Big quantum cohomology of Fano complete intersections

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References

- Hu, Xiaowen. Big quantum cohomology of Fano complete intersections. arXiv:1501.03683 (2015). v4 (2021).
- Hu, Xiaowen. Big quantum cohomology of even dimensional intersections of two quadrics. arXiv: 2109.11469.
- Packages: https://github.com/huxw06/Quantum-cohomology-of-Fano-complete-intersections

Related:

- Argüz, H., Bousseau, P., Pandharipande, R., Zvonkine, D. Gromov–Witten Theory of Complete Intersections. arXiv:2109.13323v2.
- Giosuè's localization package: https://github.com/mgemath/AtiyahBott.jl.

Gromov-Witten invariants

Let X be a smooth projective variety. The moduli stack $\overline{\mathcal{M}}_{g,k}(X,\beta)$ classifies the stable maps of degree β from nodal curves of arithmetic genus g to X. Gromov-Witten invariants is defined as intersections of the form

$$\langle \gamma_1, \ldots, \gamma_k \rangle^X_{g,k,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1{}^* \gamma_1 \cup \cdots \cup \mathrm{ev}_k^* \gamma_k,$$

where ev_i are *evaluation maps* $ev_i : \overline{\mathcal{M}}_{g,k}(X,\beta) \to X$, and $\gamma_i \in H^*(X)$.

- It is a *virtual counting* of genus g stable maps passing through the cycles in general positions representing the classes $\gamma_1, \ldots, \gamma_k$. (When genus $g = 0$, the invariants and the associated quantum product are called *quantum cohomology*).
- $\{\gamma_0, \ldots, \gamma_N\}$:= a basis of $H^*(X)$.
- $\{T^0, \ldots, T^N\}$:= the dual basis with respect to $\gamma_0, \ldots, \gamma_N$.

The generating function of genus g GW invariants:

$$\mathcal{F}_g(T^0, \ldots, T^N, \mathsf{q}) = \sum_{k \geq 0} \sum_{\beta} \frac{1}{k!} \Big\langle \sum_{i=0}^{N} \gamma_i T^i, \ldots, \sum_{i=0}^{N} \gamma_i T^i \Big\rangle_{g,k,\beta} \mathsf{q}^{\beta}.$$

Frobenius manifolds

The genus 0 generating function $\mathrm{F} = \mathcal{F}_0$ satisfies the WDVV equation

$$\sum_{e=0}^{N}\sum_{f=0}^{N} \frac{\partial^3 \mathrm{F}}{\partial T^a \partial T^b \partial T^e} g^{ef} \frac{\partial^3 \mathrm{F}}{\partial T^f \partial T^c \partial T^d}$$

$$= \sum_{e=0}^{N}\sum_{f=0}^{N} (\pm) \frac{\partial^3 \mathrm{F}}{\partial T^a \partial T^c \partial T^e} g^{ef} \frac{\partial^3 \mathrm{F}}{\partial T^f \partial T^b \partial T^d}.$$

- If $\deg_{\mathbb{R}} \gamma_i$ is odd, T^i is a *Grassmann variable*.

Data for a Frobenius manifold:

- A family of Frobenius algebra.
- Flat coordinates.
- Euler vector field $E = \sum_{i=0}^{N} (1 - \frac{|\gamma_i|}{2}) \frac{\partial}{\partial T^i} + \sum_{i=0}^{N} a_i \frac{\partial}{\partial T^i}$.

$$EF = (3-n)F + \sum_{i=0}^{N} a_i \frac{\partial}{\partial T^i} c,$$

with

$$c(T_0, \cdots, T^{n+m}) = \sum_a \sum_b \sum_c \frac{T^a T^b T^c}{6} \int_X \gamma_a \gamma_b \gamma_c.$$

Gromov-Witten invariants of complete intersections

Let $\iota : X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection of dimension n.

$$H^*_{\mathrm{amb}}(X) := \iota^* H^*(\mathbb{P}^{n+r}),\ H^*(X) = H^*_{\mathrm{amb}}(X) \oplus H^n_{\mathrm{prim}}(X).$$

- Physicists predicted quantum cohomology of quintic 3-folds in $\mathbb{P}^4$ as the beginning of mirror symmetry in 1991.
- Givental and Lian-Liu-Yau proved the predictions and extended it to Fano complete intersections in around 1996-1997.
- Genus 1 GW invariants of Calabi-Yau complete intersections, by A. Zinger, and A. Popa.
- BCOV conjecture for quintic 3-folds in higher genera is proved by Chang-Guo-Li-Li.

Quantum cohomology with primitive classes

Let $\iota : X \hookrightarrow \mathbb{P}^N$ be a smooth complete intersection.

- 3-point genus 0 invariants, with multidegree $\mathbf{d}$ of X in certain range, were computed first by Beauville for hypersurfaces, and extended to complete intersections by Collino-Jinzenji.
- The computation of quantum cohomology with primitive insertions cannot be done by torus localization or the usual degeneration formula.
- Quite recently, Argüz-Bousseau-Pandharipande-Zvonkine show a new degeneration formula, and give an algorithm to compute GW invariants of all genera of complete intersections.
- No predictions from physics.
- The direct enumerative sense in algebraic geometry is missing in general.

Quantum cohomology with primitive classes: significance

- Knowledge of (genus 0) Gromov-Witten invariants with primitive insertions is necessary for Dubrovin-type conjecture.
- Necessary for establishing a full (numerical) mirror symmetry for Fano complete intersections.
- They are needed for recursions for higher genus GW invariants, even one concerns only with the GW invariants with ambient insertions.
- They *Do* have interesting structures!

WDVV equation: essentially linear recursions

$$\sum_e \sum_f (\partial_{t^a}\partial_{t^b}\partial_{t^e} F) g^{ef} (\partial_{t^f}\partial_{t^c}\partial_{t^d} F)$$
$$= \sum_e \sum_f (\partial_{t^a}\partial_{t^c}\partial_{t^e} F) g^{ef} (\partial_{t^f}\partial_{t^b}\partial_{t^d} F).$$

Traditional way to use WDVV equations: expand the leading terms to get recursions. E.g.

$$\begin{aligned} & \mathrm{Coeff}_{t^I}(\partial_{t^a}\partial_{t^b}\partial_{t^e} F) g^{ef} (\partial_{t^f}\partial_{t^c}\partial_{t^d} F)(0) \\ & +(\partial_{t^a}\partial_{t^b}\partial_{t^e} F)(0) g^{ef} \mathrm{Coeff}_{t^I}(\partial_{t^f}\partial_{t^c}\partial_{t^d} F) \\ & -\mathrm{Coeff}_{t^I}(\partial_{t^a}\partial_{t^c}\partial_{t^e} F) g^{ef} (\partial_{t^f}\partial_{t^b}\partial_{t^d} F)(0) \\ & -(\partial_{t^a}\partial_{t^c}\partial_{t^e} F)(0) g^{ef} \mathrm{Coeff}_{t^I}(\partial_{t^f}\partial_{t^b}\partial_{t^d} F) \\ = \quad & \text{lower order terms.} \end{aligned}$$

More generally, we can use invariants of any *fixed* length 4, 5, ...

Monodromy groups

Let X be a complete intersection in $\mathbb{P}^{n+r}$ of multidegree $\mathbf{d} = (d_1, \ldots, d_r)$. We call X *exceptional* if the monodromy group as a group acting on $\mathrm{H}^n_{\mathrm{prim}}(X)$ is a finite group. The exceptional complete intersections are classified by Deligne:

- $\mathbf{d} = (2)$, i.e X is a quadric hypersurface.
- $\mathbf{d} = (3)$ and $n = 2$, i.e. X is a cubic curface.
- $\mathbf{d} = (2, 2)$ and n is even.

In all the other cases the Zariski closure of the monodromy group is

- ($n = \dim X$ is even) the orthogonal group $\mathrm{O}(H^n_{\mathrm{prim}}(X))$;
- ($n = \dim X$ is odd) the symplectic group $\mathrm{Sp}(H^n_{\mathrm{prim}}(X))$.

Symmetric reduction

Suppose X is a non-exceptional complete intersection in a projective space.

- $n := \dim X$. Assume $n \geq 3$.
- $m := \mathrm{rank} H^n_{\mathrm{prim}}(X)$.
- $\mathrm{a} = n + r + 1 - \sum_{i=1}^{r} d_i$.

Let $t^0, \ldots, t^n$ be flat coordinates on of the Frobenius manifold associated to the ambient quantum cohomology of X. Suppose n is even. Let $t^{n+1}, \ldots, t^{n+m}$ be the basis dual to an orthonormal basis of $H^n_{\mathrm{prim}}(X)$. Let

$$s = \frac{1}{2} \sum_{i=n+1}^{n+m} (t^i)^2.$$

By the theory of polynomial invariants of orthogonal groups, the generating function F of quantum cohomology of X is a function of $t^0, \ldots, t^n$ and s. When n is odd, the variable s is defined similary by a symplectic basis of $H^n_{\mathrm{prim}}(X)$:

$$s = -\sum_{i=n+1}^{n+\frac{m}{2}} t^i t^{i+\frac{m}{2}}.$$

Symmetric reduction of WDVV

Symmetric reduction of the WDVV equations of F:

$$F_{abe}g^{ef}F_{sf} + 2sF_{sab}F_{ss} = F_{sa}F_{sb}, \quad 0 \leq a, b \leq n,$$

$$F_{se}g^{ef}F_{sf} + 2sF_{ss}F_{ss} = 0.$$

In odd dimensions,

$$F_{abe}g^{ef}F_{sf} + 2sF_{sab}F_{ss} \equiv F_{sa}F_{sb} \mod s^{\frac{m}{2}}, \quad 0 \leq a, b \leq n,$$

$$F_{se}g^{ef}F_{sf} + 2sF_{ss}F_{ss} \equiv 0 \mod s^{\frac{m}{2}}.$$

System of equations

- For even n,

$$\begin{cases} F_{abe}g^{ef}F_{sf} + 2sF_{sab}F_{ss} = F_{sa}F_{sb}, & \text{for } 0 \le a, b \le n, \\ F_{se}g^{ef}F_{sf} + 2sF_{ss}F_{ss} = 0, \\ EF = (3-n)F + \mathrm{a}\frac{\partial}{\partial t^1}c, \end{cases}$$

- For odd n,

$$\begin{cases} F_{abe}g^{ef}F_{sf} + 2sF_{sab}F_{ss} = F_{sa}F_{sb} \mod s^{\frac{m}{2}}, & \text{for } 0 \le a, b \le n, \\ F_{se}g^{ef}F_{sf} + 2sF_{ss}F_{ss} = 0 \mod s^{\frac{m}{2}}, \\ EF = (3-n)F + \mathrm{a}\frac{\partial}{\partial t^1}c. \end{cases}$$

Aim: Solve F, with $F|_{s=0} = F^{(0)}$ as initial given data.

Reconstruction I

$$F^{(k)}(t^0, \cdots, t^n) := \left(\frac{\partial^k}{\partial s^k} F\right)\Big|_{s=0},$$

Expand

$$F = F^{(0)} + sF^{(1)} + \frac{s^2}{2} F^{(2)} + \dots$$

Then $F^{(0)}$ is the generating function of ambient quantum cohomology.

Theorem

- $\Theta := \sum_{e=0}^{n} \sum_{f=0}^{n} F_e^{(1)} g^{ef} \gamma_f$ *is a common eigenvector by the quantum multiplications by all cohomology classes. This determines* $F^{(1)}$.
- *For* $k \geq 2$, $F^{(k)}$ *can be reconstructed from* $F^{(i)}$ *for* $0 \leq i < k$, *and the constant leading term* $F^{(k)}(0)$.

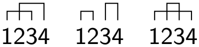
The remaining task is to compute $F^{(k)}(0)$ for $k \geq 2$.

$F^{(l)}(0)$ as ratios

Let A_{2l} be the set

$$A_{2l} = \Big\{ \big((i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l) \big) \big| \{i_1, j_1, i_2, j_2, \ldots, i_l, j_l\} = \{1, \ldots, 2l\}, \\ i_k < j_k \text{ for } 1 \leq k \leq l, i_1 < i_2 < \cdots < i_l \Big\}.$$

In other words, the elements of A_{2l} parametrize the unordered pairings in a set of cardinality $2l$. For example, the elements of A_4 can be depicted as



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For $\sigma = \big((i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l) \big) \in A_{2l}$, and $G = (g_{i,j})_{1 \leq i,j \leq 2l}$ a $2l \times 2l$ symmetric matrix (resp. a $2l \times 2l$ skew-symmetric matrix), we define

$$\mathrm{P}_\sigma(G) := \prod_{k=1}^{l} g_{i_k, j_k}. \ \big(\text{resp. } \mathrm{Pf}_\sigma(G) := \mathrm{sgn}(\sigma) \prod_{k=1}^{l} g_{i_k, j_k}.\big)$$

Then define

$$\mathrm{P}(G) := \sum_{\sigma \in A_{2l}} \mathrm{P}_\sigma(G). \ \big(\text{resp. } \mathrm{Pf}(G) := \sum_{\sigma \in A_{2l}} \mathrm{Pf}_\sigma(G).\big)$$

$F^{(l)}(0)$ as ratios

- For skew-symmetric G, $\mathrm{Pf}(G)$ is the Pfaffian of G.
- For symmetric G, we call $\mathrm{P}(G)$ the *permanent Pfaffian* of G.

For $\alpha_1, \dots, \alpha_{2l} \in H^*_{\mathrm{prim}}(X)$, we define $G(\alpha_1, \dots, \alpha_{2l})$ to be the matrix $G = (g_{i,j})_{1 \le i,j \le 2l}$ with $g_{i,j} = (\alpha_i, \alpha_j)$.

(i) When n is even,

$$\begin{aligned} & \langle \alpha_1, \dots, \alpha_{2l} \rangle_{0,2l} \\ = \quad & F^{(l)}(0) \cdot \mathrm{P}\big(G(\alpha_1, \dots, \alpha_{2l})\big); \end{aligned}$$

(ii) When n is odd,

$$\begin{aligned} & \langle \alpha_1, \dots, \alpha_{2l} \rangle_{0,k+2l} \\ = \quad & F^{(l)}(0) \cdot \mathrm{Pf}\big(G(\alpha_1, \dots, \alpha_{2l})\big). \end{aligned}$$

So $F^{(l)}(0) \in \mathbb{Q}$.

Expansions of symmetric-reduced WDVV

Expand

$$\begin{cases} F_{abe}g^{ef}F_{sf} + 2sF_{sab}F_{ss} = F_{sa}F_{sb}, & \text{for } 0 \leq a, b \leq n, \\ F_{se}g^{ef}F_{sf} + 2sF_{ss}F_{ss} = 0, \\ EF = (3-n)F + \mathrm{a}\frac{\partial}{\partial t^1}c, \end{cases}$$

with respect to s.

$$\sum_{j=0}^{k} \frac{F_{abe}^{(j)} g^{ef} F_f^{(k-j+1)}}{j!(k-j)!} + \sum_{j=1}^{k} \frac{2F_{ab}^{(j)} F^{(k-j+2)}}{(j-1)!(k-j)!} = \sum_{j=1}^{k+1} \frac{F_a^{(j)} F_b^{(k-j+2)}}{(j-1)!(k-j+1)!},$$

(resp. for $k \leq \frac{m}{2} - 1$ when n is odd)

$$\sum_{j=1}^{k+1} \frac{F_e^{(j)} g^{ef} F_f^{(k+2-j)}}{(j-1)!(k+1-j)!} + 2\sum_{j=2}^{k+1} \frac{F^{(j)} F^{(k+3-j)}}{(j-2)!(k+1-j)!} = 0,$$

(resp. for $k \leq \frac{m}{2} - 1$ when n is odd)

where $0 \leq a, b \leq n$.

Equations of constant terms of $F^{(l)}(0)$

For $I = (i_0, i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, we define

$$\partial_{\tau^I} := (\partial_{\tau^0})^{i_0} \circ \cdots \circ (\partial_{\tau^n})^{i_n}.$$

Let $I = (i_0, i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ be given.

$$\sum_{k=1}^{l} \sum_{0 \leq J \leq I} \sum_{a=0}^{n} \sum_{b=0}^{n} \binom{l-1}{k-1} \binom{I}{J} \partial_{\tau^J} \partial_{\tau^a} F^{(k)}(0) \eta^{ab} \partial_{\tau^{I-J}} \partial_{\tau^b} F^{(l+1-k)}(0)$$
$$+2(l-1) \sum_{k=2}^{l} \sum_{0 \leq J \leq I} \binom{l-2}{k-2} \binom{I}{J} F^{(k)}(0) F^{(l+2-k)}(0) = 0.$$

$(2 \leq l \leq \frac{m}{2}$ when n is odd).

Computation of $F^{(2)}(0)$

Take $k = 1$ in

$$F^{(0)}_{abe} g^{ef} F^{(k+1)}_f + 2k F^{(1)}_{ab} F^{(k+1)} - F^{(k+1)}_a F^{(1)}_b - F^{(1)}_a F^{(k+1)}_b$$
$$= \sum_{j=2}^{k} \binom{k}{j-1} F^{(j)}_a F^{(k-j+2)}_b - \sum_{j=1}^{k} \binom{k}{j} F^{(j)}_{abe} g^{ef} F^{(k-j+1)}_f$$
$$-2k \sum_{j=2}^{k} \binom{k-1}{j-1} F^{(j)}_{ab} F^{(k-j+2)}.$$

And use

$$F^{(1)}_e g^{ef} F^{(2)}_f + F^{(2)} F^{(2)} = 0.$$

And the Euler vector field gives, for $k \geq 1$,

$$E_{\mathrm{amb}} F^{(k)} + (2-n) k F^{(k-1)} = (3-n) F^{(k)}.$$

Computation of $F^{(2)}(0)$

Let X be a complete intersection in $\mathbb{P}^{n+r}$ of multidegree $\mathbf{d} = (d_1, \ldots, d_r)$.

$$\mathrm{h}_i := \underbrace{\mathrm{h} \cup \cdots \cup \mathrm{h}}_{i \text{ factors}}.$$

$$\ell(\mathbf{d}) := \prod_{i=1}^{r} d_i!, \ \ \mathrm{b}(\mathbf{d}) := d_1^{d_1} \cdots d_r^{d_r}.$$

$$\tilde{\mathrm{h}} = \begin{cases} \mathrm{h}, & \mathrm{a}(n, \mathbf{d}) \geq 2, \\ \mathrm{h} + \ell(\mathbf{d})\mathrm{q}, & \mathrm{a}(n, \mathbf{d}) = 1. \end{cases}$$

$$\tilde{\mathrm{h}}_i := \underbrace{\tilde{\mathrm{h}} \diamond \cdots \diamond \tilde{\mathrm{h}}}_{i \text{ factors}} \ \text{(small quantum product)}$$

Computation of $F^{(2)}(0)$

Let M and W be the transition matrices between h_i and $\tilde{\mathsf{h}}_i$:

$$\mathsf{h}_i = \sum_{j=0}^{n} M_i^j \tilde{\mathsf{h}}_j, \;\; \tilde{\mathsf{h}}_i = \sum_{j=0}^{n} W_i^j \mathsf{h}_j.$$

The symmetric-reduced WDVV yields

$$\begin{cases} \left(F^{(2)}(0)-1\right)^2 = 0, & \text{if } n \text{ is odd and } \mathbf{d}=(2,2); \\ \left(F^{(2)}(0)-1\right)\left(F^{(2)}(0)-4\right) = 0, & \text{if } \mathbf{d}=(3); \\ \left(F^{(2)}(0) - \frac{-\sum_{j=0}^{n} jM_j^1 W_n^j + \mathsf{b}(\mathbf{d}) \sum_{j=0}^{n} jM_j^1 W_{n-\mathsf{a}}^j}{\mathsf{a}\prod_{i=1}^{r} d_i}\right)^2 = 0\ , & \text{if } l = \frac{n-1}{\mathsf{a}} \in \mathbb{Z}_{\geq 2}; \\ 0, & \text{otherwise.} \end{cases}$$

The expression

$$-\sum_{j=0}^{n} jM_j^1 W_n^j + \mathsf{b}(\mathbf{d}) \sum_{j=0}^{n} jM_j^1 W_{n-\mathsf{a}}^j$$

comes from the Euler vector field written in the basis $\tilde{\mathsf{h}}_i$'s.

Coordinates dual to small quantum cohomology

Beauville-Givental:

$$\tilde{\mathrm{h}}^{n+1} = \mathrm{b}(\mathbf{d})\tilde{\mathrm{h}}^{n+1-\mathrm{a}(n,\mathbf{d})}.$$

This suggests us to use the coordinates τ^i dual to $\tilde{\mathrm{h}}_i$.

- Length 3 genus 0 invariants in τ-coordinates has a closed formula.
- The essentially linear recursion in τ-coordinates is simple:

$$\begin{aligned} & (\partial_{\tau^1} \diamond \partial_{\tau^{i-1}}) \circ (\partial_{\tau^j} \circ \partial_{\tau^k}) + (\partial_{\tau^1} \circ \partial_{\tau^{i-1}}) \circ (\partial_{\tau^j} \diamond \partial_{\tau^k}) \\ & -(\partial_{\tau^1} \diamond \partial_{\tau^j}) \circ (\partial_{\tau^{i-1}} \circ \partial_{\tau^k}) - (\partial_{\tau^1} \circ \partial_{\tau^j}) \circ (\partial_{\tau^{i-1}} \diamond \partial_{\tau^k}) \\ = \quad & \partial_{\tau^i}\partial_{\tau^j}\partial_{\tau^k} + \partial_{\tau^1}\partial_{\tau^i}\partial_{\tau^{j+k}} - \partial_{\tau^{i-1}}\partial_{\tau^{j+1}}\partial_{\tau^k} - \partial_{\tau^1}\partial_{\tau^j}\partial_{\tau^{i+k-1}}. \end{aligned}$$

- Application: we develop an algorithm to effectively compute $F^{(0)}$ from the mirror formula.
- A byproduct: a simple proof of Zinger's convergence theorem for complete intersections.

Square root recursion

Our remaining task is to compute $z_k := F^{(k)}(0)$ for $k \geq 3$. We write a package to extract algebraic equations for $F^{(k)}(0)$ from

$$\sum_{k=1}^{l} \sum_{0 \leq J \leq l} \sum_{a=0}^{n} \sum_{b=0}^{n} \binom{l-1}{k-1} \binom{l}{J} \partial_{\tau^l} \partial_{\tau^a} F^{(k)}(0) \eta^{ab} \partial_{\tau^{l-J}} \partial_{\tau^b} F^{(l+1-k)}(0)$$
$$+2(l-1) \sum_{k=2}^{l} \sum_{0 \leq J \leq l} \binom{l-2}{k-2} \binom{l}{J} F^{(k)}(0) F^{(l+2-k)}(0) = 0.$$

($2 \leq l \leq \frac{m}{2}$ when n is odd).
We take a quintic 4-fold as an example.

$$2\, z_2^2 - 8352000\, z_2 + 8719488000000,$$

which factors as

$$2\, (z_2 - 2088000)^2\, .$$

So $F^{(2)}(0) = 2088000$.

Square root recursion

$$46080\, z_2^2 + 8\, z_2 z_3 + 3119454720000\, z_2$$
$$-16704000\, z_3 - 6714318458880000000.$$

Substituting $z_2 = 2088000$ we get 0, i.e. a trivial equation.

$$-586224\, z_2^3 + 3190863801600\, z_2^2 + 1644480\, z_2 z_3 + 12\, z_3^2$$
$$+12\, z_2 z_4 - 7369983201945600000\, z_2$$
$$+6501980160000\, z_3 - 25056000\, z_4$$
$$+8870266887085670400000000.$$

Substituting $z_2 = 2088000$ we get

$$12\,(z_3 + 413985600000)^2\,,$$

again a quadratic equation with two equal roots! So $F^{(3)}(0) = -413985600000$.

Square root recursion conjecture

Conjecture

(non-precise form) Suppose the multidegree $\mathbf{d} \neq (3)$. *Recall* $m = \mathrm{rank}\ H^n(X)$.

- *In even dimesions* $F^{(k)}(0)$ *can be recursively computed by square root recursion.*
- *In odd dimensions* $F^{(k)}(0)$ *for* $k \leq \frac{m}{4} + 1$ *can be recursively computed by square root recursion.*
- *All the other equations are trivial.*

We have also a conjectural way to compute $F^{(k)}(0)$ for $k > \frac{m}{4} + 1$ when n is odd, which suggests the existence of a new theory of invariants.

Odd dimension puzzle

Recall

$$\sum_{k=1}^{l}\sum_{0\le J\le l}\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{l-1}{k-1}\binom{l}{J}\partial_{\tau^l}\partial_{\tau^a}F^{(k)}(0)\eta^{ab}\partial_{\tau^{l-J}}\partial_{\tau^b}F^{(l+1-k)}(0)$$

$$+2(l-1)\sum_{k=2}^{l}\sum_{0\le J\le l}\binom{l-2}{k-2}\binom{l}{J}F^{(k)}(0)F^{(l+2-k)}(0)=0.$$

($2\le l\le \frac{m}{2}$ when n is odd).

Conjecture (Sqrt recursion conjecture in odd dim)

We do not use $F^{(k)}(0)=0$ for $k>\frac{m}{2}$. Then formally solving the symmetric-reduced WDVV yields the correct $F^{(l)}(0)$ for $l\le\frac{m}{2}$.

Example

$n = 3$, $\mathbf{d} = (2,2,2)$. $m = \dim H^3_{\mathrm{prim}}(X) = 28$.

$$F^{(2)}(0) = 4 = 2^2,\ F^{(3)}(0) = -8 = -2^3,\ F^{(4)} = 32 = 2^5,$$

$$F^{(5)}(0) = -200 = -2^3 5^2, F^{(6)}(0) = 1728 = 2^6 3^3,$$

$$F^{(7)}(0) = -19208 = -2^3 7^4,\ F^{(8)}(0) = 262144 = 2^{18},$$

$$F^{(9)}(0) = -4251528 = -2^3 3^{12},\ F^{(10)} = 80000000 = 2^{10} 5^7,$$

$$F^{(11)}(0) = -1714871048 = -2^3 11^8,\ F^{(12)}(0) = 41278242816 = 2^{21} 3^9,$$

$$F^{(13)}(0) = -1102867934792 = -2^3 13^{10},$$

$$F^{(14)}(0) = 32396521357312 = 2^{14} 7^{11}.$$

Conjecture

When $n = 3$, $\mathbf{d} = (2,2,2)$,

$$F^{(k)}(0) = 8(-1)^k k^{k-3},\ \textit{for } 1 \le k \le 14.$$

Square root recursion

- We have shown the conjecture for $F^{(2)}(0)$.
- The last statement on trivial equations gives a way to get a closed formula for $F^{(k)}$ in terms of lower $F^{(i)}$ for $i < k$.
- For $\mathbf{d} = (3)$, i.e. cubic hypersurface, we compute $F^{(k)}(0)$ by geometric methods: study the Fano variety of lines, and the reduce genus one Gromov-Witten invariants.

Closed fomula of $F^{(2)}$

Let Φ be the $n \times n$ matrix with entries

$$\Phi^i_j = \begin{cases} a, & \text{if } j = 1, i = 1, \\ (1-i)t^i, & \text{if } j = 1, i =\geq 2, \\ \frac{1}{\prod_{i=1}^r d_i} F^{(0)}_{1,j-1,n-i} - \delta_{i,1} F^{(1)}_{j-1} - \delta_{i,j-1} F^{(1)}_1, & \text{if } 2 \leq j \leq n. \end{cases}$$

Conjecture (= Corollary of Square root recursion conjecture)

Let $X = X_n(\mathbf{d})$ be an n-dimensional smooth non-exceptional complete intersection of multidegree $\mathbf{d}$, with $n \geq 3$ and $\mathbf{d} \neq (3)$. Then

$$F^{(2)} = \frac{1}{\prod_{i=1}^r d_i} (\partial_{t^{n-1}} F^{(1)}, \ldots, \partial_{t^0} F^{(1)}) \Phi^{-1} \begin{pmatrix} 0 \\ \partial_{t^1}\partial_{t^1} F^{(1)} \\ \cdots \\ \partial_{t^1}\partial_{t^{n-1}} F^{(1)} \end{pmatrix}.$$

For cubic hypersurfaces of dimension $n \geq 3$,

$$F^{(2)} = \frac{1}{3} (\partial_{t^{n-1}} F^{(1)}, \ldots, \partial_{t^0} F^{(1)}) \Phi^{-1} \begin{pmatrix} -\frac{n-1}{3} \\ \partial_{t^1}\partial_{t^1} F^{(1)} \\ \cdots \\ \partial_{t^1}\partial_{t^{n-1}} F^{(1)} \end{pmatrix}.$$

Cubic hypersurfaces: Fano variety of lines

Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Recall

$$\left(F^{(2)}(0) - 1\right)\left(F^{(2)}(0) - 4\right) = 0.$$

- $\overline{\mathcal{M}}_{0,0}(X,1)$ is the *Fano variety of lines* in X.
- $\iota_X : \overline{\mathcal{M}}_{0,0}(X,1) \hookrightarrow G_2(\mathbb{C}^{n+2})$. This enable us to do kind of Schubert calculus on $\overline{\mathcal{M}}_{0,0}(X,1)$.
- $\Psi : H^n_{\mathrm{prim}}(X) \xrightarrow{\sim} H^{n-2}_{\mathrm{prim}}\left(\overline{\mathcal{M}}_{0,0}(X,1)\right)$.
- Using Galkin-Shinder's result on the Betti number of $\overline{\mathcal{M}}_{0,0}(X,1)$, we determine the cohomology ring structure of $\overline{\mathcal{M}}_{0,0}(X,1)$ and by the way we get $F^{(2)}(0) = 1$ for X.

Cubic hypersurfaces: essentially linear recursion

Theorem

(i) *For the cubic threefold X, F can be reconstructed by from $F^{(0)}$ and $F^{(2)}(0)$, $F^{(4)}(0)$.*

(ii) *For cubic hypersurfaces X with $\dim X \geq 4$, F can be reconstructed from $F^{(0)}$ and $F^{(2)}(0)$.*

Cubic hypersurfaces: from genus 1 to genus 0

Let $\gamma_i = \mathrm{h}_i$ the i-th power of the hyperplane class for $0 \le i \le n$, and $\gamma_{n+1}, \cdots, \gamma_{n+m}$ a basis of $H^*_{\mathrm{prim}}(X)$. By topological recursion relation in genus 1,

$$\begin{aligned}
& \langle \psi\gamma_b, \gamma_c \rangle_{1,1} \\
= \quad & \frac{1}{\prod_{i=1}^r d_i} \langle \gamma_b, \gamma_c, \mathrm{h}_{n-1} \rangle_{0,1} \langle \mathrm{h} \rangle_{1,0} + \frac{1}{\prod_{i=1}^r d_i} \langle \gamma_b, \gamma_c, 1 \rangle_{0,3,0} \langle \mathrm{h}_n \rangle_{1,1,1} \\
& + \frac{1}{24} \sum_{e=0}^{n+m} \sum_{f=0}^{n+m} \langle \gamma_b, \gamma_e, g^{ef}\gamma_f, \gamma_c \rangle_{0,1}.
\end{aligned}$$

Then we apply Zinger's Standard versus Reduced formula:

$$\begin{aligned}
& \langle \psi^{a_1}\mu_1, \ldots, \psi^{a_k}\mu_k \rangle_{1,\beta} - \langle \psi^{a_1}\mu_1, \ldots, \psi^{a_k}\mu_k \rangle^0_{1,\beta} \\
& = \text{genus 0 Gromov-Witten invariants}
\end{aligned}$$

to $\langle \psi\gamma_b, \gamma_c \rangle_{1,1}$. This reproves $F^{(2)}(0) = 1$.

Cubic 3-folds: from genus 1 to genus 0

The cubic 3-folds are special: $F^{(4)}(0)$ cannot be computed from the symmetric-reduced WDVV.

$$
\begin{aligned}
& \langle \psi\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \rangle_{1,2} \\
= \quad & \frac{1}{3} \sum_{i=1}^{5} (\pm) \langle \gamma_0, \gamma_i, 1 \rangle_{0,3,0} \langle \mathrm{h}_3, \cdots, \hat{\gamma}_i, \cdots \rangle_{1,5,2} \\
& + \frac{1}{3} \sum_{i=1}^{5} (\pm) \langle \gamma_0, \gamma_i, \mathrm{h}_2 \rangle_{0,1} \langle \mathrm{h}, \cdots, \hat{\gamma}_i, \cdots \rangle_{1,5,1} \\
& + \frac{1}{3} \sum_{\{i,j,k\} \subset [5]} (\pm) \langle \gamma_0, \gamma_i, \gamma_j, \gamma_k, \mathrm{h} \rangle_{0,5,1} \langle \mathrm{h}_2, \cdots, \hat{\gamma}_i, \hat{\gamma}_j, \hat{\gamma}_k \cdots \rangle_{1,3,1} \\
& + \frac{1}{3} \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \mathrm{h}_2 \rangle_{0,7,2} \langle \mathrm{h} \rangle_{1,1,0} \\
& + \frac{1}{24} \sum_{a=0}^{13} \langle \gamma_0, \Gamma_a, \Gamma^a, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \rangle_{0,8,2}.
\end{aligned}
$$

Cubic hypersurfaces: vanishing of certain reduced genus 1 GW invariants

Theorem

Let X be a smooth subvariety of $\mathbb{P}^N$. Let $\beta \in H_2(X;\mathbb{Z})$ such that $\mathrm{h} \cdot \beta = 1$, where h is the hyperplane class restricted to X. Then any reduced genus one invariant of degree β is 0.

Theorem

Let X be a cubic hypersurface in $\mathbb{P}^N$. Let $\alpha_1, \ldots, \alpha_k \in H^*(X)$. Then

$$\langle \alpha_1, \ldots, \alpha_k \rangle^0_{1,2} = 0 = \langle \psi\alpha_1, \alpha_2, \ldots, \alpha_k \rangle^0_{1,2}.$$

Cubic 3-folds: $F^{(4)}(0)$

Idea: we have a factorization of the evaluation maps $\mathrm{ev}_{[k]} = \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_k$

$$\begin{array}{ccc} \overline{\mathcal{M}}^0_{1,k}(X,2) & \xrightarrow{\ \mathrm{ev}_{[k]}\ } & X^k \\ \Phi_k \big\downarrow & \nearrow_{\mathrm{ev}_{[k]}} & \\ \overline{\mathcal{M}}_{0,[k]}(X,1) & & \end{array}$$

where

$$\overline{\mathcal{M}}_{0,[k]}(X,1) := \underbrace{\overline{\mathcal{M}}_{0,1}(X,1) \times_{\overline{\mathcal{M}}_{0,0}(X,1)} \times \cdots \times_{\overline{\mathcal{M}}_{0,0}(X,1)} \overline{\mathcal{M}}_{0,1}(X,1)}_{k \text{ factors}}.$$

Theorem

For cubic 3-folds, $F^{(4)}(0) = 0$.

Overview

We sketch our knowledge and tools on the leading terms $F^{(k)}(0)$ of non-exceptional smooth complete intersections of dimension ≥ 3.

$(n, \mathbf{d})$ \ $F^{(k)}(0)$	1	2	$3 \leq k \leq \lfloor \frac{m}{4} \rfloor + 1$	$k > \frac{m}{4} + 1$
$\mathbf{d} = (3)$, $n = 3$	eigen vector	geometric method	geometric method	
$\mathbf{d} = (3)$, $n \geq 4$	eigen vector	geometric method	essentially linear recursion	
$\mathbf{d} \neq (3)$, even n	eigen vector	sqrt recursion	sqrt recursion	
$\mathbf{d} \neq (3)$, odd n	eigen vector	sqrt recursion	sqrt recursion	sqrt recursion

An algorithm is implemented in our Macaulay2 package `QuantumCohomologyFanoCompleteIntersection`.

- Exceptional complete intersections: essentially linear recursions work (the even (2,2)-type case will be shown in the following).
- Border cases of Fano complete intersections (i.e. odd $(2,2)$-type, cubic hypersurfaces): hybrid recursions on $F^{(l)}(0)$.
- Non-exceptional, non-quasiexceptional complete intersections: the square root recursion conjecture says that essentially linear recursions NEVER do help to $F^{(l)}(0)$.

Question

Do such observations remain true for other families of Fano manifolds, e.g. Fano 3-folds?

Integrality and Positivity

Conjecture

Let X be a non-exceptional Fano complete intersections of dimension n and multidegree $\mathbf{d}$.

1. $F^{(k)}(0) \in \mathbb{Z}$.
2. $$\begin{cases} F^{(k)}(0) = 0, & \text{if } \mathbf{d} = 3, \text{ and } k = n+1; \\ F^{(k)}(0) > 0, & \text{if } k \text{ is even and } (\mathbf{d}, k) \neq (3, n+1); \\ F^{(k)}(0) < 0, & \text{if } k \text{ is odd and } (\mathbf{d}, k) \neq (3, n+1). \end{cases}$$

- The integrality: when n is odd we can deduce it from the integrality of genus 0 Gromov-Witten invariants of semipositive symplectic manifolds.
- The positivity is quite mysterious. We have no geometric interpretation.

Exceptional complete intersections: n even, $\mathbf{d} = (2,2)$

Theorem

Let X be an even dimensional complete intersection of two quadrics in $\mathbb{P}^{n+2}$, with $n \geq 4$. All the genus 0 Gromov-Witten invariants can be reconstructed from a special correlator

$$\langle \epsilon_1, \ldots, \epsilon_{n+3} \rangle_{0,n+3,\frac{n}{2}}.$$

Theorem

There exists an open (in the classical topology) neighborhood of the origin of $\mathbb{C}^{2n+4}$, on which the generating function $F(t^0, \ldots, t^{2n+3})$ is analytic and defines a semisimple Frobenius manifold.

By relating the special correlator to classical enumerative geometry, we obtain:

Theorem

For any 4-dimensional complete intersections of two quadrics in $\mathbb{P}^6$,

$$\langle \epsilon_1, \ldots, \epsilon_7 \rangle_{0,7,2} = \frac{1}{2}.$$

Monodromy group and the D_{n+3} lattice

Let $V = \mathbb{R}^{n+3}$ be the Euclidean space with the standard inner product. Let $\varepsilon_1, \ldots, \varepsilon_{n+3}$ be an orthonormal basis, and let

$$\begin{cases} \alpha_i = \varepsilon_i - \varepsilon_{i+1} \text{ for } 1 \leq i \leq n+2, \\ \alpha_{n+3} = \varepsilon_{n+2} + \varepsilon_{n+3}. \end{cases}$$

The Weyl group $D_{n+3} \subset \mathrm{GL}(n+3, \mathbb{R})$ is generated the reflections with respect to the α_i's. If one writes vectors in $\mathbb{R}^{n+3}$ in terms of the coordinates according to the basis $\varepsilon_1, \ldots, \varepsilon_{n+3}$, i.e.

$$\mathbf{v} = (v_1, \ldots, v_{n+3}) = \sum_{i=1}^{n+3} v_i \varepsilon_i,$$

then the group D_{n+3} coincides with the group generated by the permutations of the coordinates, and the change of signs

$$(v_1, \ldots, v_{n+1}, v_{n+2}, v_{n+3}) \mapsto (v_1, \ldots, v_{n+1}, -v_{n+2}, -v_{n+3}).$$

Monodromy group and the D_{n+3} lattice

From now on, let n be an even integer ≥ 4, and X be a smooth complete intersection of two quadric hypersurfaces in $\mathbb{P}^{n+2}$. By the work of Reid:

- $H^n_{\mathrm{prim}}(X)$ is a standard representation of D_{n+3}.
- The integral lattice $H^n_{\mathrm{prim}}(X) \cap H^n(X;\mathbb{Z})$ is generated by the roots α_i's of D_{n+3}.
- There is an isometry

$$V \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \left(H^n_{\mathrm{prim}}(X), (-1)^{\frac{n}{2}}(.,.)\right).$$

- $H^n(X;\mathbb{Z})$ is generated by the classes of $\frac{n}{2}$-planes in X.
- Define

$$\epsilon_i = \begin{cases} \varepsilon_i, & \text{if } n \equiv 0 \mod 4; \\ \sqrt{-1}\varepsilon_i, & \text{if } n \equiv 2 \mod 4. \end{cases}$$

  Then $\epsilon_1, \ldots, \epsilon_{n+3}$ is an *orthonormal basis* of $H^n_{\mathrm{prim}}(X)$.

Invariant theory of D_{n+3}

Let $t^{n+1}, \dots, t^{2n+3}$ be the basis of $H^*_{\mathrm{prim}}(X)^\vee$ dual to $\epsilon_1, \dots, \epsilon_{n+3}$. By the invariant theory of Weyl groups, the polynomial invariants of D_{n+3} are generated by $s_1, \dots, s_{n+3}$, where

$$s_i = \frac{1}{(2i)!} \sum_{j=n+1}^{2n+3} (t^j)^{2i}, \text{ for } 1 \leq i \leq n+2,$$

and

$$s_{n+3} = \prod_{j=n+1}^{2n+3} t^j.$$

Moreover, $s_1, \dots, s_{n+3}$ are algebraically independent.

Corollary

The genus g generating function $\mathcal{F}_g$ of X can be written in a unique way as a series of $s_1, \dots, s_{n+3}$.

Correlators of length 4

Theorem

Let X be an even dimensional complete intersection of two quadrics in $\mathbb{P}^{n+2}$, with $n \geq 4$. Then

$$\frac{\partial^2 F}{(\partial s_1)^2}(0) = 1, \ \frac{\partial F}{\partial s_2}(0) = -2.$$

Equivalently, for $1 \leq a, b \leq n+3$,

$$\langle \epsilon_a, \epsilon_a, \epsilon_b, \epsilon_b \rangle_{0,1} = 1.$$

Ingredients of the proof:

- Monodromy group;
- From genus 1 to genus 0;
- Integrality of degree 1 invariants.

Reconstruction theorem

By the invariants of length 4, an essentially linear recursion yields

Theorem

With the knowledge of the 4-point invariants, all the invariants can be reconstructed from the WDVV, the deformation invariance, and the special correlator

$$\left\langle \epsilon_1, \ldots, \epsilon_{n+3} \right\rangle_{0,n+3,\frac{n}{2}}.$$

Special correlator

- There are choices of the D_{n+3}-lattices.
- Observation:
  The WDVV equations and the knowledge of correlators of length 4 can at most determine the special correlator with a freedom of signs, unless it vanishes.

Conjecture

Set the special correlator to be an indeterminate z. Let $F(t_0, \ldots, t_{2n+3}; z)$ be the generating function of primary genus 0 Gromov-Witten invariants of X determined by the reconstruction theorem. Then $F(t_0, \ldots, t_{2n+3}; z)$ satisfies WDVV and the monodromy invariance.

Semisimplicity

Theorem (Dubrovin)

A semisimple Frobenius manifold has a unique normalized Euler field.

The cutoff of F at order 3 is a function of $t^0, \ldots, t^n$ and

$$s_1 = \sum_{i=n+1}^{2n+3} (t^i)^2.$$

It has symmetries $= \mathrm{O}\big(H^n_{\mathrm{prim}}(X)\big)$. On the contrary, C. Jordan's theorem: the degree 4 form

$$s_2 = \sum_{i=n+1}^{2n+3} (t^i)^4$$

has only finitely many automorphisms. So we can expect that the information of correlators of length 4 implies the semisimplicity.

Middle dimensional planes

Let $\lambda_0, \ldots, \lambda_{n+2} \in \mathbb{C}$ be pairwise distinct. Let

$$\varphi_1(Y_0, \ldots, Y_{n+2}) = \sum_{i=0}^{n+2} Y_i^2, \ \varphi_2(Y_0, \ldots, Y_{n+2}) = \sum_{i=0}^{n+2} \lambda_i Y_i^2,$$

and $X = \{\varphi_1 = \varphi_2 = 0\} \subset \mathbb{P}^{n+2}$. Make a change of coordinates

$$W_i = \frac{Y_i}{\sqrt{\prod_{\substack{0 \leq j \leq n+2 \\ j \neq i}} (\lambda_i - \lambda_j)}}.$$

Then X contains the plane S defined by

$$\sum_{i=0}^{n+2} \lambda_i^k W_i = 0, \ \text{for } 0 \leq k \leq \frac{n}{2} + 1.$$

- For a subset $I \subset [0, n+2]$, let S_I be the $\frac{n}{2}$-plane obtained by reversing the sign of the i-th homogeneous coordinate of the points on S for all $i \in I$.
- Denote the complement of I by $C(I)$. Then $S_I = S_{C(I)}$.

An explicit lattice

- $\varsigma_i := [S_i]$.
- For $1 \leq i \leq n+3$, we define

$$\varepsilon_i = \varsigma_{i-1} - \frac{1}{n+1}\sum_{i=0}^{n+2} \varsigma_i + \frac{1}{2(n+1)} \mathrm{h}_{n/2}.$$

- $$\begin{cases} \alpha_i = \varsigma_{i-1} - \varsigma_i \text{ for } 1 \leq i \leq n+2, \\ \alpha_{n+3} = \varsigma_{n+1} + \varsigma_{n+2} + 2\varsigma - \mathrm{h}_{n/2}. \end{cases}$$

Enumerative correlators

Denote the i-th projection from X^{n+3} to X by q_i. Consider the product of the evaluation morphisms

$$\mathrm{ev}_1 \times \cdots \mathrm{ev}_{n+3} : \overline{\mathcal{M}}_{0,n+3}(X, \frac{n}{2}) \to X^{n+3}.$$

Let $I_1, \ldots, I_{n+3} \subset [0, n+2]$. We say that the correlator

$$\langle \varsigma_{I_1}, \ldots, \varsigma_{I_{n+3}} \rangle$$

is *enumerative* if there exists an irreducible component M of $\overline{\mathcal{M}}_{0,n+3}(X, \frac{n}{2})$ satisfying the following:

Enumerative correlators

(i) $\dim M$ equals the expected dimension.

(ii) The cycles $(\mathrm{ev}_1 \times \cdots \mathrm{ev}_{n+3})(M)$ and $q_1^{-1}S_{I_1}, \ldots, q_{n+3}^{-1}S_{I_{n+3}}$ intersect *properly*, i.e. the dimension of their (scheme theoretic) intersection is 0.

(iii) Each irreducible component of $\overline{\mathcal{M}}_{0,n+3}(X, \frac{n}{2})$ other than M has empty intersection with $q_1^{-1}S_{I_1}, \ldots, q_{n+3}^{-1}S_{I_n+3}$.

Our strategy to compute the special correlator:

1. Select $I_1, \ldots, I_{n+3} \subset [0, n+2]$, such that the correlator $\langle \varsigma_{I_1}, \ldots, \varsigma_{I_{n+3}} \rangle$ is enumerative.
2. Express $\langle \varsigma_{I_1}, \ldots, \varsigma_{I_{n+3}} \rangle$ in terms of the special correlator.
3. Solve the corresponding enumerative problem by counting curves. More precisely, compute the intersection multiplicities of the intersection

$$(\mathrm{ev}_1 \times \cdots \mathrm{ev}_{n+3})_*[M] \cap q_1^*[S_{I_1}] \cap \cdots \cap q_{n+3}^*[S_{I_{n+3}}]$$

in the condition (ii) above.

Enumerative correlators

Example

The correlator

$$\langle \varsigma_0, \ldots, \varsigma_{n+3} \rangle \tag{1}$$

should not be enumerative in general. For example, let $n = 4$. Then the intersection $S \cap S_i$ is a line, for $0 \leq i \leq 6$. The moduli space of conics on S passing through the seven lines has a positive dimension. So there are infinitely many conics passing through $S_0, \ldots, S_6$. Then the conditions (ii) and (iii) in the above definition cannot be true simultaneously.

Enumerative correlators

Lemma

S is the only $\frac{n}{2}$-plane in X that has non-empty intersections with each of $S_{[i,i+\frac{n}{2}-1]}$, for $0 \leq i \leq n+2$. Moreover S meets $S_{[i,i+\frac{n}{2}-1]}$ at exactly one point.

As a consequence, we consider

$$\left\langle \varsigma_{[0,\frac{n}{2}-1]}, \cdots, \varsigma_{[n+2,n+2+\frac{n}{2}-1]} \right\rangle_{0,n+3,\frac{n}{2}}$$

as a potentially enumerative correlator.

Lemma

Let X be a 4-dim smooth complete intersection of two quadrics in $\mathbb{P}^6$. Then $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle = \frac{1}{2}$ if only if

$$\langle \varsigma_{01}, \varsigma_{12}, \varsigma_{23}, \varsigma_{34}, \varsigma_{45}, \varsigma_{56}, \varsigma_{60} \rangle_{0,7,2} = 1.$$

Lemma

For general choices of $\lambda_0, \dots, \lambda_6$, there is no conic on S passing through the 7 points $S \cap S_{01}, \dots, S \cap S_{56}, S \cap S_{60}$.

- Every conic in a projective space lies on a plane. When a conic is not a double line, it spans a unique plane.
- To find conics on X passing through the planes $S_{01}, \dots, S_{60}$, we will first find all the planes in $\mathbb{P}^6$ that meets $S_{01}, \dots, S_{60}$.
- By the above results we need to find planes $\Sigma \not\subset X$ that meets $S_{01}, \dots, S_{60}$.

Counting Conics

Theorem

Let X be the 4 dimensional smooth complete intersection of two quadrics, given by $(\lambda_0, \ldots, \lambda_6) = (1, 2, 3, 4, 5, 6, 7)$. Then

(i) *There exists a unique conic C in X that meets $S_{i,i+1}$ for $i \in [0, 6]$.*

(ii) *The conic C is a free curve in X.*

(iii) *In the ring of dual numbers $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, up to a common multiple, the system of equations for conics passing through $S_{i,i+1}$ for $i \in [0, 6]$ has a unique solution.*

Key idea: solve the Plücker coordinates of S_C, the plane spanned by C.

Counting Conics

Theorem

For any 4-dimensional complete intersections of two quadrics in $\mathbb{P}^6$,

$$\langle \varsigma_{0,1}, \ldots, \varsigma_{6,0} \rangle_{0,7,2} = 1.$$

Corollary

For any 4-dimensional complete intersections of two quadrics in $\mathbb{P}^6$,

$$\langle \epsilon_1, \ldots, \epsilon_7 \rangle_{0,7,2} = \frac{1}{2}. \quad (2)$$

By the way we obtain a result of classical flavor.

Theorem

For general 4-dimensional smooth complete intersections X *of two quadrics in* $\mathbb{P}^6$, *there exists exactly one smooth conic that meets each of the 2-planes* $S_{i,i+1}$ *in* X *for* $0 \leq i \leq 6$.

Problems

Problem

Describe explicitly the conic C for general $\lambda_0, \ldots, \lambda_6$.

Question

Is the statement the above Theorem true in an appropriate sense (e.g. allowing singular conics or double lines), for all 4-dimensional smooth complete intersections of two quadrics in $\mathbb{P}^6$?

A conjecture on the explicit conic

$$h(\lambda_0, \ldots, \lambda_6) :=$$
$$\lambda_0^2\lambda_1\lambda_3 - \lambda_0^2\lambda_1\lambda_5 - \lambda_0^2\lambda_2\lambda_3 + \lambda_0^2\lambda_2\lambda_6 + \lambda_0^2\lambda_4\lambda_5 - \lambda_0^2\lambda_4\lambda_6 + \lambda_0\lambda_1\lambda_2\lambda_5$$
$$-\lambda_0\lambda_1\lambda_2\lambda_6 - \lambda_0\lambda_1\lambda_3\lambda_4 - \lambda_0\lambda_1\lambda_3\lambda_6 + \lambda_0\lambda_1\lambda_4\lambda_6 + \lambda_0\lambda_1\lambda_5\lambda_6 + \lambda_0\lambda_2\lambda_3\lambda_4$$
$$+\lambda_0\lambda_2\lambda_3\lambda_5 - \lambda_0\lambda_2\lambda_4\lambda_5 - \lambda_0\lambda_2\lambda_5\lambda_6 - \lambda_0\lambda_3\lambda_4\lambda_5 + \lambda_0\lambda_3\lambda_4\lambda_6 - \lambda_1\lambda_2\lambda_3\lambda_5$$
$$+\lambda_1\lambda_2\lambda_3\lambda_6 + \lambda_1\lambda_3\lambda_4\lambda_5 - \lambda_1\lambda_4\lambda_5\lambda_6 - \lambda_2\lambda_3\lambda_4\lambda_6 + \lambda_2\lambda_4\lambda_5\lambda_6,$$

$$\mu_i(\lambda_0, \ldots, \lambda_6) := h(\lambda_i, \lambda_{i+1}, \lambda_{i+2}, \lambda_{i+3}, \lambda_{i+4}, \lambda_{i+5}, \lambda_{i+6}) \cdot \prod_{j=i+1}^{i+6} (\lambda_i - \lambda_j)$$

for $0 \leq i \leq 6$, where the subscripts are understood in the mod 7 sense. We define a quadric hypersurface Q by

$$\sum_{i=0}^{6} \mu_i(\lambda_0, \ldots, \lambda_6) W_i^2 = 0.$$

Then the 2-plane S_C spanned by the conic C is contained in Q.

Let X be a non-exceptional Fano complete intersection in a projetive space. Let $G(t^0, \ldots, t^{n+m})$ be the generating function of genus 1 primary GW invariants of X. Define

$$G^{(k)} = \frac{\partial^k G}{(\partial s)^k}|_{s=0}.$$

By the monodromy symmetric reduction of Getzler relations, we get:

Theorem

$G^{(0)}$ can be reconstructed from $\frac{\partial G^{(0)}}{\partial t^i}(0)$, for $1 \leq i \leq n$, and genus zero GW invariants of X.

Then we compute the initial values $\frac{\partial G^{(0)}}{\partial t^i}(0)$ via Zinger's reduced genus 1 GW invariants.

Series associated with (modified) hypergeometric series

Let X be a Fano complete intersection of multidegree $\mathbf{d} = (d_1, \ldots, d_r)$ in $\mathbb{P}^{n-1}$.

$$L_0(q) := \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{k-1}(k|\mathbf{d}| + 1 - in)}{k!} (\frac{\mathbf{d}^{\mathbf{d}} q}{n})^k,$$

$$\Phi_0(q) := L_0(q)^{\frac{r+1}{2}} \cdot \left(1 + \mathbf{d}^{\mathbf{d}}(1 - \frac{|\mathbf{d}|}{n}) q L_0(q)^{|\mathbf{d}|}\right)^{-\frac{1}{2}},$$

where

$$|\mathbf{d}| := \sum_{i=1}^{r} d_i, \ \mathbf{d}^{\mathbf{d}} := \prod_{i=1}^{r} d_i^{d_i}.$$

Series associated with (modified) hypergeometric series

$$
\begin{aligned}
\Phi_1(q) \quad := \quad & \frac{L_0(q)^{\frac{r-1}{2}} \cdot \left(1 + \mathbf{d}^{\mathbf{d}}(1 - \frac{|\mathbf{d}|}{n}) q L_0(q)^{|\mathbf{d}|}\right)^{-\frac{7}{2}}}{24|\mathbf{d}|n^3} \times \left(|\mathbf{d}|^3 \left(|\mathbf{d}|n - |\mathbf{d}| - 3r^2 + 1\right) L_0(q) \right. \\
& + |\mathbf{d}|^2 n \left(2|\mathbf{d}|^2 - 6|\mathbf{d}|n - 6|\mathbf{d}|r + 3n^2 + 6nr + n + 3r^2 - 1\right) L_0(q)^n \\
& + 3|\mathbf{d}|^2 (n - |\mathbf{d}|) \left(|\mathbf{d}|n - |\mathbf{d}| - 3r^2 + 1\right) L_0(q)^{n+1} \\
& + |\mathbf{d}|n(n - |\mathbf{d}|) \left(4|\mathbf{d}|^2 - 5|\mathbf{d}|n - 12|\mathbf{d}|r - 2n^2 + 6nr + n + 6r^2 - 2\right) L_0(q)^{2n} \\
& + 3|\mathbf{d}|(n - |\mathbf{d}|)^2 \left(|\mathbf{d}|n - |\mathbf{d}| - 3r^2 + 1\right) L_0(q)^{2n+1} \\
& + n(n - |\mathbf{d}|)^2 \left(2|\mathbf{d}|^2 + |\mathbf{d}|n - 6|\mathbf{d}|r + 3r^2 - 1\right) L_0(q)^{3n} \\
& \left. + (n - |\mathbf{d}|)^3 \left(|\mathbf{d}|n - |\mathbf{d}| - 3r^2 + 1\right) L_0(q)^{3n+1} \right) \\
& + \frac{3r^2 - 2|\mathbf{d}| \sum_{k=1}^{r} \frac{1}{d_k} - 1}{24|\mathbf{d}|} L_0(q)^{\frac{r-1}{2}} \left(L_0(q) - 1\right) \left(1 + \mathbf{d}^{\mathbf{d}}(1 - \frac{|\mathbf{d}|}{n}) q L_0^{|\mathbf{d}|}\right)^{-\frac{1}{2}}.
\end{aligned}
$$

Constants associated with hypergeometric series

Denote the Fano index by $\nu_{\mathbf{d}}$. Following Popa-Zinger, we define $c_{p,l}^{(\beta)}$, $\tilde{c}_{p,l}^{(\beta)} \in \mathbb{Q}$ with $p, \beta, l \geq 0$ by

$$\sum_{\beta=0}^{\infty} \sum_{l=0}^{\infty} c_{p,l}^{(\beta)} w^l q^\beta = \sum_{\beta=0}^{\infty} q^\beta \frac{(w+\beta)^p \prod_{k=1}^{r} \prod_{i=1}^{d_k \beta} (d_k w + i)}{\prod_{j=1}^{\beta} (w+j)^n},$$

$$\sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2 \geq 0}} \sum_{k=0}^{p-\nu_{\mathbf{d}}\beta_1} \tilde{c}_{p,k}^{(\beta_1)} c_{k,l}^{(\beta_2)} = \delta_{\beta,0}\delta_{p,l}, \text{ for } \beta, l \in \mathbb{Z}_{\geq 0},\ l \leq p - \nu_{\mathbf{d}}\beta.$$

Series associated with (modified) hypergeometric series

Define

$$\Theta_p^{(0)}(q) \quad := \quad \Phi_0(q) \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{p,p-\nu_{\mathbf{d}}\beta} q^{\beta} L(q)^{p-\nu_{\mathbf{d}}\beta}.$$

$$\begin{aligned}\Theta_p^{(1)}(q) \quad := \quad & \Phi_0(q) \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{p,p-\nu_{\mathbf{d}}\beta-1} q^{\beta} L(q)^{p-\nu_{\mathbf{d}}\beta-1} \\ & +\Phi_1(q) \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{p,p-\nu_{\mathbf{d}}\beta} q^{\beta} L(q)^{p-\nu_{\mathbf{d}}\beta} \\ & +\Phi_0'(q) \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{p,p-\nu_{\mathbf{d}}\beta} q^{\beta+1} (p-\nu_{\mathbf{d}}\beta) L(q)^{p-\nu_{\mathbf{d}}\beta-1} \\ & +L(q)'\Phi_0(q) \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{p,p-\nu_{\mathbf{d}}\beta} q^{\beta+1} \binom{p-\nu_{\mathbf{d}}\beta}{2} L(q)^{p-\nu_{\mathbf{d}}\beta-2}.\end{aligned}$$

Genus 1 GW invariant with 1 marked point

Theorem

Let X be a smooth complete intersection of multidegree $\mathbf{d}$ in $\mathbb{P}^{n-1}$, with Fano index $\nu_{\mathbf{d}} \geq 1$. For $0 \leq b \leq \frac{n-1}{\nu_{\mathbf{d}}}$,

$$\begin{aligned}
&\langle \mathsf{h}_{1+\nu_{\mathbf{d}}b} \rangle_{1,b} \\
= \quad & -\frac{\prod_{k=1}^{r} d_k}{24} \mathrm{Res}_{w=0} \left\{ \frac{(1+w)^n (\tilde{c}^{(b)}_{1+\nu_{\mathbf{d}}b,0} + \tilde{c}^{(b)}_{1+\nu_{\mathbf{d}}b,1} w)}{w^{n-r} \prod_{k=1}^{r} (d_k w + 1)} \right\} \\
& + \frac{1}{2} \mathrm{Coeff}_{q^b} \left\{ \frac{\Theta^{(0)}_{1+\nu_{\mathbf{d}}b}(q) \left(\sum_{p=0}^{n-1-r} \Theta^{(1)}_p(q) \Theta^{(0)}_{n-1-r-p}(q) + \sum_{p=1}^{r} \Theta^{(1)}_{n-p}(q) \Theta^{(0)}_{n-1-r+p}(q) \right)}{\Phi_0(q)} \right\} \\
& + \frac{n}{24} \mathrm{Coeff}_{q^b} \left\{ \left(\frac{n-1}{2} - \sum_{k=1}^{r} \frac{1}{d_k} \right) \left(1 - \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{1+\nu_{\mathbf{d}}b, 1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta} q^{\beta} \left(L(q)^{1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta} - 1 \right) \right) \right. \\
& - L(q)' \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{1+\nu_{\mathbf{d}}b, 1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta} q^{\beta+1} \binom{1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta}{2} L(q)^{1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta-2} \\
& - \frac{\Phi_0'(q)}{\Phi_0(q)} \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{1+\nu_{\mathbf{d}}b, 1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta} q^{\beta+1} (1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta) L(q)^{1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta-1} \\
& \left. - \sum_{\beta=0}^{\infty} \tilde{c}^{(\beta)}_{1+\nu_{\mathbf{d}}b, 1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta-1} q^{\beta} \left(L(q)^{1+\nu_{\mathbf{d}}b-\nu_{\mathbf{d}}\beta-1} - 1 \right) \right\}.
\end{aligned}$$

Conclusion

Corollary

Assuming the square root recursion conjecture, we have an effective algorithm for Genus 1 GW invariants of non-exceptional Fano complete intersections, with only ambient insertions.

This is covered by the work of Argüz-Bousseau-Pandharipande-Zvonkine.

Question

What can we say about a cohomological field theory with a sufficiently large group of symmetries (typically coming from monodromies)?

Thank You!