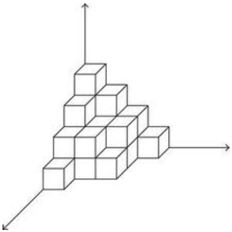

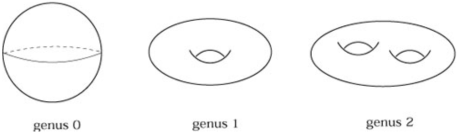

Algebraic geometry of moduli spaces

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§I. PARTITIONS

How can we write n as a sum of positive numbers?

The full list of partitions of $n = 3$ is

$$3, \quad 2+1, \quad 1+1+1$$

and the full list of partitions of $n = 4$ is

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

There are 3 partitions of 3 and 5 partitions of 4.

$$\boxed{p(n) \;=\; \textit{Number of partitions of } n}$$

so $p(3) = 3$ and $p(4) = 5$.

A formula for $p(n)$?

There is no direct formula for $p(n)$, but there is a formula for the generating series:

$$\boxed{\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)}$$

Expand the right side

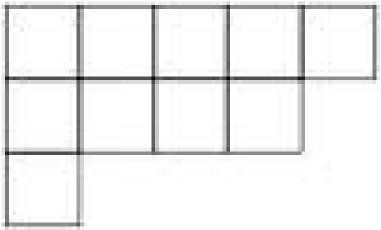
$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)\cdots \\ &= 1+q^1+2q^2+3q^3+5q^4+7q^5+\ldots\end{aligned}$$

The product formula for the counting of partitions was found by Leonhard Euler (1707-1783):

Express partitions as diagrams:

$$10 = 5 + 4 + 1$$

can be pictured as



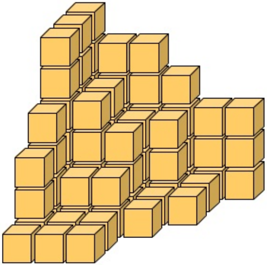
Such a diagram may be viewed as stacking squares in the corner of a 2-d room (stable for both possible directions of gravity).

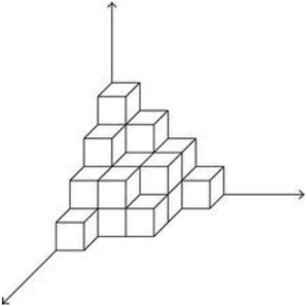
What about 3-dimensions?

We would like to stack 3-dimensional boxes in the corner of a 3-dimensional room.

The above is a photo of the installation, *Five Boxes*, by the Icelandic artist Egill Sæbjörnsson.

A 3-dimensional partition is a stacking of boxes in the corner of a room (which is stable for any of the three possible directions of gravity).





$$\boxed{P(n) \; = \; \textit{Number of 3-dimensional partitions of } n \; .}$$

We see $P(1) = 1$, $P(2) = 3$, $P(3) = 6$, $\ldots$.

A formula for $P(n)$?

Again, there is no direct formula for $P(n)$, but there is a formula for the generating series:

$$\boxed{\sum_{n=0}^{\infty} P(n)q^n = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)^k}$$

The formula is due to Percy MacMahon (1854-1929). Before his mathematical career, he was a Lieutenant in the British army. He was said to be at least partially inspired by stacking cannon balls.

A formula for counting partitions in 4-dimensions?

2-dim
$$\boxed{\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right)}$$

3-dim
$$\boxed{\sum_{n=0}^{\infty} P(n) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right)^k}$$

%pause

MacMahon proposed $\prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right)^{\binom{k+1}{2}}$ for the generating series of 4-dimensional partitions. %pause

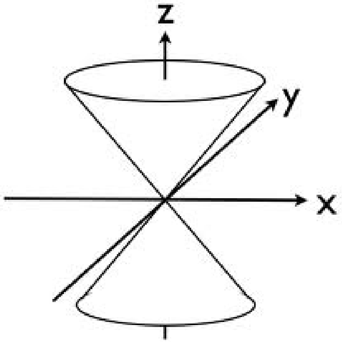
He was wrong! Formulas for dimensions 4 and higher are unknown.

§II. IDEALS

Algebraic geometry is the study of zeros of polynomial equations.

$\mathbb{C}[x, y, z]$ is the ring of polynomials in the variables x, y, z.

The zeros of $x^2 + y^2 - z^2$ form the cone:



An ideal is a vector subspace of polynomials $\mathcal{I} \subset \mathbb{C}[x, y, z]$ satisfying

$$f \cdot \mathcal{I} \subset \mathcal{I}$$

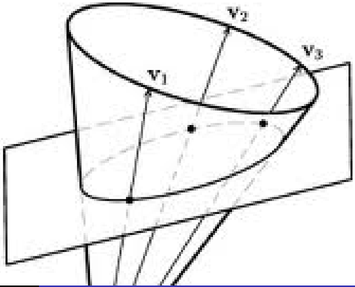
for every polynomial $f \in \mathbb{C}[x, y, z]$.

To each ideal $\mathcal{I} \subset \mathbb{C}[x, y, z]$, we associate the zero set:

$$\mathcal{I} \quad \longleftrightarrow \quad V_{\mathcal{I}} \subset \mathbb{C}^3 \ .$$

To the ideal $\mathcal{I} = (x^2 + y^2 - z^2, x - y - z + 1)$, we associate the curve defined by the intersection:
%pause

$$V_{\mathcal{I}} =$$



An idea due Alexander Grothendieck (1928-) is to parameterize all algebraic subspaces of $\mathbb{C}^3$ by another algebraic space, the Hilbert scheme.

The Hilbert scheme is an example of a moduli space in algebraic geometry:
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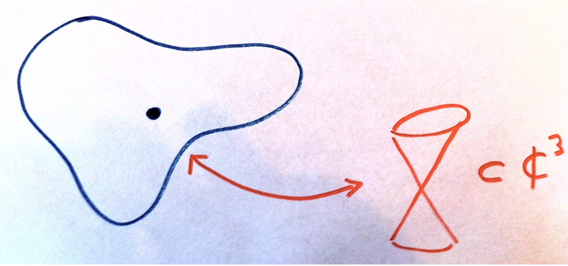
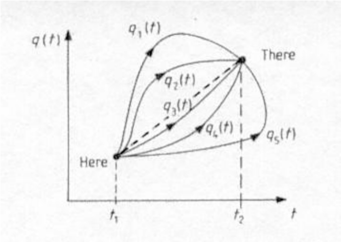


Every point of the Hilbert scheme corresponds to an algebraic subspace of $\mathbb{C}^3$.

§III. INTEGRATION

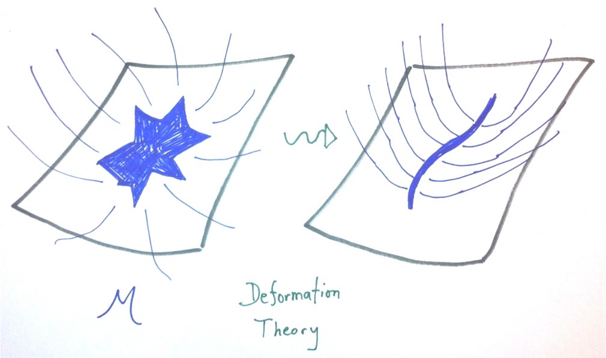
In quantum field theories (and string theory), path integrals arise: integrals over the spaces of functions.



Sometimes (in the presence of supersymmetry and further constraints), such path integrals are related to integration over finite-dimensional moduli spaces in algebraic geometry. Examples in gauge theory, topological string theory, ...

In 1990's, there was an effort made in algebraic geometry to define the integration on algebraic moduli spaces predicted by path integral techniques [Kontsevich, Li-Tian, Behrend-Fantechi].

The idea is to use deformation theory in algebraic geometry. The moduli spaces, such as the Hilbert scheme, are very singular spaces, but we have some understanding of their local structure:



The outcome is a virtual fundamental class and a well-defined theory of integration on many algebraic moduli spaces including the Hilbert scheme of $\mathbb{C}^3$.

What happens if we integrate over the Hilbert scheme of $\mathbb{C}^3$?

There are many components of the Hilbert scheme. We integrate over the components where

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x,y,z]}{\mathcal{I}} < \infty \ ,$$

%pause
Result of Maulik, Nekrasov, Okounkov, P [2003]:

$$\int_{\text{Hilbert scheme}(\mathbb{C}^3)} (-q)^{\dim\left(\frac{\mathbb{C}[x,y,z]}{\mathcal{I}}\right)} = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right)^k$$

which is MacMahon's series for counting 3-dimensional partitions.

The study of such integration over the Hilbert scheme of $\mathbb{C}^3$ is called Donaldson-Thomas theory — viewed as a counting theory of sheaves.

Donaldson-Thomas theory can be studied for any nonsingular 3-dimensional space, not just $\mathbb{C}^3$. For example the Calabi-Yau quintic,

$$(x^5 + y^5 + z^5 + w^5 = 1) \subset \mathbb{C}^4 \ .$$

The outcome is a completely non-linear generalization of the box counting of MacMahon.

What are the answers?

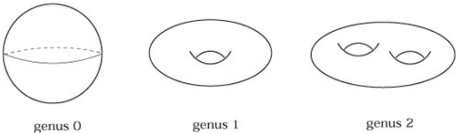
%pause In basic toric cases, new product formulae have been

found. In elliptic and K3 case, the answers are in terms of modular forms. Hypergeometric series play a role in the Calabi-Yau 3-fold cases.

§IV. CURVES

Another counting question began in the 19th century: the counting of algebraic curves.

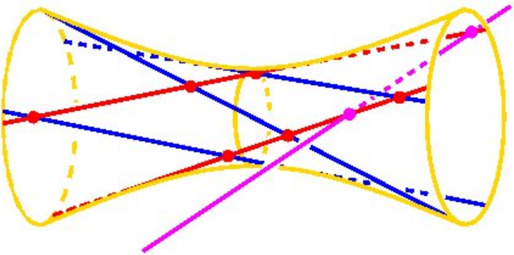
An algebraic curve is a complex 1-dimensional manifold, so a real 2-dimensional surface:



Hermann Schubert (1848-1911) teacher of of Adolf Hurwitz (1859-1919)

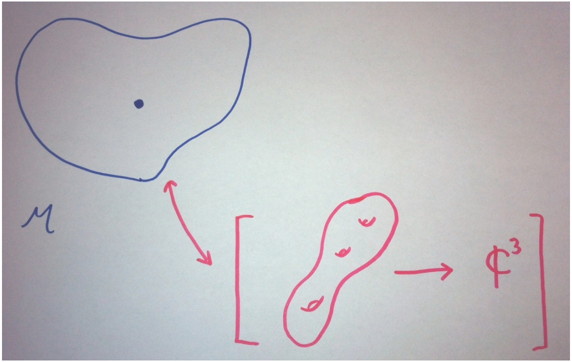
How many lines meet 4 skew lines in space?

%pause
Answer is 2:



%pause There was a long classical development of curve enumeration. But the subject has now been recast as Gromov-Witten theory.

There is a moduli space of maps from curves to algebraic spaces such as $\mathbb{C}^3$.



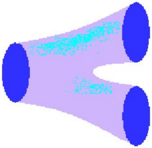
Instead of $\mathbb{C}^3$, other targets such as toric varieties or the Calabi-Yau quintic can be considered. The study of integration over the moduli of such maps is Gromov-Witten theory.

The partition function for Gromov-Witten theory is

$$\boxed{Z_{GW}(u) = \sum_{g} u^{2g-2} N_g}$$

where N_g is the count of the genus g curves in the specified geometry.

Gromov-Witten theory has origins in Gromov's study of holomorphic maps in symplectic geometry and Witten's study of topological string theory.



§V. EQUIVALENCE

Box and curve counting questions in 3-dimensions are equivalent

Let X be any nonsingular 3-fold. Let $\mathsf{Z}_{\mathsf{DT}}(q)$ be the generating series for the Hilbert scheme integrals of Donaldson-Thomas theory. Let $\mathsf{Z}_{\mathsf{GW}}(u)$ be the generating series for the moduli space of map integrals of Gromov-Witten theory.

The main conjectured correspondence [MNOP]:

$$\boxed{\mathsf{Z}_{\mathsf{DT}}(q) = \mathsf{Z}_{\mathsf{GW}}(u)}$$

after the change of variables $\boxed{-q = e^{iu}}$

The equivalence for $\mathbb{C}^3$ generalizes (and provides a proof of) the topological vertex formula of Aganagic, Klemm, Mariño, Vafa.

For the classical spaces, the correspondence unites the counting problems of Hurwitz, MacMahon, Schubert, ...

The main correspondence is proven for many geometries and is open for many geometries. Lots of work to do!

§PHOTO CREDITS

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P.02 Euler and 10 Swiss Franc Note [Wikipedia]

P.05 5+4+1 [Wikipedia]

P.06 http://www.i8.is/ and Reykjavik Museum of Art [Egill Sæbjörnsson]

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