P' = \{L\} \sqcup \{R\} \sqcup \tilde{P}$ are equal to the terms of (97) with $P' = \tilde{P}$ by Proposition 14.

The above four cases match all the terms in (96) and (96). □

6. Virasoro constraints for Hilbert schemes of points of surfaces

Let S be a nonsingular projective toric surface, and let

$$X = S \times \mathbb{P}^1\,.$$

As an immediate consequence of Theorem 4 applied to the toric variety X, we obtain the following Virasoro constraints:

$$\forall k \geqslant -1\,, \qquad \Big\langle \mathcal{L}_k^{\mathrm{PT}} \prod_{i=1}^r \mathsf{ch}_{m_i}(\gamma_i \times \mathsf{p}) \Big\rangle_{n[\mathbb{P}^1]}^{X,\mathrm{PT}} = 0\,, \tag{98}$$

where $\gamma_i \in H^*(X)$, $\mathsf{p} \in H^2(\mathbb{P}^1)$ is the point class, and $[\mathbb{P}^1] \in H_2(X)$ is the fiber class.

We can specialize the constraints (98) further to the case of the minimal possible Euler characteristic,

$$P_n(S \times \mathbb{P}^1, n[\mathbb{P}^1]) \cong \mathrm{Hilb}^n(S)\,.$$

The above isomorphism of schemes is defined as follows. A point $\xi \in \mathrm{Hilb}^n(S)$ corresponds to a 0-dimensional subscheme of S of length n. Then,

$$\xi \times \mathbb{P}^1 \subseteq S \times \mathbb{P}^1$$

is a curve embedded in $S \times \mathbb{P}^1$ with Euler characteristic n and curve class $n[\mathbb{P}^1]$. The isomorphism sends ξ to the corresponding stable pair

$$\mathcal{O}_{S\times\mathbb{P}^1} \to \mathcal{O}_{\xi\times\mathbb{P}^1}\,.$$

Since the moduli space of stable pairs is nonsingular of expected dimension

$$\int_{n[\mathbb{P}^1]} c_1(S \times \mathbb{P}^1) = 2n\,,$$

the virtual class is the standard fundamental class here. The result is a new set of Virasoro constraints for tautological classes on $\mathrm{Hilb}^n(S)$.

To write the Virasoro constraints for $\mathrm{Hilb}^n(S)$ explicitly, we first define the corresponding descendent insertions. Let

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathrm{Hilb}^n(S)\times S} \to \mathcal{O}_Z \to 0$$

be the universal sequence associated to the universal subscheme

$$Z \subset S \times \mathrm{Hilb}^n(S)\,.$$

For $\gamma \in H^*(S)$, let

$$\mathsf{ch}_k(\gamma) = -\pi_*\big(\mathsf{ch}_k(\mathcal{I}) \cdot \gamma\big)\,,$$

where π is the projection to $\mathrm{Hilb}^n(S)$. We follow as closely as possible the descendent notation for 3-folds in Section 0.1.

Let $\mathbb{D}(S)$ be the commutative algebra with generators

$$\big\{\, \mathsf{ch}_i(\gamma) \,|\, i \geqslant 0\,,\ \gamma \in H^*(S) \,\big\}$$

following Section 0.2. We define derivations R_k by their actions on the generators:

$$\mathrm{R}_k(\mathsf{ch}_i(\gamma)) = \left(\prod_{n=0}^{k}(i + d - 2 + n)\right)\mathsf{ch}_{i+k}(\gamma)\,, \quad \gamma \in H^{2d}(S)\,.$$

For $k \geqslant -1$, we define differential operators

$$\begin{aligned}\mathrm{L}_k^S \;&=\; -\sum_{a+b=k+2}(-1)^{(d^L+1)(d^R+1))}(a + d^L - 2)!(b + d^R - 2)!\mathsf{ch}_a\mathsf{ch}_b(1)\\ &\quad + \frac{1}{12}\sum_{a+b=k} a!b!\mathsf{ch}_a\mathsf{ch}_b(c_1^2 + c_2) + \mathrm{R}_k\,.\end{aligned}$$

where the sum is over ordered pairs (a, b) with $a, b \geqslant 0$.

Theorem 20. *For all $k \geqslant -1$ and $D \in \mathbb{D}(S)$, we have*

$$\int_{\mathrm{Hilb}^n(S)} \left(\mathrm{L}_k^S + (k+1)!\mathrm{R}_{-1}\mathsf{ch}_{k+1}(\mathsf{p})\right)(D) = 0$$

for all $n \geqslant 0$.

Proof. For clarity, we will use superscripts $\mathsf{ch}_i^{\mathrm{Hilb}}$ and $\mathsf{ch}_i^{\mathrm{PT}}$ here to indicate whether we are referring to descendents on the Hilbert scheme of S as defined above or to stable pairs descendents on $S \times \mathbb{P}^1$ as defined in Section 0.1.

The universal stable pair of $P_n(S \times \mathbb{P}^1, n[\mathbb{P}^1])$ is $\mathbb{F} = \mathcal{O}_{Z \times \mathbb{P}^1}$. Hence,

$$\mathsf{ch}_i(\mathbb{F} - \mathcal{O}_{S \times \mathbb{P}^1 \times \mathrm{Hilb}^n(S)}) = (\rho \times \mathrm{id})^* \mathsf{ch}_i(-\mathcal{I}) \,,$$

where ρ is the projection $\rho : S \times \mathbb{P}^1 \to S$. By the push-pull formula, for $\delta \in H^*(S \times \mathbb{P}^1)$, we have

$$\begin{aligned} \mathsf{ch}_i^{\mathrm{PT}}(\delta) &= \pi_* \left((\rho \times \mathrm{id})^* \left(\mathsf{ch}_i(-\mathcal{I}) \cdot \delta\right)\right) \\ &= \pi_* \left(\mathsf{ch}_i(-\mathcal{I}) \cdot \rho_* \delta\right) \\ &= \mathsf{ch}_i^{\mathrm{Hilb}}(\rho_* \delta) \,. \end{aligned}$$

So, $\mathsf{ch}_i^{\mathrm{PT}}(\gamma \times 1) = 0$, and $\mathsf{ch}_i^{\mathrm{PT}}(\gamma \times \mathsf{p}) = \mathsf{ch}_i^{\mathrm{Hilb}}(\gamma)$.

Since we have the Virasoro constraints (98), we must only check that the composition

$$\mathbb{D}(S) \hookrightarrow \mathbb{D}_{\mathrm{PT}}^{X+} \xrightarrow{\mathcal{L}_k^{\mathrm{PT}}} \mathbb{D}_{\mathrm{PT}}^{X+} \to \mathbb{D}(S) \tag{99}$$

is precisely

$$\mathrm{L}_k^S + (k+1)!\mathrm{R}_{-1}\mathsf{ch}_{k+1}(\mathsf{p}) \,.$$

The first inclusion in (99) is determined by sending generators $\mathsf{ch}_i^{\mathrm{Hilb}}(\gamma)$ to $\mathsf{ch}_i^{\mathrm{PT}}(\gamma \times \mathsf{p})$, and the last map of (99) sends $\mathsf{ch}_i^{\mathrm{PT}}(\delta)$ to $\mathsf{ch}_i^{\mathrm{Hilb}}(\rho_*\delta)$.

The analysis of the composition is straightforward. For the diagonal terms, we note that

$$c_1(X) = 2(1 \times \mathsf{p}) + c_1(S) \times 1$$

and

$$\frac{c_1 c_2}{24}(X) = \mathsf{td}_3(X) = \mathsf{td}_2(S) \times \mathsf{td}_2(\mathbb{P}^1) = \frac{1}{12}(c_1(S)^2 + c_2(S)) \times \mathsf{p} \,.$$

We write the Künneth decomposition of the diagonal as

$$\Delta \cdot 1 = \sum_i \theta_i^L \otimes \theta_i^R \in H^*(S \times S) \,.$$

Then, the Künneth decomposition of $\Delta \cdot c_1 \in H^*(X \times X)$ is

$$2 \sum_i (\theta_i^L \times \mathsf{p}) \otimes (\theta_i^R \times \mathsf{p}) + \cdots \,,$$

where the remaining terms in the dots are killed by ρ_*. The matching of operators then follows from the definition of $\mathcal{L}_k^{\mathrm{PT}}$. ∎

7. GW/PT descendent correspondence: review

7.1. **Vertex operators.** Our goal here is to review the results of [17] and to explain how Theorem 6 can be derived from [17]. The full derivation is postponed to Section 8.

To state the main result of [17], we require negative descendents $\{\mathfrak{a}_k\}$ for $k \in \mathbb{Z}_{<0}$ which are defined to satisfy the Heisenberg relations with positive descendents:

$$[\mathfrak{a}_k(\alpha), \mathfrak{a}_m(\gamma)] = k\delta_{k+m} \int_X \alpha \cup \gamma \,. \tag{100}$$

The descendents $\{\mathfrak{a}_k\}$ for $k \in \mathbb{Z}\backslash\{0\}$ generate the $H^*(X)$-algebra Heis_X.

For curve class $\beta \in H_2(X)$, there is a geometrically defined Gromov-Witten evaluation $\langle \cdot \rangle_\beta$ map on the algebra generated by the non-negative descendents. We can extend the evaluation map to the whole algebra Heis_X by defining

$$\left\langle \mathfrak{a}_k(\gamma)\Phi \right\rangle_\beta^{X,\mathrm{GW}} = \left[\int_X \big(-c_1\delta_{k+1} + \delta_{k+2} iu \big) \cdot \gamma \right] \left\langle \Phi \right\rangle_\beta^{X,\mathrm{GW}} , \quad k < 0 \,.$$

We assemble the operators $\mathfrak{a}_k$ in the following generating function:

$$\phi(z) = \sum_{n>0} \frac{\mathfrak{a}_n}{n} \left(\frac{\imath z c_1}{u} \right)^{-n} + \frac{1}{c_1} \sum_{n<0} \frac{\mathfrak{a}_n}{n} \left(\frac{\imath z c_1}{u} \right)^{-n} . \tag{101}$$

The main objects of study in [17] are the vertex operators

$$\mathrm{H}^{\mathsf{GW}}(x) = \sum_{k=0}^{\infty} \mathrm{H}_k^{\mathsf{GW}} x^{k+1} = \mathrm{Res}_{w=\infty} \left(\frac{\sqrt{dydw}}{y-w} : e^{\theta\phi(y) - \theta\phi(w)} : \right) , \tag{102}$$

where y, w, and x satisfy the constraints

$$ye^y = we^w e^{-x/\theta} , \quad \theta^{-2} = -c_2(T_X) \,. \tag{103}$$

Here, $\mathrm{Res}_{w=\infty}$ denotes $\frac{1}{2\pi\imath}$ times the integral along a small loop around $w = \infty$.

Normally ordered monomials

$$\mathfrak{a}_{i_1}\mathfrak{a}_{i_2} \ldots \mathfrak{a}_{i_k}, \quad i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k,$$

form a linear basis of Heis. Respectively, we use $: \cdot :$ for the normal ordering operation

$$: \prod_j \mathfrak{a}_{i_j} : \; = \mathfrak{a}_{i_1}\mathfrak{a}_{i_2} \ldots \mathfrak{a}_{i_k}, \quad i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k,$$

Extended $H^*(X)$-linearly to the whole algebra Heis_X.

Let us notice that the equation (103) as well as the vertex operator (102) have symmetry

$$y \mapsto w, \quad w \mapsto y, \quad \theta \mapsto -\theta, \quad x \mapsto x.$$

This symmetry implies that the only even powers of θ appear in the expansion of (102) (see Lemma 15 from [17] for more discussions and further properties of the vertex operator).

The operators $\mathrm{H}_k^{\mathsf{GW}}$ are mutually commutative. To obtain explicit formulas for $\mathrm{H}_k^{\mathsf{GW}}$, we use the Lambert function to solve equation (103) and express y in terms of x, w. The integral in the definition of $\mathrm{H}_k^{\mathsf{GW}}$ can be interpreted as an extraction of the coefficient of w^{-1}. The descendent classes

$$\mathrm{H}_k^{\mathsf{GW}}(\gamma) \in \mathsf{Heis}_X$$

are then obtained using the Sweedler coproduct. We also use the Sweedler coproduct conventions in

$$\mathrm{H}_{\vec{k}}^{\mathsf{GW}}(\gamma) = \prod_{i=1}^{m} \mathrm{H}_{k_i}^{\mathsf{GW}}(\gamma)\,, \qquad \vec{k} = (k_1, \ldots, k_m)\,. \tag{104}$$

In the Sweedler conventions [10], we abbreviate notation for the intersection with the small diagonal $\Delta_n \subset X^n$ with the pull-back of a class $\gamma \in H^*(X)$:

$$H^*(X^n) \ni [\Delta_n] \cdot \gamma = \sum_k \gamma_1^k \otimes \ldots \gamma_n^k = \gamma_{(1)} \otimes \cdots \otimes \gamma_{(n)}\,.$$

Thus, the formula (104) expands as

$$\prod_{i=1}^{m} \mathrm{H}_{k_i}^{\mathsf{GW}}(\gamma) = \prod_{i=1}^{m} \mathrm{H}_{k_i}^{\mathsf{GW}}(\gamma_{(i)})\,.$$

7.2. **Stable pairs.** The stable pairs analogues of the operators $\mathrm{H}_{\vec{k}}^{\mathsf{GW}}(\gamma)$ are products of $\mathrm{H}_k^{\mathrm{PT}}(\gamma)$ defined as follows.

The classes $\mathrm{H}_k^{\mathrm{PT}}(\gamma)$ are linear combinations of descendents on the moduli spaces of stable pairs. Let

$$\mathrm{H}_k^{\mathrm{PT}}(\gamma) = \pi_*\left(\mathrm{H}_k^{\mathrm{PT}} \cdot \gamma\right) \in \bigoplus_{n \in \mathbb{Z}} H^*(P_n(X, \beta))\,,$$

where the classes $\mathrm{H}_k^{\mathrm{PT}} \in \bigoplus_{n\in\mathbb{Z}} H^*(X \times P_n(X,\beta))$ are defined by

$$\begin{aligned} \mathrm{H}^{\mathrm{PT}}(x) &= \sum_{k=0}^{\infty} x^{k+1} \mathrm{H}_k^{\mathrm{PT}} \\ &= \mathcal{S}^{-1}\left(\frac{x}{\theta}\right) \sum_{k=0}^{\infty} x^k \mathrm{ch}_k(\mathbb{F} - \mathcal{O})\,, \end{aligned}$$

where

$$\theta^{-2} = -c_2(T_X)\,, \qquad \mathcal{S}(x) = \frac{e^{x/2} - e^{-x/2}}{x}\,.$$

In particular, we have

$$\mathrm{H}_k^{\mathrm{PT}} = \mathrm{ch}_{k+1}(\mathbb{F}) + \frac{c_2}{24}\mathrm{ch}_{k-1}(\mathbb{F}) + \frac{7c_2^2}{5760}\mathrm{ch}_{k-3}(\mathbb{F}) + \ldots\,.$$

7.3. **Equivariant correspondence.** All the definitions and construction introduced in Section 7.1 have canonical lifts to the equivariant setting with respect to a group action on the variety X. We review here the equivariant GW/PT descendent correspondence [25].

The most natural setting is the capped vertex formalism of [15, 25] which we review briefly here. Let the 3-dimensional torus

$$\mathsf{T} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$$

act on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ diagonally. The tangent weights of the T-action at the point

$$\mathsf{p} = 0 \times 0 \times 0 \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

are s_1, s_2, s_3. The T-equivariant cohomology ring of a point is

$$H_{\mathsf{T}}(\bullet) = \mathbb{C}[s_1, s_2, s_3]\,.$$

We have the following factorization of the restriction of class $c_1c_2 - c_3$ of X to p,

$$c_1c_2 - c_3 = (s_1 + s_2)(s_1 + s_3)(s_2 + s_3)\,,$$

where $c_i = c_i(T_X)$.

Let $U \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ be the T-equivariant 3-fold obtained by removing the three T-equivariant lines L_1, L_2, L_3 passing through the point $\infty \times \infty \times \infty$. Let $D_i \subset U$ be the divisor with i^{th} coordinate ∞. For a triple of partitions μ_1, μ_2, μ_3, let

$$\left\langle \prod_i \tau_{k_i}(\mathsf{p}) \,\Big|\, \mu_1, \mu_2, \mu_3 \right\rangle_{U,D}^{\mathsf{GW},\mathsf{T}}, \quad \left\langle \prod_i \mathrm{ch}_{k_i}(\mathsf{p}) \,\Big|\, \mu_1, \mu_2, \mu_3 \right\rangle_{U,D}^{\mathsf{PT},\mathsf{T}} \tag{105}$$

denote the generating series of the T-equivariant relative Gromov-Witten and stable pairs invariants of the pair

$$D = \cup_i D_i \subset U$$

with relative conditions μ_i along the divisor D_i.

The stable maps spaces are always taken with no contracted connected components of genus great than or equal to 2. The series (105) are the *capped descendent vertices* following the conventions of [17].

Theorem 21. [17] *After the change of variables $-q = e^{iu}$ the following correspondence between the 2-leg capped descendent vertices holds:*

$$\left\langle \prod_i \mathrm{H}^{\mathsf{GW}}_{k_i}(\mathsf{p}) \,\Big|\, \mu_1, \mu_2, \varnothing \right\rangle_{U,D}^{\mathsf{GW},\mathsf{T}} = q^{-|\mu_1|-|\mu_2|} \left\langle \prod_i \mathrm{H}^{\mathrm{PT}}_{k_i}(\mathsf{p}) \,\Big|\, \mu_1, \mu_2, \varnothing \right\rangle_{U,D}^{\mathrm{PT},\mathsf{T}}$$

mod $(s_1 + s_3)(s_2 + s_3)$.

The result of Theorem 21 has two defects. Since the third partition is empty, the result only covers the *2-leg* case. Moreover, the equality of the correspondence is not proven exactly, but only mod $(s_1 + s_3)(s_2 + s_3)$. For the 1-leg vertex with partitions $(\mu_1, \varnothing, \varnothing)$, Theorem 21 can be restricted in two ways to obtain the equality of the correspondence

$$\text{mod} \ \ (s_1 + s_3)(s_1 + s_2)(s_2 + s_3)\,.$$

7.4. **Non-equivariant limit.** By following the arguments of [25], a non-equivariant GW/PT descendent correspondence for stationary insertions is derived in [17]. For our statements, we will follow as closely as possible the notation of [17, 25].

Let Heis^c be the Heisenberg algebra with generators $\mathfrak{a}_{k\in\mathbb{Z}\setminus\{0\}}$, coefficients $\mathbb{C}[c_1, c_2]$, and relations

$$[\mathfrak{a}_k, \mathfrak{a}_m] = k\delta_{k+m}c_1c_2\,.$$

Let $\mathsf{Heis}^c_+ \subset \mathsf{Heis}^c$ be the subalgebra generated by the elements $\mathfrak{a}_{k>0}$, and define the $\mathbb{C}[c_1, c_2]$-linear map

$$\mathsf{Heis}^c \to \mathsf{Heis}^c_+\,, \quad \Phi \mapsto \widehat{\Phi} \tag{106}$$

by $\widehat{\mathfrak{a}}_k = \mathfrak{a}_k$ for $k > 0$ and

$$\widehat{\mathfrak{a}_k\Phi} = (-c_1\delta_{k+1} + \delta_{k+2}iu)\widehat{\Phi}\,, \quad \text{for } k < 0\,. \tag{107}$$

When restricted to the subalgebra Heis^c_+, the $\mathbb{C}[c_1, c_2]$-linear map (106) is an isomorphism.

For a nonsingular projective 3-fold X and classes $\gamma_1, \ldots, \gamma_l \in H^*(X)$, the hat operation make no difference inside the Gromov-Witten bracket,

$$\langle \mathrm{H}^{\mathrm{GW}}_{\vec{k}}(\gamma)\rangle^{X,\mathrm{GW}}_{\beta} = \langle \widehat{\mathrm{H}}^{\mathrm{GW}}_{\vec{k}}(\gamma)\rangle^{\mathrm{GW}}_{\beta}, \tag{108}$$

because the treatment of the negative descendents on the left side is compatible with the treatment of the negative descendents by the hat operation.

Let $\vec{k} = (k_1, \ldots, k_l)$ be a vector of non-negative integers. Following [25], we define the following element of Heis^c_+:

$$\widetilde{\mathrm{H}}_{\vec{k}} = \frac{1}{(c_1c_2)^{l-1}} \sum_{\text{set partitions } P \text{ of } \{1,\ldots,l\}} (-1)^{|P|-1}(|P|-1)! \prod_{S\in P} \widehat{\mathrm{H}}^{\mathsf{GW}}_{\vec{k}_S}\,,$$

where $\mathrm{H}^{\mathsf{GW}}_{\vec{k}_S} = \prod_{i\in S} \mathrm{H}^{\mathsf{GW}}_{k_i}$ and the element $\mathrm{H}^{\mathrm{GW}}_{k} \in \mathsf{Heis}^c$ is a linear combination of monomials of $\mathfrak{a}_i$, the expression is given by (102).

For classes $\gamma_1, \ldots, \gamma_l \in H^*(X)$ and a vector $\vec{k} = (k_1, \ldots, k_l)$ of non-negative integers, we define

$$\overline{\mathrm{H}_{k_1}(\gamma_1)\ldots\mathrm{H}_{k_l}(\gamma_l)} = \sum_{\text{set partitions } P \text{ of } \{1,\ldots,l\}} \prod_{S\in P} \widetilde{\mathrm{H}}_{\vec{k}_S}(\gamma_S)\,,$$

where $\gamma_S = \prod_{i\in S}\gamma_i$.

Theorem 22. [17] *Let X be a nonsingular projective toric 3-fold, and let $\gamma_i \in H^{\geqslant 2}(X, \mathbb{C})$. After the change of variables $-q = e^{iu}$, we have*

$$\Big\langle \overline{\mathrm{H}_{k_1}(\gamma_1)\ldots\mathrm{H}_{k_l}(\gamma_l)} \Big\rangle^{\mathsf{GW}}_{\beta} = q^{-d/2} \Big\langle \mathrm{H}^{\mathrm{PT}}_{k_1}(\gamma_1)\ldots\mathrm{H}^{\mathrm{PT}}_{k_l}(\gamma_l)\Big\rangle^{\mathsf{PT}}_{\beta}\,,$$

where $d = \int_\beta c_1$.

7.5. **Examples for** $X = \mathbb{P}^3$**.** The prefactor $\mathcal{S}^{-1}\left(\frac{x}{\theta}\right)$ in front of $\sum_{k=0}^{\infty} x^k \mathsf{ch}_k(\mathbb{F}-\mathcal{O})$ in the formula for $\mathrm{H}^{\mathrm{PT}}(x)$ has an expansion which the following initial terms:

$$1 + \frac{c_2}{24}x^2 + \frac{7c_2^2}{5760}x^4 + \dots .$$

Therefore, the non-equivariant limit of $\mathrm{H}_k^{\mathrm{PT}}(\gamma)$ is

$$\left(\mathrm{ch}_{k+1}(\gamma) + \frac{1}{24}\mathrm{ch}_{k-1}(\gamma \cdot c_2)\right) .$$

On the Gromov-Witten side of the correspondence, we have

$$\langle \mathrm{H}_1^{\mathrm{GW}}(\gamma)\Phi\rangle = \langle \mathfrak{a}_1(\gamma)\Phi\rangle , \qquad \langle \mathrm{H}_2^{\mathrm{GW}}(\gamma)\Phi\rangle = \frac{1}{2}\langle \mathfrak{a}_2(\gamma)\Phi\rangle ,$$

$$\langle \mathrm{H}_3^{\mathrm{GW}}(\gamma)\Phi\rangle = \frac{1}{6}\langle \mathfrak{a}_3(\gamma)\Phi\rangle + \frac{1}{24u^2}\langle c_1^2 c_2 \cdot \Phi\rangle ,$$

$$\langle \mathrm{H}_4^{\mathrm{GW}}(\gamma)\Phi\rangle = \frac{1}{24}\langle \mathfrak{a}_4(\gamma)\Phi\rangle - \frac{i}{12u}\langle \mathfrak{a}_1^2(c_1\cdot\gamma)\Phi\rangle - \frac{5i}{144u^3}\langle c_1^3 c_2 \cdot \Phi\rangle ,$$

$$\begin{aligned}\langle \mathrm{H}_5^{\mathrm{GW}}\Phi\rangle = \frac{1}{120}\langle \mathfrak{a}_5(\gamma)\Phi\rangle - \frac{i}{24u}\langle \mathfrak{a}_1\mathfrak{a}_2(c_1\cdot\gamma)\Phi\rangle - \frac{1}{48u^2}\langle \mathfrak{a}_1^2(c_1^2\cdot\gamma)\Phi\rangle \\ + \frac{1}{24u^2}\langle \mathfrak{a}_1(c_1^2c_2\cdot\gamma)\Phi\rangle - \frac{1}{64u^4}\langle c_1^4 c_2\cdot\Phi\rangle .\end{aligned}$$

The operators $\mathfrak{a}_k$ are expressed in terms of standard descendents[29]

$$\begin{aligned} \mathfrak{a}_1 &= \tau_0 - \frac{c_2}{24} , \\ iu\mathfrak{a}_2/2 &= \tau_1 + c_1\cdot\tau_0 , \\ -u^2\mathfrak{a}_3/3 &= 2\tau_2 + 3c_1\cdot\tau_1 + c_1^2\cdot\tau_0 , \\ -iu^3\mathfrak{a}_4/4 &= 6\tau_3 + 11c_1\cdot\tau_2 + 6c_1^2\tau_1 + c_1^3\cdot\tau_0 , \\ u^4\mathfrak{a}_5/5 &= 24\tau_4 + 50c_1\cdot\tau_3 + 35c_1^2\cdot\tau_2 + 10c_1^3\cdot\tau_1 + c_1^4\cdot\tau_0 . \end{aligned} \tag{109}$$

The descendent correspondence of Theorem 22 implies relations for stable pairs and Gromov-Witten invariants of $\mathbf{P}^3$. For example, for β of degree 1,

$$\begin{aligned} -iq^{-2}\langle \mathrm{ch}_5(\mathsf{L})\rangle &= \left(\frac{1}{u^3}\langle\tau_3(\mathsf{L})\rangle + \frac{22}{3u^3}\langle\tau_2(\mathsf{p})\rangle - \frac{1}{3u}\langle\tau_0\tau_0(\mathsf{p})\rangle\right) , \\ q^{-2}\left(\langle \mathrm{ch}_6(\mathsf{H})\rangle + \frac{1}{4}\langle \mathrm{ch}_4(\mathsf{p})\rangle\right) &= \left(\frac{1}{u^4}\langle\tau_4(\mathsf{H})\rangle + \frac{25}{3u^4}\langle\tau_3(\mathsf{L})\rangle + \frac{70}{3u^4}\langle\tau_2(\mathsf{p})\rangle \right. \\ &\quad \left. -\frac{1}{3u^2}\langle\tau_0\tau_1(\mathsf{L})\rangle + \frac{5}{3u^2}\langle\tau_0\tau_0(\mathsf{p})\rangle\right) . \end{aligned} \tag{110}$$

Here, p is the class of point, L is the class of line and H is the class of hyperplane.[30]

[29] For $\mathfrak{a}_1(\gamma)$, the term $-\frac{c_2}{24}$ on the right is the constant $-\frac{1}{24}\int_X c_2\gamma$.

[30] We can also check the relations (110) numerically up to u^8 with the help of Gathmann's code on the Gromov-Witten side and previously known computations for stable pairs [20].

7.6. **Residues.** To complete our proof of Theorem 6, we will compute the residues (102). More precisely, we will prove the following result.

Proposition 23. *For* $k_i \in \mathbb{Z}_{\geqslant 0}$ *and* $\gamma_i \in H^{\geqslant 2}(X)$ *such that* $\widetilde{\mathsf{ch}}_{k_i+2}(\gamma_i) \in \mathbb{D}^{X\star}_{\mathrm{PT}}$, *we have:*

$$\begin{aligned}
\widetilde{\mathrm{H}}_{k_1+1}(\gamma_1) &= \mathfrak{C}^{\circ}(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1))\,,\\
\widetilde{\mathrm{H}}_{k_1+1,k_2+1}(\gamma_1\cdot\gamma_2) &= \mathfrak{C}^{\circ}(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2))\,,\\
\widetilde{\mathrm{H}}_{k_1+1,k_2+1,k_3+1}(\gamma_1\cdot\gamma_2\cdot\gamma_3) &= \mathfrak{C}^{\circ}(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma_1)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma_2)\widetilde{\mathsf{ch}}_{k_3+2}(\gamma_3))\,,
\end{aligned}$$

where the right side is defined by (13)-(15).

8. Residue computation

8.1. **Preliminary computations.** Before starting the proof of the Proposition 23, we compute the expansion of the terms of the residue formula (102).

Consider first the constraint equation (103). Solutions of the equation are formal power series in the variable

$$r = 1/\theta\,, \qquad \theta^{-2} = -c_2(T_X)\,.$$

We can solve the constraint equation iteratively in powers of r. Indeed, modulo r^1, the constraint equation implies $w = y$, and we start the expansion by

$$w(x,y) = y + O(r)\,.$$

To find the next term of r in the expansion of $w(x,y)$, we substitute

$$w(x,y) = y + f_1(x,y)r$$

into (103) and expand the result of the substitution in powers of r. The coefficient of r^1 in the expansion gives a linear equation which determines f_1. After iterating the above procedure three times, we obtain

$$w(x,y) = y - xr\frac{y}{y+1} + (xr)^2\frac{y}{2(y+1)^3} + (xr)^3\frac{2y-1}{6(y+1)^5} + O(r^4)\,. \tag{111}$$

To see the expansion of the residue (102) has positive powers of $t = c_1$, we use a change variables:

$$y = v/t\,. \tag{112}$$

The residue with respect to w on right side of (102) is converted to a residue with respect to y via (111). Using (112), we will compute the residue with respect to v.

In the new variables, we have

$$\sqrt{dwdy} = \left(1 - \frac{xrt}{2(v+t)} - \frac{(xr)^2t^3(4v-t)}{8(v+t)^4}\right)\frac{dv}{t} + O(r^3)\,.$$

After we normal order the elements of the Heisenberg algebra in the expression for the vertex operator $\mathrm{H}^{\mathrm{GW}}(x)$, the negative Heisenberg operators end up next to the vacuum

$\langle\,|$ inside the bracket $\langle\cdot\rangle^{\mathrm{GW}}$. Relation (107), which governs interaction with $\langle\,|$, yields the following factor in the expression under the residue:

$$\mathrm{E} = \exp\left(-\frac{t}{2u}\left(\frac{w(y)^2-y^2}{r}\right)-\frac{t}{u}\left(\frac{w(y)-y}{r}\right)\right) = \exp\left(\frac{xv}{u}\right)\left(1-\frac{trx^2v}{2u(v+t)}+\frac{t^2r^2(3xv^2+3txv+4t^2u)}{24u(v+t)^3}\right)+O(r^3)\,. \tag{113}$$

The inverse of $y-w$ in (102) becomes the factor:

$$\mathrm{D} = -\frac{r}{w(y)-y} = \frac{v+t}{v}\left(1+\frac{t^2rx}{2(v+t)^2}+\frac{t^3r^2x^2(4v+t)}{12(v+t)^4}\right)+O(r^3)\,. \tag{114}$$

The elements of the Heisenberg algebra that participate in the residue formula are packed into the vertex operator:

$$\mathrm{V}=\mathrm{V}_+\cdot\mathrm{V}_-\,,\quad \mathrm{V}_+(x,y)=\exp\left(\frac{1}{r}\sum_{n>0}\frac{\mathfrak{a}_n}{n(\imath ut)^n}(y^{-n}-w(y)^{-n})\right)\,,$$
$$\mathrm{V}_-(x,y)=\exp\left(\frac{1}{rt}\sum_{n<0}\frac{\mathfrak{a}_n}{n(\imath ut)^n}(y^{-n}-w(y)^{-n})\right)\,.$$

Thus we need to compute the difference of powers in the expression for the vertex operators. Using formula for $w(y)$ (111), we obtain:

$$\frac{(yt)^{-n}-(w(y)t)^{-n}}{t^nr} = \frac{nxt}{v^n(v+t)}+nx^2rt^2\frac{((n+1)v+nt)}{v^n(v+t)^3} + nx^3r^2t^3\frac{n((n+1)(n+2)v^2+(2n^2+3n-1)tz+n^2t^2)}{6v^n(v+t)^5}+O(r^3)\,. \tag{115}$$

The above calculations yield the leading terms of all algebraic expressions occurring in formula (102) for the vertex operator $\mathrm{H}^{\mathrm{GW}}(x)$. As we will see in Section 8.2, the knowledge of these leading terms almost immediately leads to the simplest case of the descendent correspondence (13). For the other two cases (14) and (15), we must analyze the interaction of two and three vertex operators $\mathrm{H}^{\mathrm{GW}}(x)$. We apply standard vertex operator techniques to complete the proof of Proposition 23 in Section 8.2.

8.2. **Proof of Proposition 23.**

8.2.1. *Case* $\widetilde{\mathrm{H}}_{k_1+1}(\gamma_1)$. We start with the proof of the formula for the self-reaction. We must analyze the r expansion of the residue

$$\widetilde{\mathrm{H}}(x)=\widehat{\mathrm{H}}^{\mathrm{GW}}(x)=\mathrm{Res}_{v=\infty}\frac{1}{t}\mathrm{E}\cdot\mathrm{D}\cdot\mathrm{V}_+\,. \tag{116}$$

More precisely, we must compute the coefficients of

$$r^it^j\,,\quad i+j\leqslant 2\,.$$

By the argument of [17, Section 3.2], the coefficient of rt^j vanishes. From the computations of the v expansions (111), (113), (114) and (115), the terms in front of r^i, $i > 0$ are proportional to t. The expression under the residue sign becomes:

$$\exp\left(\frac{xv}{u}\right)\left(\frac{v+t}{t} + x\Sigma + \frac{x^2 t}{v+t}\Sigma^2 + \frac{x^3t^2}{(v+t)^2}\Sigma^3\right) + O(t^3) + tO(r^2)\,, \quad \Sigma = \sum_{n>0}\frac{\mathfrak{a}_n}{(\imath uv)^n}\,.$$

After applying the residue operation to the last expression, we obtain the terms of formula (13) in the coefficients of the x-expansion. □

8.2.2. *Case* $\widetilde{\mathrm{H}}_{k_1+1,k_2+1}(\gamma_1\cdot\gamma_2)$. We show next that the double interaction term yields formula (14). The new computation that is needed for understanding the interaction term is $\widehat{\mathrm{H}}_{k_1,k_2}$. It is convenient to assemble the expressions into a generating series $\widehat{\mathrm{H}}(x_1,x_2)$.

To compute $\widehat{\mathrm{H}}(x_1,x_2)$, we must move all negative Heisenberg operators in the product of the vertex operators $\mathrm{H}^{\mathrm{GW}}(x_1)\mathrm{H}^{\mathrm{GW}}(x_2)$ to the left, next to the vacuum $\langle\,|$. We use the standard vertex operator commutation relation to perform this reshuffling:

$$\mathrm{V}_+(x_1,y_1)\mathrm{V}_-(x_2,y_2) = \mathrm{B}(x_1,y_1,x_2,y_2)\mathrm{V}_-(x_2,y_2)\mathrm{V}_+(x_1,y_1)\,, \tag{117}$$

$$\mathrm{B} = \frac{(w_2-y_1)(y_2-w_1)}{(y_2-y_1)(w_2-w_1)}\,,$$

where $w_i = w(x_i,y_i)$. Using the computations of Section 8.1, we derive the following expansion:

$$\mathrm{B} = 1 - \frac{r^2y_1y_2x_1x_2}{(y_1-y_2)^2(y_1+1)(y_2+1)} + O(r^3)\,.$$

The negative Heisenberg operators interact with the vacuum $\langle\,|$. We obtain:

$$\widehat{\mathrm{H}}(x_1,x_2) = \mathrm{Res}_{y_1=\infty}(\mathrm{Res}_{y_2=\infty}(\mathrm{V}_+^{(1)}\mathrm{V}_+^{(2)}\mathrm{D}^{(1)}\mathrm{D}^{(2)}\mathrm{E}^{(1)}\mathrm{E}^{(2)}\mathrm{B}^{(12)}))\,,$$

where $\mathrm{V}_+^{(i)} = \mathrm{V}_+(x_i,y_i)$, $\mathrm{D}^{(i)} = \mathrm{D}(x_i,y_i)$, $\mathrm{E}^{(i)} = \mathrm{E}(x_i,y_i)$.

From (116), we see

$$\widehat{\mathrm{H}}(x_1)\widehat{\mathrm{H}}(x_2) = \left(\mathrm{Res}_{y_1=\infty}\,\mathrm{E}^{(1)}\cdot\mathrm{D}^{(1)}\cdot\mathrm{V}_+^{(1)}\right)\left(\mathrm{Res}_{y_2=\infty}\,\mathrm{E}^{(2)}\cdot\mathrm{D}^{(2)}\cdot\mathrm{V}_+^{(2)}\right) = \mathrm{Res}_{y_1=\infty}(\mathrm{Res}_{y_2=\infty}(\mathrm{V}_+^{(1)}\mathrm{V}_+^{(2)}\mathrm{D}^{(1)}\mathrm{D}^{(2)}\mathrm{E}^{(1)}\mathrm{E}^{(2)}))\,,$$

where the second equality holds because $V_+^{(i)}$ commute. We conclude, after the change of variables, the generating function $\widetilde{\mathrm{H}}(x_1,x_2)$ for $\widetilde{\mathrm{H}}_{k_1,k_2}$ is given by

$$\widetilde{\mathrm{H}}(x_1,x_2) = \frac{1}{r^2t}\left(\widehat{\mathrm{H}}(x_1,x_2) - \widehat{\mathrm{H}}(x_1)\widehat{\mathrm{H}}(x_2)\right) = \mathrm{Res}(\mathrm{V}_+^{(1)}\mathrm{V}_+^{(2)}\mathrm{D}^{(1)}\mathrm{D}^{(2)}\mathrm{E}^{(1)}\mathrm{E}^{(2)}\widetilde{\mathrm{B}}^{(12)})/(r^2t^3)\,,$$

where $\mathrm{Res} = \mathrm{Res}_{v_1=\infty}\mathrm{Res}_{v_2=\infty}$ and $\widetilde{\mathrm{B}}^{(12)} = \mathrm{B}^{(12)} - 1$. By expanding the scalar factor

$$\mathrm{D}^{(1)}\mathrm{D}^{(2)}\mathrm{E}^{(1)}\mathrm{E}^{(2)}\widetilde{\mathrm{B}}^{(12)}/(r^2t^3)$$

in the operator inside the residue operation, we obtain:

$$\text{(118)}\quad \frac{tv_1v_2x_1x_2}{(v_1-v_2)^2(v_1+t)(v_2+t)}\exp\left(\frac{x_1v_1+x_2v_2}{u}\right)\left(\frac{v_1+t}{t}+x_1\Sigma^{(1)}+\frac{x_1^2t}{v_2+t}\Sigma^{(1)}\Sigma^{(1)}\right)$$
$$\left(\frac{v_2+t}{t}+x_2\Sigma^{(2)}+\frac{x_2^2t}{v_2+t}\Sigma^{(2)}\Sigma^{(2)}\right)+O(t^2)+O(r^2)\,.$$

The residue of the coefficient in front of t^{-1} in (118) vanishes. The coefficient in front of t^0 is

$$\exp\left(\frac{x_1v_1+x_2v_2}{u}\right)\frac{x_1x_2}{(v_1-v_2)^2}\left(v_2(1+x_1\Sigma^{(1)})+v_1(1+x_2\Sigma^{(2)})\right).$$

After applying the Res operation, we obtain:

$$\mathrm{Res}_{v_1=\infty}\,\mathrm{Res}_{v_2=\infty}\exp\left(\frac{x_1v_1+x_2v_2}{u}\right)\frac{x_1^2x_2v_2}{(v_1-v_2)^2}\Sigma^{(1)}.$$

The coefficient in front of $x_1^{k_1+2}x_2^{k_2+2}$ in the last expression matches with the $\mathfrak{a}$-linear terms of right side of (14) that are proportional to c_1^0.

Finally, we compute the coefficient in front of t^1 in (118):

$$\frac{x_1x_2}{(v_1-v_2)^2}\exp\left(\frac{x_1v_1+x_2v_2}{u}\right)\Big[x_1x_2\Sigma^{(1)}\Sigma^{(2)}+x_1^2v_2\Sigma^{(1)}\Sigma^{(1)}+x_2^2v_1\Sigma^{(2)}\Sigma^{(2)}$$
$$+\left(\frac{1}{v_1}+\frac{1}{v_2}\right)\left(v_2(1+x_1\Sigma^{(1)})+v_1(1+x_2\Sigma^{(2)})\right)\Big]\,.$$

The residue of the terms from the first line of the last expression form the generating function of the $\mathfrak{a}$-quadratic terms of the right hand side of (14). The residue of the terms from the second line of the last expression form the generating function of the c_1-proportional $\mathfrak{a}$-linear terms of the right side of (14). □

8.2.3. *Case* $\widetilde{\mathrm{H}}_{k_1+1,k_2+1,k_3+1}(\gamma_1\cdot\gamma_2\cdot\gamma_3)$. Finally, we must analyze the triple interaction. T he computation here is parallel to computations in Sections 8.2.1 and 8.2.2. The new ingredient for the triple bumping reaction is the residue formula:

$$\widehat{\mathrm{H}}(x_1,x_2,x_3)=\mathrm{Res}\left(\mathrm{V}_+^{(1)}\mathrm{V}_+^{(2)}\mathrm{V}_+^{(3)}\mathrm{D}^{(1)}\mathrm{D}^{(2)}\mathrm{D}^{(3)}\mathrm{E}^{(1)}\mathrm{E}^{(2)}\mathrm{E}^{(3)}\mathrm{B}^{(12)}\mathrm{B}^{(23)}\mathrm{B}^{(13)}/(r^4t^5)\right)$$

for the generating function of the operators $\widehat{\mathrm{H}}_{k_1,k_2,k_3}$. Here and below Res stands for the triple residue

$$\mathrm{Res}_{v_1=\infty}\,\mathrm{Res}_{v_2=\infty}\,\mathrm{Res}_{v_3=\infty}\ .$$

The generating function $\widetilde{\mathrm{H}}(x_1,x_2,x_3)$ for the operators $\widetilde{\mathrm{H}}_{k_1,k_2,k_3}$ is given by:

$$\widehat{\mathrm{H}}(x_1,x_2,x_3)-\widehat{\mathrm{H}}(x_1,x_2)\widehat{\mathrm{H}}(x_3)-\widehat{\mathrm{H}}(x_1,x_3)\widehat{\mathrm{H}}(x_2)-\widehat{\mathrm{H}}(x_2,x_3)\widehat{\mathrm{H}}(x_1)+2\widehat{\mathrm{H}}(x_1)\widehat{\mathrm{H}}(x_2)\widehat{\mathrm{H}}(x_3)\,.$$

We expand the above as

$$\frac{1}{r^4t^5}\operatorname{Res}\Big(\mathrm{V}_+^{(1)}\mathrm{V}_+^{(2)}\mathrm{V}_+^{(3)}\mathrm{D}^{(1)}\mathrm{D}^{(2)}\mathrm{D}^{(3)}\mathrm{E}^{(1)}\mathrm{E}^{(2)}\mathrm{E}^{(3)} \Big(\widetilde{\mathrm{B}}^{(12)}\widetilde{\mathrm{B}}^{(23)}\widetilde{\mathrm{B}}^{(13)}+\widetilde{\mathrm{B}}^{(12)}\widetilde{\mathrm{B}}^{(23)}+\widetilde{\mathrm{B}}^{(12)}\widetilde{\mathrm{B}}^{(13)}+\widetilde{\mathrm{B}}^{(23)}\widetilde{\mathrm{B}}^{(13)}\Big)\Big)\,.$$

Since $\widetilde{\mathrm{B}}^{(12)}\widetilde{\mathrm{B}}^{(23)}\widetilde{\mathrm{B}}^{(13)}$ is proportional to r^6, we can write the last expression as

$$\frac{1}{r^4t^5}\operatorname{Res}\Big(\mathrm{V}_+^{(1)}\mathrm{V}_+^{(2)}\mathrm{V}_+^{(3)}\mathrm{D}^{(1)}\mathrm{D}^{(2)}\mathrm{D}^{(3)}\mathrm{E}^{(1)}\mathrm{E}^{(2)}\mathrm{E}^{(3)}\Big(\widetilde{\mathrm{B}}^{(12)}\widetilde{\mathrm{B}}^{(23)}+\widetilde{\mathrm{B}}^{(12)}\widetilde{\mathrm{B}}^{(13)}+\widetilde{\mathrm{B}}^{(23)}\widetilde{\mathrm{B}}^{(13)}\Big)\Big)$$

up to $O(r^2)$.

After expanding the expression inside Res, including the prefactor $\frac{1}{r^4t^5}$, we obtain:

$$t^2\left(\frac{v_1+t}{t}+x_1\Sigma^{(1)}\right)\left(\frac{v_2+t}{t}+x_2\Sigma^{(2)}\right)\left(\frac{v_3+t}{t}+x_3\Sigma^{(3)}\right) \times\exp\left(\frac{x_1v_1+x_2v_2+x_3v_3}{u}\right)\cdot\Big(f(12;23)+f(23;31)+f(31;12)\Big)\ +\ O(t)\ ,$$

where

$$f(ij;jk)=\frac{v_iv_j^2v_kx_ix_j^2x_k}{(v_i-v_j)^2(v_j-v_k)^2(v_i+t)(v_j+t)^2(v_k+t)}\,.$$

The application of Res to the coefficient in front of t^{-1} in the last expression yields zero. On the other hand, the coefficient in front of t^0 equals

$$x_1x_2x_3\left(v_2v_3(1+x_1\Sigma^{(1)})+v_1v_3(1+x_2\Sigma^{(2)})+v_1v_2(1+x_3\Sigma^{(3)})\right)$$

$$\times\exp\left(\frac{x_1v_1+x_2v_2+x_3v_3}{u}\right)$$

$$\times\left(\frac{x_2}{(v_1-v_2)^2(v_2-v_3)^2}+\frac{x_3}{(v_1-v_3)^2(v_3-v_2)^2}+\frac{x_1}{(v_3-v_1)^2(v_1-v_2)^2}\right)\,.$$

The result of application of Res is therefore equal to the generating function of the right side of (15). ☐

9. Degree 1 series for $\mathbb{P}^3$

9.1. **Stationary descendent series.** We provide a complete table of the stationary stable pair descendent series for projective $\mathbb{P}^3$ in degree 1. Our notation is given by three vectors $V_{\mathsf{p}}, V_{\mathsf{L}}, V_{\mathsf{H}}$ of non-negative integers which specify the stationary descendents with cohomology insertions

$$\mathsf{p},\mathsf{L},\mathsf{H}\in H^*(\mathbb{P}^3)$$

corresponding to the point, line, and hyperplane classes respectively. For example, the data $[1,2],[4,9],[6]$ corresponds to the descendent

$$\mathsf{ch}_3(\mathsf{p})\mathsf{ch}_4(\mathsf{p})\mathsf{ch}_6(\mathsf{L})\mathsf{ch}_{11}(\mathsf{L})\mathsf{ch}_8(\mathsf{H})\,.$$

In the table, below the full descendent series is given as rational function in q.

$[], [0,1], [1]$	$q(3q^2 - 5\,q + 3)$
$[1], [0], []$	$q(q^2 - 1)/2$
$[0], [0,0], []$	$q(q+1)^2$
$[0], [1], []$	$3q(q^2 - 1)/2$
$[], [0,0,1], []$	$2q(q^2 - 1)$
$[], [1,1], []$	$5q(q-1)^2/2$
$[], [0,2], []$	$q(5q^2 - 14q + 5)/6$
$[1], [], [1]$	$3q(q-1)^2/4$
$[], [0,0,0], [1]$	$3q(q^2 - 1)$
$[], [2], [1]$	$\frac{5q(q-1)^3}{4(1+q)}$
$[0], [], [1,1]$	$3q(3q^2 - 2q + 3)/4$
$[], [0,0], [1,1]$	$q(9q^2 - 10q + 9)/2$
$[], [1], [1,1]$	$\frac{q(q-1)(9q^2-2q+9)}{2(1+q)}$
$[], [0], [1,1,1]$	$\frac{q(q-1)(27q^2+14q+27)}{4(1+q)}$
$[0], [], [2]$	$q(5q^2 - 2q + 5)/4$
$[], [0,0], [2]$	$2q(q^2 - q + 1)$
$[], [1], [2]$	$\frac{q(q-1)(9q^2-2q+9)}{4(1+q)}$
$[], [], [1,1,2]$	$q(9q^2 - 14q + 9)/2$
$[], [], [2,2]$	$q(17q^2 - 30q + 17)/8$
$[], [0], [3]$	$\frac{q(q-1)(9q^2-2q+9)}{12(1+q)}$
$[], [], [1,3]$	$q(9q^2 - 22q + 9)/8$
$[0], [0], [1]$	$3q(q^2 - 1)/2$
$[], [], [4]$	$q(q^2 - 5q + 1)/6$
$[], [3], []$	$\frac{q(q-1)(q^2-8q+1)}{6(1+q)}$
$[2], [], []$	$q(q^2 - 10q + 1)/12$
$[], [0], [1,2]$	$\frac{q(q-1)(3q^2+q+3)}{(1+q)}$
$[0,0], [], []$	$q(q+1)^2$
$[], [0,0,0,0], []$	$2q(q+1)^2$
$[], [], [1,1,1,1]$	$q(81q^2 - 102q + 81)/2$

The symmetry in the above series is a consequence of the functional equation, see [20, Section 1.7]. In the stationary case, the stable pairs series are equal to the corresponding descendent series for the Donaldson-Thomas theory of ideal sheaves, see [17, Theorem 22].

9.2. **Descendents of 1.** We tabulate here descendent series of $\mathbb{P}^3$ in degree 1 with descendents of the identity $1 \in H^*(\mathbb{P}^3)$ together with stationary descendents specified as before by a triple of vectors.

- With $\mathsf{ch}_4(1)$ and the rest stationary:

$[], [1], [1]$	$\frac{q\left(21\,q^4+37\,q^3-88\,q^2+37\,q+21\right)}{6(1+q)^2}$
$[], [0,1], []$	$7\,q\,(q-1)\,(1+q)\,/3$
$[], [], [1,2]$	$\frac{q(q-1)\left(21\,q^4+79\,q^3+86\,q^2+79\,q+21\right)}{6(1+q)^3}$
$[0], [], [1]$	$7\,q\,(q-1)\,(1+q)\,/4$
$[], [0,0], [1]$	$7\,q\,(q-1)\,(1+q)\,/2$
$[], [0], [1,1]$	$\frac{q\left(63\,q^4+116\,q^3-134\,q^2+116\,q+63\right)}{12(1+q)^2}$
$[0], [0], []$	$q\,(7\,q^2+2\,q+7)\,/6$
$[1], [], []$	$7\,q\,(q-1)\,(1+q)\,/12$
$[], [0,0,0], []$	$q\,(7\,q^2+2\,q+7)\,/3$
$[], [2], []$	$\frac{q\left(35\,q^4+56\,q^3-318\,q^2+56\,q+35\right)}{36(1+q)^2}$
$[], [], [3]$	$\frac{q(q-1)\left(63\,q^4+232\,q^3+218\,q^2+232\,q+63\right)}{72(1+q)^3}$
$[], [0], [2]$	$\frac{q\left(7\,q^4+13\,q^3-18\,q^2+13\,q+7\right)}{3(1+q)^2}$

- With $\mathsf{ch}_5(1)$ and the rest stationary:

$[0], [], []$	$3\,q\,(q-1)\,(1+q)\,/4$
$[], [0,0], []$	$4\,q\,(q-1)\,(1+q)\,/3$
$[], [1], []$	$\frac{q\left(17\,q^4+24\,q^3-106\,q^2+24\,q+17\right)}{12(1+q)^2}$
$[], [], [1,1]$	$\frac{q(q-1)\left(9\,q^4+31\,q^3+14\,q^2+31\,q+9\right)}{3(1+q)^3}$
$[], [], [2]$	$\frac{q(q-1)\left(33\,q^4+112\,q^3+38\,q^2+112\,q+33\right)}{24(1+q)^3}$
$[], [0], [1]$	$\frac{q(3\,q+1)(q+3)\left(4\,q^2-7\,q+4\right)}{6(1+q)^2}$

- With $\mathsf{ch}_4(1)\mathsf{ch}_4(1)$ and the rest stationary:

$[], [0], [1]$	$\frac{q(q-1)\left(49\,q^4+196\,q^3+534\,q^2+196\,q+49\right)}{12(1+q)^3}$
$[0], [], []$	$q\,(49+2\,q+49\,q^2)\,/36$
$[], [0,0], []$	$q\,(49+2\,q+49\,q^2)\,/18$
$[], [1], []$	$\frac{q(q-1)\left(49\,q^4+196\,q^3+654\,q^2+196\,q+49\right)}{18(1+q)^3}$
$[], [], [1,1]$	$\frac{q\left(441+1754\,q+4007\,q^2-3252\,q^3+4007\,q^4+1754\,q^5+441\,q^6\right)}{72(1+q)^4}$
$[], [], [2]$	$\frac{q\left(49+195\,q+459\,q^2-454\,q^3+459\,q^4+195\,q^5+49\,q^6\right)}{18(1+q)^4}$

- With $\mathsf{ch}_6(1)$ and the rest of stationary:

$[],[0],[]$	$\frac{q\left(17\,q^4+20\,q^3-114\,q^2+20\,q+17\right)}{36(1+q)^2}$
$[],[],[1]$	$\frac{q(q-1)\left(17\,q^4+48\,q^3-58\,q^2+48\,q+17\right)}{24(1+q)^3}$

- With $\mathsf{ch}_4(1)\mathsf{ch}_4(1)\mathsf{ch}_4(1)$ and the rest stationary:

$[],[0],[]$	$\frac{q\left(343\,q^6+1374\,q^5+249\,q^4+11396\,q^3+249\,q^2+1374\,q+343\right)}{108(1+q)^4}$
$[],[],[1]$	$\frac{q(q-1)\left(343\,q^6+2058\,q^5+3705\,q^4+29900\,q^3+3705\,q^2+2058\,q+343\right)}{72(1+q)^5}$

- With $\mathsf{ch}_5(1)\mathsf{ch}_4(1)$ and the rest stationary:

$[],[],[1]$	$\frac{q\left(84+331\,q+928\,q^2-1878\,q^3+928\,q^4+331\,q^5+84\,q^6\right)}{36(1+q)^4}$
$[],[0],[]$	$\frac{2q(q-1)\left(7+28\,q+87\,q^2+28\,q^3+7\,q^4\right)}{9(1+q)^3}$

- Without stationary descendents:

$\mathsf{ch}_7(1)$	$\frac{q(q-1)\left(2+3\,q-28\,q^2+3\,q^3+2\,q^4\right)}{18(1+q)^3}$
$\mathsf{ch}_5(1)\mathsf{ch}_5(1)$	$\frac{5q\left(13+50\,q+179\,q^2-580\,q^3+179\,q^4+50\,q^5+13\,q^6\right)}{72(1+q)^4}$
$\mathsf{ch}_4(1)\mathsf{ch}_6(1)$	$\frac{q\left(119+462\,q+1737\,q^2-5852\,q^3+1737\,q^4+462\,q^5+119\,q^6\right)}{216(1+q)^4}$
$\mathsf{ch}_4(1)\mathsf{ch}_4(1)\mathsf{ch}_5(1)$	$\frac{q\left(-49-245\,q-81\,q^2-6365\,q^3+6365\,q^4+81\,q^5+245\,q^6+49\,q^7\right)}{27(1+q)^5}$
$\mathsf{ch}_4(1)\mathsf{ch}_4(1)\mathsf{ch}_4(1)\mathsf{ch}_4(1)$	$\frac{q\left(2401+14405\,q+55690\,q^2-594229\,q^3+1834570\,q^5-594229\,q^5+55690\,q^6+14405\,q^7+2401\,q^8\right)}{648(1+q)^6}$

9.3. **Examples of the Virasoro relations.**

9.3.1. $\mathcal{L}_2^{\mathrm{PT}}$. Examples of the Virasoro relations for $\mathcal{L}_1^{\mathrm{PT}}$ were given in [20, Section 3]. We consider here the operator $\mathcal{L}_2^{\mathrm{PT}}$ for $X=\mathbb{P}^3$.

The Chern classes of the tangent bundle of $\mathbb{P}^3$ are

$$c_1 = 4\mathsf{H}\,, \quad c_1c_2 = 24\mathsf{p}\,,$$

The constant term for $k=2$ is

$$\begin{aligned}\mathrm{T}_2 &= -\frac{1}{2}\sum_{a+b=4}(-1)^{d^Ld^R}(a+d^L-3)!(b+d^R-3)!\,\mathsf{ch}_a\mathsf{ch}_b(c_1)+\frac{1}{24}\sum_{a+b=2}a!b!\,\mathsf{ch}_a\mathsf{ch}_b(c_1c_2)\\ &= -8\mathsf{ch}_4(\mathsf{H})+8\mathsf{ch}_2(\mathsf{H})\mathsf{ch}_2(\mathsf{p})-2\mathsf{ch}_2(\mathsf{L})^2-4\mathsf{ch}_2(\mathsf{p}),\end{aligned}$$

where we used the evaluation $\mathsf{ch}_0(\gamma) = -\int_X \gamma$ and dropped all the terms with ch_1. The Virasoro operator for $k=2$ is then

$$\begin{aligned} \mathcal{L}_2^{\mathrm{PT}} &= \mathrm{T}_2 + \mathrm{R}_2 + 3!\mathrm{R}_{-1}\mathsf{ch}_3(p) \\ &= -8\mathsf{ch}_4(\mathsf{H}) + 8\mathsf{ch}_2(\mathsf{H})\mathsf{ch}_2(\mathsf{p}) - 2\mathsf{ch}_2(\mathsf{L})^2 - 4\mathsf{ch}_2(\mathsf{p}) + \mathrm{R}_2 + 3!\mathrm{R}_{-1}\mathsf{ch}_3(\mathsf{p})\,. \end{aligned}$$

Since our examples will be for curves of degree 1 in $\mathbb{P}^3$ and since

$$\mathsf{ch}_2(\mathsf{H}) = \mathsf{H}\cdot\beta\,,$$

we can simplify the operator even further:

$$\mathcal{L}_{2,\beta=\mathsf{L}}^{\mathrm{PT}} = -8\mathsf{ch}_4(\mathsf{H}) + 10\mathsf{ch}_2(\mathsf{p}) - 2\mathsf{ch}_2(\mathsf{L})^2 + \mathrm{R}_2 + 6\mathsf{ch}_3(\mathsf{p})\mathrm{R}_{-1}.$$

9.3.2. *Stationary example.* Let us check the Virasoro constraints of Theorem 4 for $k=2$ and

$$D = \mathsf{ch}_3(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\,.$$

The constant term part of the relation has three summands:

$$\begin{aligned} -8\langle\mathsf{ch}_4(\mathsf{H})\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} &= -\frac{8q(q-1)(3q^2+q+3)}{1+q}\,, \\ 10\langle\mathsf{ch}_2(\mathsf{p})\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} &= 15q(q^2-1)\,, \\ -2\langle\mathsf{ch}_2(\mathsf{L})^2\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} &= -6q(q^2-1)\,. \end{aligned}$$

The rest of the relation can be divided into two parts. The first part is $\mathrm{R}_2(D)$ which has two terms:

$$\begin{aligned} 6\langle\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_4(\mathsf{L})\rangle_{\mathsf{L}} &= \frac{15q(q-1)^3}{2(1+q)}\,, \\ 6\langle\mathsf{ch}_5(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} &= \frac{q(q-1)(9q^2-2q+9)}{2(1+q)}\,. \end{aligned}$$

The second part is

$$\begin{aligned} 6\langle\mathsf{ch}_3(\mathsf{p})\mathrm{R}_{-1}(D)\rangle_{\mathsf{L}} &= 6\langle\mathsf{ch}_3(\mathsf{p})\mathsf{ch}_2(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} + 6\langle\mathsf{ch}_3(\mathsf{p})\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_1(\mathsf{L})\rangle_{\mathsf{L}} \\ &= 6\langle\mathsf{ch}_3(\mathsf{p})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} \\ &= 3q(q^2-1)\,. \end{aligned}$$

Using the cancellation of poles

$$-8\langle\mathsf{ch}_4(\mathsf{H})\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} + 6\langle\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_4(\mathsf{L})\rangle_{\mathsf{L}} + 6\langle\mathsf{ch}_5(\mathsf{H})\mathsf{ch}_2(\mathsf{L})\rangle_{\mathsf{L}} = -12q(q^2-1)\,,$$

we easily verify the Virasoro relation

$$\Big\langle \mathcal{L}_2^{\mathrm{PT}}(\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_2(\mathsf{L}))\Big\rangle_{\mathsf{L}}^{X,\mathrm{PT}} = 0\,.$$

9.3.3. *Non-stationary example.* Let us check the Virasoro relation $\mathcal{L}^{\mathrm{PT}}_{2,\beta=\mathsf{L}}$ for

$$D = \mathsf{ch}_5(1)\,,$$

a non-stationary case (not covered by Theorem 4, but implied by Conjecture 3).

The constant term part of the relation has three summands:

$$\begin{aligned}
-8\langle \mathsf{ch}_4(H)\mathsf{ch}_5(1)\rangle_{\mathsf{L}} &= -\frac{q\,(q-1)\,(33\,q^4+112\,q^3+38\,q^2+112\,q+33)}{3\,(1+q)^3}\,,\\
10\langle \mathsf{ch}_2(\mathsf{p})\mathsf{ch}_5(1)\rangle_{\mathsf{L}} &= \frac{15}{2}\,q\,(q-1)\,(1+q)\,,\\
-2\langle \mathsf{ch}_2^2(\mathsf{L})\mathsf{ch}_5(1)\rangle_{\mathsf{L}} &= -\frac{8}{3}\,q\,(q-1)\,(1+q)\,.
\end{aligned}$$

The rest of the relation can be divided into two parts:

$$\begin{aligned}
24\langle \mathsf{ch}_7(1)\rangle_{\mathsf{L}} &= \frac{4q\,(q-1)\,(2+3\,q-28\,q^2+3\,q^3+2\,q^4)}{3\,(1+q)^3}\,,\\
6\langle \mathsf{ch}_3(\mathsf{p})\mathsf{ch}_4(1)\rangle_{\mathsf{L}} &= \frac{7}{2}\,q\,(q-1)\,(1+q)\,.
\end{aligned}$$

After a remarkable cancellation of poles,

$$-8\langle \mathsf{ch}_4(H)\mathsf{ch}_5(1)\rangle_{\mathsf{L}} + 24\langle \mathsf{ch}_7(1)\rangle_{\mathsf{L}} = -\frac{25}{3}q(q-1)(1+q)\,,$$

we verify the Virasoro relation

$$\Big\langle \mathcal{L}^{\mathrm{PT}}_2(\mathsf{ch}_5(1))\Big\rangle^{X,\mathrm{PT}}_{\mathsf{L}} = 0\,.$$

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