

Zero cycles on the moduli space of curves

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Abstract

While the Chow groups of 0-dimensional cycles on the moduli spaces of Deligne-Mumford stable pointed curves can be very complicated, the span of the 0-dimensional tautological cycles is always of rank 1. The question of whether a given moduli point

$$[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$$

determines a tautological 0-cycle is subtle. Our main results address the question for curves on rational and $K3$ surfaces. If C is a nonsingular curve on a nonsingular rational surface of positive degree with respect to the anticanonical class, we prove $[C, p_1, \dots, p_n]$ is tautological if the number of markings does not exceed the virtual dimension in Gromov-Witten theory of the moduli space of stable maps. If C is a nonsingular curve on a $K3$ surface, we prove $[C, p_1, \dots, p_n]$ is tautological if the number of markings does not exceed the genus of C and every marking is a Beauville-Voisin point. The latter result provides a connection between the rank 1 tautological 0-cycles on the moduli of curves and the rank 1 tautological 0-cycles on $K3$ surfaces.

Several further results related to tautological 0-cycles on the moduli spaces of curves are proven. Many open questions concerning the moduli points of curves on other surfaces (Abelian, Enriques, general type) are discussed.

0 Introduction

0.1 Moduli of curves

Let (C, p_1, \dots, p_n) be a Deligne-Mumford stable curve of genus g with n marked points defined over \mathbb{C} . Let

$$[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$$

be the associated *moduli point* in the moduli space.¹ As a Deligne-Mumford stack, $\overline{\mathcal{M}}_{g,n}$ is nonsingular, irreducible, and of \mathbb{C} -dimension $3g-3+n$. Though the moduli spaces $\overline{\mathcal{M}}_{g,n}$ can be irrational and complicated, their study has been marked by the discovery of beautiful mathematical structures.

Fundamental to the geometry of the moduli spaces of stable pointed curves are three basic types of morphisms:

- (i) forgetful morphisms

$$p : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$$

defined by dropping a marking,

- (ii) irreducible boundary morphisms

$$q : \overline{\mathcal{M}}_{g-1,n+2} \rightarrow \overline{\mathcal{M}}_{g,n}$$

defined by identifying two markings to create a node,

- (iii) reducible boundary morphisms

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \rightarrow \overline{\mathcal{M}}_{g,n},$$

where $n = n_1 + n_2$ and $g = g_1 + g_2$, defined by identifying the markings of separate pointed curves.

Following [13, Section 0.1], the tautological rings²

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$$

are defined as the smallest system of \mathbb{Q} -subalgebras (with unit) closed under push-forward by all morphisms (i)-(iii). We denote the group of tautological k -cycles by

$$R_k(\overline{\mathcal{M}}_{g,n}) = R^{3g-3+n-k}(\overline{\mathcal{M}}_{g,n}).$$

For an introduction to the current study of tautological classes, we refer the reader to [14, 30].

¹Stability requires $2g - 2 + n > 0$ which we always impose when we write $\overline{\mathcal{M}}_{g,n}$.

²Chow groups will be taken with \mathbb{Q} -coefficients unless explicitly stated otherwise.

0.2 0-cycles in the tautological ring

Whenever the moduli space $\overline{\mathcal{M}}_{g,n}$ is rationally connected, we have

$$A_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}.$$

Rational connectedness is known at least in the cases appearing in Figure 1. For genus 23 and higher, $\overline{\mathcal{M}}_{g,n}$ is never rationally connected.

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n_{\max}	∞	10	12	14	15	12	15	11	8	9	3	10	1	0	2	0

Figure 1: $\overline{\mathcal{M}}_{g,n}$ is rationally connected for $n \leq n_{\max}$, see [3, 7, 9, 15, 26, 37].

On the other hand, the Chow groups of 0-cycles are of infinite rank as \mathbb{Q} -vector spaces at least in the following genus 1 and 2 cases (due to the existence³ of holomorphic p -forms):

$$A_0(\overline{\mathcal{M}}_{1,n \geq 11}), \quad A_0(\overline{\mathcal{M}}_{2,n \geq 14}).$$

Moreover, such forms⁴ and infinite ranks are expected in the following genus 3 and 4 cases:

$$A_0(\overline{\mathcal{M}}_{3,n \geq 15}), \quad A_0(\overline{\mathcal{M}}_{4,n \geq 16}).$$

While the data is insufficient for a general prediction, the following speculation would not be surprising.

Speculation 1 *For $g \geq 1$, the Chow group $A_0(\overline{\mathcal{M}}_{g,n})$ is of infinite rank except for finitely many (g, n) .*

On the other hand, the group $R_0(\overline{\mathcal{M}}_{g,n})$ of tautological 0-cycles is much better behaved. The following result was proven by Graber and Vakil in [20] and also in [13, 21].

Proposition 2 *For all (g, n) , we have $R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$.*

Since the proof is so short (and depends only upon structural properties of tautological classes), we present the argument here.⁵ Consider the moduli

³By results of Mumford and Srinivas (see [29, 32, 36] and [20, Remark 1.1]), the existence of a holomorphic p -form for $p \geq 1$ forces $A_0(\overline{\mathcal{M}}_{g,n})$ to have infinite rank. Constructions of such forms in $g = 1$ and $g = 2$ are well-known, see [14].

⁴There are no written proofs for the genus 3 and 4 claims, but these expectations, based on geometric calculations, have been communicated to us by C. Faber (in genus 3) and G. Farkas (in genus 4).

⁵We follow the path of the proof [13, 21]. See [13, Section 4] and [21, Section 5.1].

space $\overline{\mathcal{M}}_{0,2g+n}$ together with the boundary morphism

$$\iota : \overline{\mathcal{M}}_{0,2g+n} \rightarrow \overline{\mathcal{M}}_{g,n}$$

defined by pairing the first $2g$ markings to create g nodes. Since $\overline{\mathcal{M}}_{0,2g+n}$ is a rational variety,

$$R_0(\overline{\mathcal{M}}_{0,2g+n}) = A_0(\overline{\mathcal{M}}_{0,2g+n}) \cong \mathbb{Q}.$$

Therefore, all the moduli points in the image of ι are tautological and span a \mathbb{Q} -subspace of $R_0(\overline{\mathcal{M}}_{g,n})$ of rank 1. We will prove that the span equals $R_0(\overline{\mathcal{M}}_{g,n})$.

Using the additive generators of the tautological ring constructed in [19, Appendix], we need only consider 0-cycles on $\overline{\mathcal{M}}_{g,n}$ which are of a special form. The strata of $\overline{\mathcal{M}}_{g,n}$ are indexed by stable graphs Γ of genus g with n markings,

$$\iota_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,n}.$$

We need only consider 0-cycles

$$\iota_{\Gamma*} \left[\prod_{v \in \text{Vert}(\Gamma)} P(v) \right] \in R_0(\overline{\mathcal{M}}_{g,n}), \quad (1)$$

where $P(v)$ is a monomial in ψ and κ classes on the the moduli space $\overline{\mathcal{M}}_{g(v),n(v)}$ associated to the vertex v . Let $\deg(P(v))$ be the degree of the vertex class. Using the Getzler-Ionel vanishing in the strong form proven⁶ in [13, Proposition 2], we can impose the following additional restriction on (1):

$$g(v) > 0 \quad \Rightarrow \quad \deg(P(v)) < g - \delta_{0,n(v)}. \quad (2)$$

Suppose we have a vertex v of Γ with $g(v) > 0$. Using the vertex stability condition $2g(v) - 2 + n(v) > 0$, we deduce

$$g(v) - \delta_{0,n(v)} \leq 3g(v) - 3 + n(v).$$

But then we obtain

$$\deg(P(v)) < 3g(v) - 3 + n(v),$$

⁶See [12] for a much more effective approach to the boundary terms than provided by the argument of [13].

which is impossible since (1) is a 0-cycle. Therefore, we must have $g(v) = 0$ for all $v \in \text{Vert}(\Gamma)$.

The 0-cycle (1) is now easily seen to be in the image of

$$\iota_* : R_0(\overline{\mathcal{M}}_{0,2g+n}) \rightarrow R_0(\overline{\mathcal{M}}_{g,n}). \quad (3)$$

We conclude that the push-forward (3) is surjective. \diamond

0.3 Tautological points

Our central question here is how to decide whether a given moduli point

$$[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$$

determines a tautological 0-cycle.

While our focus is on the geometry of C , there is an interesting connection to arithmetic: Bloch and Beilinson have conjectured⁷ that for a nonsingular proper variety X defined over $\overline{\mathbb{Q}}$, the complex Abel-Jacobi map

$$\Phi_{k,\mathbb{Q}} : A_{\text{hom}}^k(X/\overline{\mathbb{Q}})_{\mathbb{Q}} \rightarrow J^k(X(\mathbb{C}))_{\mathbb{Q}}$$

to the intermediate Jacobian $J^k(X(\mathbb{C}))$ is injective (after tensoring with \mathbb{Q}). The map above factors through the usual Abel-Jacobi map of $X(\mathbb{C})$, and the image of $A_{\text{hom}}^k(X/\overline{\mathbb{Q}})_{\mathbb{Q}}$ in $A_{\text{hom}}^k(X(\mathbb{C}))_{\mathbb{Q}}$ is the set of k -cycles in $X(\mathbb{C})$ defined over $\overline{\mathbb{Q}}$ which are homologous to 0. If the Bloch-Beilinson conjecture holds for

$$X = \overline{\mathcal{M}}_{g,n},$$

the map

$$\Phi_{3g-3+n,\mathbb{Q}} : A_0(\overline{\mathcal{M}}_{g,n})_{\text{hom}} \rightarrow \text{Alb}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$$

would be injective on the set of 0-cycles defined over $\overline{\mathbb{Q}}$. But since $\overline{\mathcal{M}}_{g,n}$ is simply connected [6, Proposition 1.1], the Albanese variety is trivial. Since a tautological class in $A_0(\overline{\mathcal{M}}_{g,n})$ can be represented by a curve defined over $\overline{\mathbb{Q}}$, we would obtain the following consequence.

Speculation 3 *If the pointed curve (C, p_1, \dots, p_n) is defined over $\overline{\mathbb{Q}}$, then the associated moduli point in $A_0(\overline{\mathcal{M}}_{g,n})$ is tautological.*

⁷See [4, 5] for the original papers by Bloch and Beilinson and [25] for a detailed account. See [25, Conjecture 9.12] and the remark thereafter for the particular form of the conjecture that we have used.

A first step in the study of Speculation 3 is perhaps to use Belyi's Theorem to express the curve as a Hurwitz covering

$$C \rightarrow \mathbb{P}^1$$

ramified only over 3 points of \mathbb{P}^1 . Unfortunately, there has not been much progress in the direction of Speculation 3. However, we will present a result about cyclic covers of \mathbb{P}^1 in Section 5.

0.4 Curves on surfaces

Instead of studying the moduli points of special Hurwitz covers of \mathbb{P}^1 , our main results here concern the moduli points of curves on special surfaces.

Rational surfaces

Let S be a nonsingular projective rational surface over \mathbb{C} , and let $C \subset S$ be an irreducible nonsingular curve of genus g . The *virtual dimension* in Gromov-Witten theory of the moduli space of stable maps $\overline{\mathcal{M}}_g(S, [C])$ is given by the following formula

$$\text{vdim } \overline{\mathcal{M}}_g(S, [C]) = \int_{[C]} c_1(S) + g - 1.$$

Our first result gives a criterion for curves on rational surfaces in terms of the virtual dimension.

Theorem 4 *Let $C \subset S$ be an irreducible nonsingular curve of genus g on a nonsingular rational surface satisfying $\int_{[C]} c_1(S) > 0$. Let $p_1, \dots, p_n \in C$ be distinct points. If*

$$n \leq \text{vdim } \overline{\mathcal{M}}_g(S, [C]),$$

then $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ determines a tautological 0-cycle in $R_0(\overline{\mathcal{M}}_{g,n})$.

For Theorem 4, we always assume (g, n) is in the stable range

$$2g - 2 + n > 0.$$

If positivity

$$\int_{[C]} c_1(S) > 0 \tag{4}$$

holds, then Theorem 4 can be applied with $n = 0$ to obtain

$$[C] \in R_0(\overline{\mathcal{M}}_g).$$

In case S is toric, positivity (4) always holds for nonsingular curves of genus $g \geq 1$ since there exists an effective toric anticanonical divisor with affine complement. Whether positivity (4) can be avoided in Theorem 4 is an interesting question.⁸

As an example, consider a nonsingular curve

$$C_4 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$$

of genus 4 and bidegree (3, 3). Positivity (4) holds, and the virtual dimension here is 15, so all moduli points

$$[C_4, p_1, \dots, p_{15}] \in \overline{\mathcal{M}}_{4,15}$$

are tautological. Since the general curve of genus 4 is of the form C_4 , but not all points of $\overline{\mathcal{M}}_{4,16}$ are expected to be tautological, the virtual dimension bound on n in Theorem 4 should not have room for improvement here.

***K3* surfaces**

Let S be a nonsingular projective *K3* surface over \mathbb{C} . Unlike the case of a rational surface, the Chow group $A_0(S, \mathbb{Z})$ of 0-cycles of S is very complicated. However, there is a beautiful rank 1 subspace

$$\mathbf{BV} \subset A_0(S, \mathbb{Z})$$

spanned by points lying on rational curves of S . Following [1], define $p \in S$ to be a *Beauville-Voisin* point if $[p] \in \mathbf{BV}$.

Let $C \subset S$ be an irreducible nonsingular curve of genus g . The *virtual dimension* of the moduli space of stable maps $\overline{\mathcal{M}}_g(S, [C])$ is now

$$\text{vdim } \overline{\mathcal{M}}_g(S, [C]) = g - 1.$$

Important for us, however, will be the *reduced* virtual dimension g . Our second result gives a criterion for curves on *K3* surfaces.

⁸The issue is not unrelated to the Harbourne-Hartshorne conjecture and will be discussed in Section 2.2.

Theorem 5 *Let $C \subset S$ be an irreducible nonsingular curve of genus g on a $K3$ surface. Let*

$$p_1, \dots, p_n \in C$$

be distinct Beauville-Voisin points of S . If $n \leq g$, then $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ determines a tautological 0-cycle in $R_0(\overline{\mathcal{M}}_{g,n})$.

For example, consider a nonsingular curve of genus 11

$$(C_{11}, p_1, \dots, p_{11}) \subset S \tag{5}$$

in a primitive class on a $K3$ surface S with 11 distinct points. By Theorem 5,

$$[C_{11}, p_1, \dots, p_{11}] \in R_0(\overline{\mathcal{M}}_{11,11})$$

in case *all* the points p_i are Beauville-Voisin. By the Mukai correspondence [28], we can obtain the general moduli point of $\overline{\mathcal{M}}_{11,11}$ by varying the data (5) in the moduli space of polarized $K3$ surfaces of genus 11 with 11 points. Since $\overline{\mathcal{M}}_{11,11}$ is of Kodaira dimension 19 by [16, Theorem 5.1], the Chow group of 0-cycles is expected (but not known) to be complicated. In particular, the general moduli point of $\overline{\mathcal{M}}_{11,11}$ is not expected to be tautological. The geometry of $K3$ surfaces in genus 11 therefore suggests that a condition on the points is necessary.

The condition of Theorem 5 exactly links the rank 1 Beauville-Voisin subspace

$$\text{BV} \subset A_0(S, \mathbb{Z})$$

to the rank 1 tautological subspace

$$R_0(\overline{\mathcal{M}}_{g,n}) \subset A_0(\overline{\mathcal{M}}_{g,n}).$$

Other surfaces

Since every nonsingular curve lies on a nonsingular algebraic surface, results along the lines of Theorems 4 and 5 will always require special surface geometries. For nonsingular curves lying on Enriques and Abelian surfaces, we hope for results parallel to those in the rational and $K3$ surface cases. However, the questions are, at the moment, open. For the Enriques surface, there is a clear path, but the argument depends upon currently open questions about the nonemptiness of certain Severi varieties. For Abelian

surfaces, the matter appears more subtle (and there is no obvious line of argument that we can see).

For surfaces of general type, canonical curves play a very special role from the perspective of Gromov-Witten and Seiberg-Witten theories. A natural question to ask is whether a nonsingular canonical curve on a surface of general type always determine a tautological 0-cycle. We expect new strategies will be required to resolve such questions in the general type case.

0.5 Further results on tautological 0-cycles

We have seen that a moduli point

$$[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$$

need not determine a tautological 0-cycle. We can measure *how far away from tautological* moduli points of $\overline{\mathcal{M}}_{g,n}$ are by considering sums. Let

$$T(g, n) \in \mathbb{Z}_{>0}$$

be the smallest number satisfying the following condition: *for every point $Q_1 \in \overline{\mathcal{M}}_{g,n}$, there exist $T(g, n) - 1$ other points $Q_2, \dots, Q_{T(g,n)} \in \overline{\mathcal{M}}_{g,n}$ which together have a tautological sum*

$$[Q_1] + [Q_2] + \dots + [Q_{T(g,n)}] \in R_0(\overline{\mathcal{M}}_{g,n}).$$

An easy proof of the existence of $T(g, n)$ is given in Section 6. Finding good bounds for $T(g, n)$ appears much harder. Our main result here states that the growth of $T(g, n)$ for fixed g as $n \rightarrow \infty$ is at most linear in n . Can better asymptotics be found? For example, could $T(g, n)$ for fixed g be bounded independent of n ?

0.6 T -numbers for $K3$ surfaces

For comparison, we can consider the parallel question for a $K3$ surface S , namely: what is the smallest positive integer T such that for any given $p \in S$ we find $q_2, \dots, q_T \in S$ such that the sum

$$[p] + [q_2] + \dots + [q_T] \in A_0(S, \mathbb{Z})$$

lies in the Beauville-Voisin subspace $\text{BV} \subset A_0(S, \mathbb{Z})$?

On the one hand, we have $T \geq 2$, since $T = 1$ would be the statement that for every $p \in S$ we have $[p] \in \mathbf{BV}$, a contradiction since $A_0(S, \mathbb{Z})$ is infinite-dimensional and spanned by the classes $[p]$. On the other hand, since we have families of elliptic curves which sweep out S , the given point p must lie on a (possibly singular) genus 1 curve $E \subset S$. Let $R \subset S$ be a rational curve in an ample class. Since

$$R \cap E \neq \emptyset,$$

E contains a Beauville-Voisin point $z \in E$. We can always solve the equation

$$[p] + [q] = 2[z] \in A_0(E, \mathbb{Z})$$

for $q \in E$. We conclude that for any $p \in S$, there exists a $q \in S$ satisfying

$$[p] + [q] \in \mathbf{BV}.$$

The T -number for $K3$ surfaces is therefore just 2.

The Hilbert scheme $S^{[n]}$ of n points on S also has a holomorphic form and a distinguished Beauville-Voisin subspace in $A_0(S^{[n]}, \mathbb{Z})$. The holomorphic form shows that the T -number of $S^{[n]}$ is greater than 1. Using families of elliptic curves on S , the T -number of $S^{[n]}$ is proven to be at most $n + 1$ in the upcoming paper [34], again a linear bound. Whether the T -number is exactly $n + 1$ is an interesting question.

0.7 Plan of the paper

We start in Section 1 with basic results about cycles and curves which we will use throughout the paper. Theorem 4 for rational surfaces is proven in Section 2 and Theorem 5 for $K3$ surfaces is proven in Section 3. Open questions for Enriques surfaces, Abelian surfaces, and surfaces of general type are discussed in Section 4. A result concerning cyclic covers of \mathbb{CP}^1 is proven in Section 5. The paper ends with results about the number $T(g, n)$ in Section 6.

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1 Basic results about cycles and curves

We start by recalling the following useful (and well-known) result about families of algebraic cycles, see [38, Proposition 2.4].

Proposition 6 *Let $\pi : \mathcal{X} \rightarrow B$ be a flat morphism of algebraic varieties where B is nonsingular of dimension r and let $\mathcal{Z} \in A_N(\mathcal{X})$ be a cycle. Then, the set $B_{\mathcal{Z}}$ of points $t \in B$ satisfying*

$$\mathcal{Z}_t = \mathcal{Z}|_{\mathcal{X}_t} = 0 \in A_{N-r}(\mathcal{X}_t)$$

is a countable union of proper closed algebraic subsets of B .

Proposition 7 *Let $X \subset \overline{\mathcal{M}}_{g,n}$ be an irreducible algebraic set such that the generic point of X is tautological. Then, every point of X is tautological.*

Proof. Consider the trivial family

$$\pi : \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$$

defined by projection on the second factor. Let $\Delta \subset \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n}$ be the diagonal, and let S be the section of π determined by a fixed tautological point of $\overline{\mathcal{M}}_{g,n}$. By applying⁹ Proposition 6 to the relative 0-cycle

$$\mathcal{Z} = \Delta - S,$$

⁹We leave the standard movement of scheme results to stacks for the reader.

the set of points in $\overline{\mathcal{M}}_{g,n}$ whose class is tautological is a countable union of closed algebraic sets. Since the generic point of X is contained in this union, X must also be contained. \diamond

Let S be a nonsingular projective surface which is either rational or $K3$. In both cases,

$$\text{Pic}(S, \mathbb{Z}) = H_2(S, \mathbb{Z}).$$

Let $L \in \text{Pic}(S)$ be an effective divisor class. Let $|L| = \mathbb{P}(H^0(S, L))$ be the associated linear system of divisors with hyperplane class $H \in A^1(|L|)$. There exists a natural Hilbert-Chow morphism

$$c : \overline{\mathcal{M}}_{g,n}(S, c_1(L)) \rightarrow |L|, \quad (6)$$

sending a stable map $(f : (C, p_1, \dots, p_n) \rightarrow S)$ to the effective divisor $f_*[C]$.

In the stable range $2g - 2 + n > 0$, let

$$\epsilon : \overline{\mathcal{M}}_{g,n}(S, c_1(L)) \rightarrow \overline{\mathcal{M}}_{g,n}$$

be the natural forgetful morphism. Let

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(S, c_1(L)) \rightarrow S$$

be the evaluation map corresponding to the i th marking.

Lemma 8 *Let S be a rational surface with $L \in \text{Pic}(S)$. Let $C \subset S$ be a nonsingular irreducible curve of genus g contained in $|L|$. Assume*

$$\dim |L| = \text{vdim } \overline{\mathcal{M}}_g(S, [C]) = g - 1 + \int_{[C]} c_1(S).$$

Then, for $0 \leq n \leq \text{vdim } \overline{\mathcal{M}}_g(S, [C])$ satisfying $2g - 2 + n > 0$ and pairwise distinct points $p_1, \dots, p_n \in C$, we have

$$\epsilon_* \left(c^* H^{\dim |L| - n} \cdot \prod_{i=1}^n \text{ev}_i^*[p_i] \cap [\overline{\mathcal{M}}_{g,n}(S, [C])]^{\text{vir}} \right) = [C, p_1, \dots, p_n] \quad (7)$$

in $A_0(\overline{\mathcal{M}}_{g,n})$.

Proof. We first prove the Lemma for general points

$$p_1, \dots, p_n \in C.$$

For general points p_i , the set of curves in $|L|$ passing through the p_i is a linear subspace H_1 of codimension n . We choose a complementary linear subspace $H_2 \subset |L|$ of codimension $r - n$ satisfying

$$H_1 \cap H_2 = \{[C]\}.$$

Therefore, on $\overline{\mathcal{M}}_{g,n}(S, [C])$, the cycle $c^*[H_2] \cap \prod_{i=1}^n \text{ev}_i^*[p_i]$ is supported on the point

$$[(C, p_1, \dots, p_n) \hookrightarrow S] \in \overline{\mathcal{M}}_{g,n}(S, [C]) \quad (8)$$

Near the point (8) in $\overline{\mathcal{M}}_{g,n}(S, [C])$, the map $\Phi = (c, \text{ev}_1, \dots, \text{ev}_n)$ defines a local isomorphism¹⁰ to the incidence variety

$$\mathcal{I} = \{(D, q_1, \dots, q_n) : D \in |L|, q_1, \dots, q_n \in D\} \subset |L| \times S^n.$$

Since near (8) \mathcal{I} is nonsingular of dimension $\dim |L| + n$ and since this is the virtual dimension of $\overline{\mathcal{M}}_{g,n}(S, [C])$, the virtual fundamental class restricts to the standard fundamental class near (8). Since $H_2 \times \prod_{i=1}^n [p_i]$ intersects \mathcal{I} transversally in the point $([C], p_1, \dots, p_n)$, we obtain the equality (7).

We finish the proof by going from the case of general points $p_1, \dots, p_n \in C$ to the case of any pairwise distinct set of points. Consider the complement $B = C^n \setminus \Delta$ of the diagonals inside the product C^n . The difference of the two sides of equation (7) defines a natural cycle \mathcal{Z} inside $\overline{\mathcal{M}}_{g,n} \times B$. For $b \in B$ general, we have

$$\mathcal{Z}|_{\overline{\mathcal{M}}_{g,n} \times \{b\}} = 0.$$

By Proposition 6, the set of such b is a countable union of closed algebraic sets, and so must be all of B . \diamond

For S a nonsingular projective $K3$ surface, we need a variant of Lemma 8 involving the reduced virtual fundamental class (see [8, 27]).

Lemma 9 *Let S be a $K3$ surface with $L \in \text{Pic}(S)$. Let $C \subset S$ be a nonsingular irreducible curve of genus g contained in $|L|$. Then for*

$$0 \leq n \leq g \text{ satisfying } 2g - 2 + n > 0$$

¹⁰Since all the curves D near C are irreducible and nonsingular, the inverse map is well-defined.

and distinct points $p_1, \dots, p_n \in C$, we have

$$\epsilon_* \left(c^* H^{g-n} \cdot \prod_{i=1}^n \text{ev}_i^*[p_i] \cap [\overline{\mathcal{M}}_{g,n}(S, [C])]^{\text{red}} \right) = [C, p_1, \dots, p_n] \quad (9)$$

in $A_0(\overline{\mathcal{M}}_{g,n})$.

Proof. Since $L = \mathcal{O}_S(C)$, the exact sequence

$$0 \rightarrow H^0(S, \mathcal{O}_S) \rightarrow H^0(S, \mathcal{O}_S(C)) \rightarrow H^0(C, \mathcal{O}_C(C)) \rightarrow H^1(S, \mathcal{O}_S) \rightarrow \dots$$

together with the ranks

$$h^0(C, \mathcal{O}_C(C)) = h^0(C, \omega_C) = g, \quad h^1(S, \mathcal{O}_S) = 0$$

shows $\dim |L| = g$. Hence, we have

$$\dim |L| + n = g + n = \dim [\overline{\mathcal{M}}_{g,n}(S, c_1(L))]^{\text{red}}.$$

The proof of Lemma 8 can then be exactly followed for the reduced class here to conclude the result. \diamond

2 Rational surfaces

2.1 Proof of Theorem 4

If C is of genus $g = 0$, Theorem 4 is trivial (since the moduli space $\overline{\mathcal{M}}_{0,n}$ is rational and all 0-cycles are tautological). We will assume $g \geq 1$. The argument proceeds in three steps:

- (1) We apply Lemma 8 to express the 0-cycle

$$[C, p_1, \dots, p_n] \in A_0(\overline{\mathcal{M}}_{g,n})$$

in terms of a push-forward involving the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(S, [C])$.

- (2) We deform the rational surface S to a nonsingular projective toric surface \widehat{S} over a base which is rationally connected.

(3) We apply virtual localization [18] to the toric surface \widehat{S} to conclude the desired class is tautological.

Step 1. To apply Lemma 8, we must check the hypothesis

$$\dim |L| = g - 1 + \int_{[C]} c_1(S), \quad (10)$$

where $L = \mathcal{O}_S(C)$. Condition (10) is equivalent to $h^0(L) = g + \int_{[C]} c_1(S)$.

Since C is nonsingular of genus g , the adjunction formula yields

$$\langle [C], [C] - c_1(S) \rangle = 2g - 2,$$

where $\langle \cdot, \cdot \rangle$ is the intersection product and S . On the other hand, by Riemann-Roch we have

$$\begin{aligned} \chi(L) &= \frac{1}{2} \langle [C], [C] + c_1(S) \rangle + \chi(\mathcal{O}_S) \\ &= \frac{1}{2} \langle [C], [C] - c_1(S) \rangle + \langle [C], c_1(S) \rangle + 1 \\ &= g - 1 + \langle [C], c_1(S) \rangle + 1 \\ &= g + \langle [C], c_1(S) \rangle. \end{aligned}$$

Furthermore, we have

$$h^2(L) = h^2(S, \mathcal{O}_S(C)) = h^0(S, \omega_S(-C)) \leq h^0(S, \omega_S) = 0,$$

where the last equality holds since S is rational. So, we see

$$h^0(L) \geq \chi(L) = g + \langle [C], c_1(S) \rangle.$$

To prove the vanishing of $h^1(L)$, we use the sequence

$$H^1(S, \mathcal{O}_S) \rightarrow H^1(S, \mathcal{O}_S(C)) \rightarrow H^1(C, \mathcal{O}_C(C)) \rightarrow H^2(S, \mathcal{O}_S).$$

Since the higher cohomologies of \mathcal{O}_S on S vanish,

$$H^1(S, L) = H^1(C, \mathcal{O}_C(C)).$$

By Serre duality and adjunction, we have

$$h^1(C, \mathcal{O}_C(C)) = h^0(C, \omega_C(-C)) = h^0(C, \omega_S).$$

However, by the positivity hypothesis,

$$\langle [C], c_1(\omega_S) \rangle < 0,$$

so $h^0(C, \omega_S) = 0$.

Since the hypotheses of Lemma 8 hold, we may apply the conclusion: for $r = g - 1 + \int_{[C]} c_1(S)$ and pairwise distinct $p_1, \dots, p_r \in C$, we have

$$\epsilon_* \left(\prod_{i=1}^r \text{ev}_i^*[\text{pt}] \cap [\overline{\mathcal{M}}_{g,r}(S, c_1(L))]^{\text{vir}} \right) = [C, p_1, \dots, p_r] \in A_0(\overline{\mathcal{M}}_{g,r}), \quad (11)$$

where $[\text{pt}] \in A_0(S, \mathbb{Z})$ is the class of (any) point as S is rational.

Step 2. The rational surface S can be deformed to a toric surface \widehat{S} in a smooth family

$$S \rightarrow B$$

over a rationally connected variety B containing S, \widehat{S} as special fibres.¹¹ The line bundle L can be deformed along with S to a line bundle

$$\widehat{L} \rightarrow \widehat{S}.$$

Since the virtual fundamental class is constructed in families [2],

$$\epsilon_* \left(\prod_{i=1}^r \text{ev}_i^*[\text{pt}] \cap [\overline{\mathcal{M}}_{g,r}(S, c_1(L))]^{\text{vir}} \right) = \epsilon_* \left(\prod_{i=1}^r \text{ev}_i^*[\text{pt}] \cap [\overline{\mathcal{M}}_{g,r}(\widehat{S}, c_1(\widehat{L}))]^{\text{vir}} \right).$$

We have therefore moved the calculation to the toric setting.

Step 3. The virtual localization formula of [18] applied to the toric surface \widehat{S} immediately shows

$$\epsilon_* \left(\prod_{i=1}^n \text{ev}_i^*[\text{pt}] \cap [\overline{\mathcal{M}}_{g,r}(\widehat{S}, c_1(\widehat{L}))]^{\text{vir}} \right) \in R_0(\overline{\mathcal{M}}_{g,r}).$$

We have proven that the 0-cycle $[C, p_1, \dots, p_r] \in A_0(\overline{\mathcal{M}}_{g,r})$ is tautological. If $0 \leq n \leq r$,

$$[C, p_1, \dots, p_n] \in A_0(\overline{\mathcal{M}}_{g,n})$$

must also be tautological (by applying the forgetful map). \diamond

¹¹There is no difficulty in finding such a deformation. The minimal model of S is toric. The exceptional divisors can then be moved to toric fixed points.

2.2 Variations

Let S be a nonsingular projective rational surface, and let

$$C \subset S$$

be an reduced, irreducible, nodal curve of arithmetic genus g satisfying the positivity condition

$$\int_{[C]} c_1(S) > 0. \quad (12)$$

The statements and proofs of Lemma 8 and Theorem 4 are still valid for such curves¹² : *the 0-cycle*

$$[C, p_1, \dots, p_n] \in A_0(\overline{\mathcal{M}}_{g,n})$$

is tautological if $n \leq \text{vdim } \overline{\mathcal{M}}_g(S, [C])$.

Can the positivity condition (12) be relaxed? Positivity was used in the proof of Theorem 4 only to prove that the associated linear series has the expected dimension. If $C \subset S$ is an irreducible nodal curve of arithmetic genus g satisfying

$$H^1(S, \mathcal{O}_S(C)) = 0, \quad (13)$$

then we can *still* conclude that the 0-cycle

$$[C, p_1, \dots, p_n] \in A_0(\overline{\mathcal{M}}_{g,n})$$

is tautological if $n \leq \text{vdim } \overline{\mathcal{M}}_g(S, [C])$.

According to the Harbourne-Hirschowitz conjecture [22, 23], the vanishing (13) should always hold if S is sufficiently general. We therefore expect an affirmative answer to the following question.

Question 10 *Let $C \subset S$ be an irreducible nonsingular (or an irreducible nodal) curve with no positivity assumption on $\int_{[C]} c_1(S)$. Is the 0-cycle*

$$[C, p_1, \dots, p_n] \in A_0(\overline{\mathcal{M}}_{g,n})$$

tautological for $n \leq \text{vdim } \overline{\mathcal{M}}_g(S, [C])$?

¹²The points p_i here are distinct and lie in the nonsingular locus of C .

On the other hand, if $C \subset S$ is a *reducible* nodal curve, we obtain a parallel statement by applying the results above for each irreducible component separately. Here, each component C_v with arithmetic genus g_v must satisfy the positivity condition (12), and the number of markings plus the number of preimages of nodes must be bounded by the virtual dimension $\text{vdim } \overline{\mathcal{M}}_{g_v}(S, [C_v])$.

3 K3 surfaces

3.1 Beauville-Voisin classes

On a nonsingular projective K3 surface S , there exists a canonical zero cycle $c_S \in A_0(S, \mathbb{Z})$ of degree 1 satisfying the following three properties [1]:

- all points in S lying on a (possibly singular) rational curve have class $c_S \in A_0(S, \mathbb{Z})$,
- the image of the intersection product $A_1(S, \mathbb{Z}) \otimes A_1(S, \mathbb{Z}) \rightarrow A_0(S, \mathbb{Z})$ lies in $\mathbb{Z} \cdot c_S$,
- the second Chern class $c_2(S)$ is equal to $24c_S$.

The *Beauville-Voisin* subspace is defined by

$$\text{BV} = \mathbb{Z} \cdot c_S \subset A_0(S, \mathbb{Z}).$$

A point $p \in S$ is a *Beauville-Voisin* point if $[p] \in \text{BV}$.

3.2 Proof of Theorem 5

The claim is trivial for genus $g = 1$ since $\overline{\mathcal{M}}_{1,1}$ is rational. We can therefore assume $g \geq 2$. By Lemma 9, we have

$$\epsilon_* \left(c^* H^{g-n} \cap \prod_{i=1}^n \text{ev}_i^* [p_i] \cap [\overline{\mathcal{M}}_{g,n}(S, [C])]^{\text{red}} \right) = [C, p_1, \dots, p_n] \quad (14)$$

in $A_0(\overline{\mathcal{M}}_{g,n})$.

We briefly recall the notation used in (14). For $L = \mathcal{O}_S(C)$,

$$c : \overline{\mathcal{M}}_{g,n}(S, [C]) \rightarrow \mathbb{P}(H^0(S, L))$$

is the map sending

$$f : (\widehat{C}, \widehat{p}_1, \dots, \widehat{p}_n) \rightarrow S$$

to $f_*[\widehat{C}] \in |L|$ and H is the hyperplane class of $|L|$. Since the points p_i are all Beauville-Voisin, equality (9) immediately implies that the right hand side depends *only* upon the surface S and the class

$$[C] \in \text{Pic}(S).$$

By Lemma 2.3 of [24, Chapter 2], the line bundle L is base point free and hence nef. Let

$$L = L_0^{\otimes k}, \quad k \geq 1$$

for L_0 primitive of degree

$$d = 2g' - 2 > 0.$$

Then, L_0 is still nef, so (S, L_0) is a quasi-polarized $K3$ surface of degree d . Consider the moduli stack \mathcal{F}_d of quasi-polarized $K3$ surfaces $(\widehat{S}, \widehat{L}_0)$ of degree d . Let

$$\pi : \mathcal{S} \rightarrow \mathcal{F}_d$$

be the universal $K3$ surface over \mathcal{F}_d with universal polarization $\mathcal{L}_0 \in A^1(\mathcal{S})$. The restriction of $(\mathcal{S}, \mathcal{L}_0)$ to the fibre over $(\widehat{S}, \widehat{L}_0) \in \mathcal{F}_d$ is isomorphic to $(\widehat{S}, \widehat{L}_0)$, see [31].

Consider furthermore the projective bundle

$$\mathcal{P} = \mathbb{P}(R^0 \pi_*((\mathcal{L}_0)^{\otimes k})) \rightarrow \mathcal{F}_d$$

parametrizing elements in the linear system $(\mathcal{L}_0)^{\otimes k}$ on the fibres of \mathcal{S} . The projective bundle \mathcal{P} is of fibre dimension g by Theorem 1.8 of [24, Chapter 2].

We can then obtain the left hand side of (9) as a fibre in a family of cycles parametrized by \mathcal{F}_d . Indeed, denote by \mathcal{S}^n the n -fold self product of \mathcal{S} over \mathcal{F}_d and consider the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(\pi, c_1(\mathcal{L}_0^k)) & \xrightarrow{c \times \text{ev}} & \mathcal{P} \times_{\mathcal{F}_d} \mathcal{S}^n \\ \swarrow \epsilon & \downarrow \pi & \swarrow \\ \overline{\mathcal{M}}_{g,n} & & \mathcal{F}_d \end{array} \quad . \quad (15)$$

Here, $\overline{\mathcal{M}}_{g,n}(\pi, c_1(\mathcal{L}_0^k))$ is the moduli space of stable maps to the fibres of π of curve class equal to $c_1(\mathcal{L}_0^k)$ on the fibres of π . The map c is the version of the previous map c in families, and

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_n)$$

is the evaluation map corresponding to the n points. Let

$$\mathcal{H} = c_1(\mathcal{O}_{\mathcal{P}}(1))$$

be the hyperplane class of the projective bundle \mathcal{P} , and let

$$c_{\mathcal{S}} = \frac{1}{24} c_2(\Omega_{\pi}) \in A^2(\mathcal{S})$$

be the relative Beauville-Voisin class of the family

$$\pi : \mathcal{S} \rightarrow \mathcal{F}_d.$$

Consider the cycle $\mathcal{Z} \in A^{3g-3+n}(\overline{\mathcal{M}}_{g,n} \times \mathcal{F}_d)$ defined by

$$\mathcal{Z} = (\epsilon, \pi)_* \left(c^* \mathcal{H}^{g-n} \cdot \prod_{i=1}^n \text{ev}_i^* c_{\mathcal{S}} \cap [\overline{\mathcal{M}}_{g,n}(\pi, c_1(\mathcal{L}_0^k))]^{\text{red}} \right). \quad (16)$$

The fibre of \mathcal{Z} over (S, L_0) is equal to the left hand side of (9).

By Proposition 6, we need only show that the fibre of \mathcal{Z} over the general point of \mathcal{F}_d is tautological. So let

$$(\widehat{S}, \widehat{L}_0) \in \mathcal{F}_d$$

be a general quasi-polarized $K3$ of degree d . By the existence result of [10], the linear system $|\widehat{L}_0^{\otimes k}|$ contains an irreducible nodal rational curve

$$R \subset \widehat{S}.$$

Furthermore, since $(\widehat{S}, \widehat{L})$ is general, we can assume that \widehat{L}_0 and thus $\widehat{L}_0^{\otimes k}$ are basepoint free (see Theorem 4.2 of [24, Chapter 2]). By Bertini's theorem, the general member \widehat{C} of the linear system $|\widehat{L}_0^{\otimes k}|$ intersects the rational curve R only in reduced points. The number of these intersection points is exactly

$$\langle [\widehat{C}], [\widehat{C}] \rangle = 2g - 2,$$

which is at least g (since we assume $g \geq 2$). Choose distinct points

$$q_1, \dots, q_n \in R \cap \widehat{C}.$$

Certainly all the q_i are Beauville-Voisin points since they lie on the rational curve R . Since

$$R, \widehat{C} \in |\widehat{L}_0^k|,$$

there exists a pencil of curves connecting $(\widehat{C}, q_1, \dots, q_n)$ and (R, q_1, \dots, q_n) . The 0-cycle $[(R, q_1, \dots, q_n)] \in A^0(\overline{\mathcal{M}}_{g,n})$ is clearly tautological, since the point lies in the image of

$$\overline{\mathcal{M}}_{0,n+2g} \rightarrow \overline{\mathcal{M}}_{g,n}.$$

Therefore, $[(\widehat{C}, q_1, \dots, q_n)]$ is tautological. \diamond

We isolate part of the above proof as a separate Corollary for later application.

Corollary 11 *Let S be a K3 surface with $L \in \text{Pic}(S)$. There exists a \mathbb{Q} -linear map*

$$\Phi : A_0(S^n) \rightarrow A_0(\overline{\mathcal{M}}_{g,n})$$

defined by

$$\Phi(\alpha) = \epsilon_* (c^* H^{g-n} \cdot \text{ev}^* \alpha \cap [\overline{\mathcal{M}}_{g,n}(S, c_1(L))]^{\text{red}}).$$

For an irreducible nonsingular projective curve $C \subset S$ of genus $g \geq 1$ in the linear series L and distinct points $p_1, \dots, p_n \in C$ we have

$$[(C, p_1, \dots, p_n)] = \Phi([(p_1, \dots, p_n)]). \quad (17)$$

Moreover, $\Phi((c_S)^{\times n})$ is tautological.

3.3 Quotients

The symmetric group S_n acts on $\overline{\mathcal{M}}_{g,n}$ by permuting the markings. For a partition $\mu = (n_1, \dots, n_\ell)$ of n , let

$$S_\mu = S_{n_1} \times \dots \times S_{n_\ell} \subset S_n$$

be the subgroup permuting elements within the blocks defined by μ . The stack quotient

$$\overline{\mathcal{M}}_{g,\mu} = \overline{\mathcal{M}}_{g,n}/S_\mu$$

parametrizes curves

$$(C, (\{p_{i,1}, \dots, p_{i,n_i}\})_{i=1, \dots, \ell})$$

together with ℓ pairwise disjoint sets of marked points with sizes n_i according to the partition μ . The quotient map

$$\pi : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,\mu}$$

allows us to define the tautological ring $R^*(\overline{\mathcal{M}}_{g,\mu})$ as the image of $R^*(\overline{\mathcal{M}}_{g,n})$ via push-forward by π . The composition

$$\pi_*\pi^* : A^*(\overline{\mathcal{M}}_{g,\mu}) \rightarrow A^*(\overline{\mathcal{M}}_{g,\mu})$$

is given by multiplication by $|S_\mu|$. Therefore, to check if a cycle α on $\overline{\mathcal{M}}_{g,\mu}$ is tautological, it suffices to check that $\pi^*(\alpha)$ is tautological on $\overline{\mathcal{M}}_{g,n}$.

The following results for the quotient moduli spaces $\overline{\mathcal{M}}_{g,\mu}$ is parallel to Theorem 5 for $\overline{\mathcal{M}}_{g,n}$.

Theorem 12 *Let $C \subset S$ be an irreducible nonsingular curve of genus g on a K3 surface. Let $0 \leq n \leq g$ and fix a partition $\mu = (n_1, \dots, n_\ell)$ of n . Let*

$$(p_{i,j})_{\substack{i=1, \dots, \ell \\ j=1, \dots, n_i}}$$

be a collection of distinct points $p_{i,j} \in C$ satisfying

$$[p_{i,1}] + [p_{i,2}] + \dots + [p_{i,n_i}] \in \text{BV} \tag{18}$$

for all $1 \leq i \leq \ell$. Then, the 0-cycle

$$[C, (\{p_{i,1}, \dots, p_{i,n_i}\})_{i=1, \dots, \ell}] \in A_0(\overline{\mathcal{M}}_{g,\mu})$$

is tautological.

Proof. It suffices to show that the pullback $\pi^*([C, (p_{i,j})_i])$ is tautological. Fix an ordering $\mathbf{p} = (p_{i,j})_i$ of all the markings. The pullback is exactly given by

$$\pi^*([C, (p_{i,j})_i]) = \sum_{\sigma \in S_\mu} [\sigma(C, \mathbf{p})].$$

Using Corollary 11, we can write the result as $\Phi(\Sigma(\mathbf{p}))$ for the sum

$$\Sigma(\mathbf{p}) = \sum_{\sigma \in S_\mu} [\sigma \cdot \mathbf{p}] \in A_0(S^n),$$

where we have used the natural permutation action of S_n on S^n .

We claim that the cycle $\Sigma(\mathbf{p})$ only depends on the blockwise sums

$$\Sigma_i(\mathbf{p}) = \sum_j [p_{i,j}] \in A_0(S)$$

for $i = 1, \dots, \ell$. Blockwise dependence together with the hypothesis

$$\Sigma_i(\mathbf{p}) \in \text{BV}$$

immediately yields the result of Theorem 12 (since we can exchange all the $p_{i,j}$ for Beaville-Voisin points).

It remains only to prove the blockwise dependence. We first observe that we can write $\Sigma(\mathbf{p})$ as a product

$$\Sigma(\mathbf{p}) = \sum_{\sigma \in S_\mu} [\sigma \cdot \mathbf{p}] = \prod_{i=1}^{\ell} \sum_{\sigma_i \in S_{n_i}} [\sigma_i \cdot (p_{i,j})_{j=1, \dots, n_i}],$$

where we recall that S_μ is the product of the groups S_{n_i} . It suffices then to show that the i th factor in the above product only depends on the sum $\Sigma_i(\mathbf{p})$. The latter claim amounts to a reduction to the case of the partition $\mu = (n)$ where all the markings are permuted.

Let $P = \{p_1, \dots, p_n\}$. We will write

$$\Sigma(\mathbf{p}) = \sum_{\sigma \in S_n} [(p_{\sigma(1)}, \dots, p_{\sigma(n)})]$$

as a sum of terms depending only upon

$$\theta = \Sigma_1(\mathbf{p}) = [p_1] + \dots + [p_n]$$

using a simple inclusion-exclusion strategy.

We illustrate the strategy in the case of $n = 3$. We start with the formula

$$\theta^{\times 3} = \sum_{q_1, q_2, q_3 \in P} [(q_1, q_2, q_3)].$$

To obtain $\Sigma(\mathbf{p})$, we must subtract all summands where there is a pair $i \neq j$ with $q_i = q_j$. Let

$$\Delta_{12,3}, \Delta_{13,2}, \Delta_{23,1} : S^2 \rightarrow S^3$$

be the three diagonal maps. The cycle

$$\theta^{\times 3} - (\Delta_{12,3})_*(\theta^{\times 2}) - (\Delta_{13,2})_*(\theta^{\times 2}) - (\Delta_{23,1})_*(\theta^{\times 2})$$

is equal to $\Sigma(\mathbf{p})$ minus 2 times the cycle

$$[(p_1, p_1, p_1)] + [(p_2, p_2, p_2)] + [(p_3, p_3, p_3)].$$

We can cancel the error term by adding a correction $2(\Delta_{123})_*(\theta)$ by the small diagonal:

$$\Sigma(\mathbf{p}) = \theta^{\times 3} - (\Delta_{12,3})_*(\theta^{\times 2}) - (\Delta_{13,2})_*(\theta^{\times 2}) - (\Delta_{23,1})_*(\theta^{\times 2}) + 2(\Delta_{123})_*(\theta).$$

Such an inclusion-exclusion strategy is valid for all $n \geq 1$. \diamond

4 Other surface geometries

4.1 Enriques

An Enriques surface E is a free \mathbb{Z}_2 quotient of a nonsingular projective $K3$ surface S :

$$E = S/\mathbb{Z}_2.$$

Conjecture 13 *The moduli point of an irreducible nonsingular curve $C \subset E$ of genus $g \geq 2$ determines a tautological 0-cycle in $\overline{\mathcal{M}}_g$.*

There is a clear strategy for the proof of Conjecture 13. The curve C is expected to move in a linear series $|L|$ on E of dimension $g - 1$. We therefore expect to find irreducible curves $\widehat{C} \in |L|$ with $g - 1$ nodes. The issue can be formulated as the nonemptiness of certain Severi varieties for linear systems

on Enriques surfaces which is currently being studied, see [11]. Once it is shown that the linear series $|L|$ contains an irreducible $(g - 1)$ -nodal curve $\widehat{C} \subset E$, the final step is to prove that the 0-cycle

$$[\widehat{C}] \in A_0(\overline{\mathcal{M}}_g)$$

is always tautological. In fact, the following stronger result holds.

Proposition 14 *The locus of irreducible $(g - 1)$ -nodal curves in $\overline{\mathcal{M}}_{g,1}$ is rational. In particular, every such curve defines a tautological cycle*

$$[\widehat{C}, p] \in R_0(\overline{\mathcal{M}}_{g,1}).$$

Proof. The closure of the locus of $(g - 1)$ -nodal curves is parametrized by the gluing map

$$\xi : \overline{\mathcal{M}}_{1,1+2(g-1)} \rightarrow \overline{\mathcal{M}}_g$$

taking a curve $(X, p, q_1, q'_1, \dots, q_{g-1}, q'_{g-1})$ of genus 1 with $1+2(g-1)$ markings and identifying the $g - 1$ pairs q_j, q'_j of points. The group

$$G = (\mathbb{Z}/2\mathbb{Z})^{g-1} \rtimes S_{g-1}$$

acts on $\overline{\mathcal{M}}_{1,1+2(g-1)}$: the j th factor $\mathbb{Z}/2\mathbb{Z}$ switches the two points q_j, q'_j and the group S_{g-1} permutes the n pairs of points among each other. Since the gluing map ξ is invariant under this action, it factors through the map

$$\tilde{\xi} : \overline{\mathcal{M}}_{1,1+2(g-1)}/G \rightarrow \overline{\mathcal{M}}_{g,1},$$

which is birational onto its image.

To prove $\mathcal{M} = \overline{\mathcal{M}}_{1,1+2(g-1)}/G$ is rational, we take a modular reinterpretation. Instead of remembering the $2(g - 1)$ points q_j, q'_j on X individually, we only remember the set

$$\{D_j = q_j + q'_j : j = 1, \dots, g - 1\}$$

of $g - 1$ effective divisors of degree 2 on the curve X . We therefore have a birational identification

$$\mathcal{M} \leftrightarrow \left\{ (X, p, (D_j)_{j=1}^{g-1}) : \begin{array}{l} X \text{ nonsingular elliptic curve with origin } p, \\ D_j \subset X \text{ effective degree 2 divisors} \end{array} \right\} / S_{g-1},$$

where S_{g-1} acts by permuting the divisors D_1, \dots, D_{g-1} .

An effective divisor $D_j \subset X$ is equivalent to the data of the degree 2 line bundle

$$\mathcal{L}_j = \mathcal{O}(D_j)$$

together with an element

$$s_j \in \mathbb{P}(H^0(X, \mathcal{L}_j)) \cong \mathbb{P}^1.$$

Furthermore, the class of the line bundle \mathcal{L}_j is equivalent to specifying a point $l_j \in X$, by the correspondence sending l_j to $\mathcal{O}(p + l_j)$, where $p \in X$ is origin. We define

$$\mathcal{P} = \left\{ \begin{array}{l} X \text{ nonsingular elliptic curve with origin } p, \\ (X, p, (l_j)_{j=1}^{g-1}, (s_j)_{j=1}^{g-1}) : \begin{array}{l} l_j \in E \\ s_j \in \mathbb{P}(H^0(X, \mathcal{O}(p + l_j))) \end{array} \end{array} \right\}.$$

We have a birational identification

$$\mathcal{M} \longleftrightarrow \mathcal{P}/S_{g-1}.$$

By forgetting the projective sections s_j , we obtain a map

$$\mathcal{P} \rightarrow \mathcal{S}$$

to the space \mathcal{S} parametrizing tuples $(X, p, (l_j)_j)$ as above. The above forgetful map is a $(\mathbb{P}^1)^{g-1}$ -bundle which descends (birationally) to a $(\mathbb{P}^1)^{g-1}$ -bundle

$$\mathcal{P}/S_{g-1} \rightarrow \mathcal{S}/S_{g-1}$$

on the quotient. The base, the moduli space parameterizing the data

$$(X, p, (l_j)_j)$$

up to permutations of the l_j by S_{g-1} , is easily seen to be rational using, to start, the rationality of the universal family of Jac_2 over $\overline{\mathcal{M}}_{1,1}$. \diamond

Using the rationality of $\overline{\mathcal{M}}_{1,10}$, Proposition 14 can be easily strengthened to show that the locus of irreducible $(g-1)$ -nodal curves in $\overline{\mathcal{M}}_{g,9}$ is rational. In particular, every such curve defines a tautological cycle

$$[\widehat{C}, p_1, \dots, p_9] \in R_0(\overline{\mathcal{M}}_{g,9}).$$

4.2 Abelian

Let A be a nonsingular projective Abelian surface. An irreducible nonsingular curve

$$C \subset A$$

is expected to move in a linear series $|L|$ of dimension $g - 2$. We therefore expect to find curves $\widehat{C} \in |L|$ with $g - 2$ nodes. Unfortunately the strategy that we have outlined in the case of Enriques surfaces fails here! The locus of irreducible $(g-2)$ -nodal curves in $\overline{\mathcal{M}}_g$ is *not* always rational. The irrationality of the locus of 7 nodal curves in $\overline{\mathcal{M}}_9$ was proven with C. Faber using the non-triviality (and representation properties) of $H^{14,0}(\overline{\mathcal{M}}_{2,14})$. A study of the Kodaira dimensions of the loci of curves with multiple nodes in many (other) cases can be found in [35].

Nevertheless, an affirmative answer to the following question appears likely.

Question 15 *Does every irreducible nonsingular curve $C \subset A$ of genus g determine a tautological 0-cycle $[C] \in A_0(\overline{\mathcal{M}}_g)$?*

Another approach to Question 15 is to use curves on $K3$ surfaces via the Kummer construction. Using the involution

$$\iota : A \rightarrow A, \quad a \mapsto -a,$$

we obtain a $K3$ surface S by resolving the singular points of the quotient A/ι . If C does not meet any of these 16 points (which are the fixed-points of ι), the corresponding rational map

$$A \rightarrow A/\iota \dashrightarrow S$$

is defined around $C \subset A$ and sends C to a curve $C' \subset S$. The map $C \rightarrow C'$ is either a double cover (in which case it must be étale with C' smooth) or birational. In the first case, $[C']$ is tautological by Theorem 5 which may help in proving that $[C]$ is tautological. In the second case, the curve C is the normalization of C' , and we would require a variant of Theorem 5 to show that, under suitable conditions, the normalization of an irreducible, nodal curve in a $K3$ surface is tautological.

4.3 General type

Let S be a nonsingular projective surface of general type. A curve $C \subset S$ is *canonical* if

$$\omega_S \cong \mathcal{O}_S(C).$$

The most basic question which can be asked is the following.

Question 16 *Does every irreducible nonsingular canonical curve $C \subset S$ of genus g determine a tautological 0-cycle $[C] \in A_0(\overline{\mathcal{M}}_g)$?*

For surfaces S arising as complete intersections in projective space, the answer to Question 16 is yes (since complete intersection curves are easily seen to determine tautological 0-cycles by degenerating their defining equations to products of linear factors). However, even for surfaces of general type arising as double covers of \mathbb{P}^2 , the issue does not appear trivial (even though the canonical curves there are realized as concrete double covers of plane curves). In fact, Question 16 is completely open in almost all cases.

5 Cyclic covers

If a nonsingular projective complex curve C admits a Hurwitz covering of \mathbb{P}^1 ramified over only 3 points of \mathbb{P}^1 , then C can be defined over $\overline{\mathbb{Q}}$ by Belyi's Theorem. Speculation 3, for $n = 0$, then suggests that the moduli point of C is tautological. The following result proves a special case for cyclic covers.¹³

Theorem 17 *Let C be a nonsingular projective curve of genus g admitting a cyclic cover*

$$\varphi : C \rightarrow \mathbb{P}^1$$

ramified over exactly three points of \mathbb{P}^1 and with total ramification over at least one of them. Let $p_1, \dots, p_n \in C$ be the ramification points of φ (in some order). Then, the 0-cycle

$$[C, p_1, \dots, p_n] \in A_0(\overline{\mathcal{M}}_{g,n})$$

is tautological.

¹³Following the notation of [33], Theorem 17 shows that the 0-cycle

$$[\overline{\mathcal{H}}_{g, \mathbb{Z}/k\mathbb{Z}, (a,b,c)}] \in A_0(\overline{\mathcal{M}}_{g,n})$$

is tautological for $a, b, c \in \mathbb{Z}/k\mathbb{Z}$ where at least one of a, b, c is coprime to k .

Proof The basic idea is that a cyclic cover of \mathbb{P}^1 can (essentially) be cut out by a single equation in a projective bundle over \mathbb{P}^1 . Indeed, after a change of coordinates, we can assume that the branch points of φ are given by

$$0, 1, 2 \in \mathbb{P}^1.$$

Let k be the degree of φ , and let $a, b, c \in \mathbb{Z}/k\mathbb{Z}$ be the monodromies of φ at the branch points $0, 1, 2$ satisfying

$$a + b + c = 0 \in \mathbb{Z}/k\mathbb{Z}.$$

Assume that the total ramification occurs over 0 . Then a is coprime to k , and, by applying an automorphism of $\mathbb{Z}/k\mathbb{Z}$, we may assume $a = 1$. We can then choose representatives

$$b, c \in \{1, \dots, k-1\}$$

such that $a + b + c = k$.

With these choices in place, we see that (birationally) the curve C is cut out in the projectivization of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ over \mathbb{P}^1 by the equation

$$y^k = x \cdot (x-1)^b \cdot (x-2)^c, \tag{19}$$

where x is a coordinate on the base \mathbb{P}^1 . We view the right hand side of (19) as a section of

$$\mathcal{O}_{\mathbb{P}^1}(1 + b + c) = \mathcal{O}_{\mathbb{P}^1}(k)$$

where y is the coordinate on (the total space of) the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ over \mathbb{P}^1 .

We say that C is cut out birationally since, for $b, c \neq 1$, the above curve will have singularities at

$$(x, y) = (1, 0), (2, 0).$$

The singularities can be resolved by performing a specific sequence of iterated blowups (as will be explained in the next paragraph). After finitely many steps, we will obtain C sitting inside a blowup S of

$$\mathbb{P} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}),$$

which is a nonsingular rational surface. In order to conclude by applying Theorem 4, we will have to check that

$$\int_{[C]} c_1(S) > 0$$

holds and that the number n of ramification points of φ is at most equal to $\text{vdim } \overline{\mathcal{M}}_g(S, [C])$.

The original curve C_0 in \mathbb{P} is easily seen to be of class $\beta = kc_1(\mathcal{O}_{\mathbb{P}}(1))$, and we have

$$\int_{\beta} c_1(\mathbb{P}) = 3k.$$

If $b > 1$, then C_0 has a singularity of multiplicity b at $(x, y) = (1, 0)$. For the coordinate $z = y/(x - 1)$ on the blowup of \mathbb{P} at $(1, 0)$, the strict transform of C_0 is locally cut out by $z^{k-b} = (x - 1)^b$. The relevant intersection number

$$\int_{\beta - bE_1} c_1(\text{Bl}_{(1,0)}\mathbb{P}) = 3k - b$$

has exactly decreased by the multiplicity b of C_0 at $(1, 0)$.

We can continue the process of blowing-up the singular point and taking the strict transform. After j steps, the curve still has a local equation of the form $z_1^{e_j} = z_2^{f_j}$. We started with $(e_0, f_0) = (k, b)$ and obtained

$$(e_1, f_1) = (k - b, b)$$

in the first step. In general, the pairs (e_j, f_j) are then obtained by performing a Euclidean algorithm starting from (k, b) . The multiplicity of the singular point after the j th step is exactly $\min(e_j, f_j)$. The process terminates after finitely many steps (when the minimum of e_j, f_j is either 0 or 1). Then, the local equation is $z^g = 1$ or $z^g = z'$, which is nonsingular.

Denote by $\text{ms}(e, f)$ the sum of the multiplicities of the singular points that occur in the desingularization of $z_1^e = z_2^f$ in the above manner. The function is uniquely determined by the axioms

- $\text{ms}(e, f) = \text{ms}(f, e)$,
- $\text{ms}(e, 0) = \text{ms}(e, 1) = 0$,
- $\text{ms}(e, f) = f + \text{ms}(e - f, f)$, for $e \geq f$.

By the above analysis, the curve $C \subset S$ obtained by desingularizing $C_0 \subset \mathbb{P}$ satisfies

$$\int_{[C]} c_1(S) = \int_{\beta} c_1(\mathbb{P}) - \text{ms}(k, b) - \text{ms}(k, c) = 3k - \text{ms}(k, b) - \text{ms}(k, c).$$

In order to show positivity, we must bound $\text{ms}(e, f)$ from above. By induction, for $(e, f) \neq (1, 1)$, we obtain:

$$\text{ms}(e, f) \leq e + f - R(e, f), \text{ with } R(e, f) = \begin{cases} \gcd(e, f) & \text{if } \gcd(e, f) > 1, \\ 3 & \text{otherwise.} \end{cases} \quad (20)$$

Then, we have

$$\int_{[C]} c_1(S) \geq 3k - k - b - k - c + R(k, b) + R(k, c) = 1 + R(k, b) + R(k, c) \geq 1. \quad (21)$$

For the virtual dimension we obtain

$$\text{vdim } \overline{\mathcal{M}}_g(S, [C]) = \int_{[C]} c_1(S) + g - 1 \geq g + R(k, b) + R(k, c).$$

On the other hand, the number of ramification points equals

$$n = 1 + \gcd(k, b) + \gcd(k, c)$$

so we have

$$\text{vdim} - n \geq g - 1 + \underbrace{(R(k, b) - \gcd(k, b))}_{\geq 0} + \underbrace{(R(k, c) - \gcd(k, c))}_{\geq 0} \geq g - 1,$$

which we can assume to be nonnegative. We have thus verified the assumptions of Theorem 4. \diamond

Without the assumption of total ramification over one of the three points, the proof technique above no longer works. Indeed, for $k = 30$ and

$$(a, b, c) = (2, 3, 25),$$

a desingularization procedure over $x = 0, 1, 2$ as in the above proof would result in a curve C in S satisfying

$$\int_{[C]} c_1(S) = -20,$$

which cannot be remedied by applying an automorphism of $\mathbb{Z}/30\mathbb{Z}$. Nevertheless, we expect Theorem 17 to hold without the assumption of total ramification and even without the assumption of the cover being cyclic.

6 Summing to tautological cycles

6.1 Existence

As the examples $\overline{\mathcal{M}}_{1,n \geq 11}$ show, the Chow group of 0-cycles on $\overline{\mathcal{M}}_{g,n}$ can be infinite dimensional over \mathbb{Q} . The general point of $\overline{\mathcal{M}}_{g,n}$ may not determine a tautological 0-cycle. However, by adding additional points (with the number of points uniformly bounded in terms of g, n), we can arrive at a tautological 0-cycle. For technical reasons, we formulate the result for the coarse moduli space $\overline{M}_{g,n}$.

Proposition 18 *Given g, n with $2g - 2 + n > 0$, there exists an integer $T = T(g, n) \geq 1$ satisfying the following property: for any point*

$$Q_1 = (C, p_1, \dots, p_n) \in \overline{M}_{g,n},$$

we can find $Q_2, \dots, Q_T \in \overline{M}_{g,n}$ such that

$$[Q_1] + \dots + [Q_T] \in A_0(\overline{M}_{g,n})$$

is tautological.

Proof (suggested by A. Kresch). By standard arguments using the results of Section 1, we may take $Q = Q_1$ to be a general point of $\overline{M}_{g,n}$. We then choose a very ample divisor class

$$H \in A^1(\overline{M}_{g,n}).$$

Since Q is a nonsingular point of $\overline{M}_{g,n}$, general hyperplane sections

$$H_1, \dots, H_{3g-3+n} \in |H|$$

through Q will intersect transversely in a union of reduced points

$$\alpha = [Q_1] + \dots + [Q_T],$$

with $T = \deg(\overline{M}_{g,n}, H)$. On the other hand, since all divisor classes on $\overline{M}_{g,n}$ are tautological, the class α is also tautological. \diamond

Remark 19 Since the push-forward along the basic map

$$\overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$$

is an isomorphism of \mathbb{Q} -Chow groups, we can derive a version of Proposition 18 with $\overline{M}_{g,n}$ replaced by $\overline{\mathcal{M}}_{g,n}$. However, $T(g, n)$ for $\overline{M}_{g,n}$, may differ from the corresponding number for $\overline{\mathcal{M}}_{g,n}$: if $Q_i \in \overline{\mathcal{M}}_{g,n}$ has nontrivial automorphisms, then the cycle $[Q_i] \in A_0(\overline{M}_{g,n})$ corresponds to the cycle

$$|\text{Aut}(Q_i)| \cdot [Q_i] \in A_0(\overline{M}_{g,n}).$$

6.2 Minimality

We denote by $T(g, n)$ the minimal integer having the property described in Proposition 18. The proof of Proposition 18 used the degree of $\overline{\mathcal{M}}_{g,n}$, but there are several other geometric approaches to bounding $T(g, n)$. For example, we could use instead the Hurwitz cycle results of [13]. After fixing a degree $d \geq 1$, points $q_1, \dots, q_b \in \mathbb{P}^1$, and partitions $\lambda_1, \dots, \lambda_b$ of d , the sum of all points $[(C, (p_i)_i)]$ satisfying

- \exists a degree d map $C \rightarrow \mathbb{P}^1$ with ramification profile λ_j over $q_j \in \mathbb{P}^1$,
- with $(p_i)_i$ the set of preimages of the points q_1, \dots, q_b

is tautological by [13]. Since every genus g curve C admits *some* map $C \rightarrow \mathbb{P}^1$, the result above implies that adding to $[C] \in A_0(\overline{\mathcal{M}}_g)$ all cycles $[C']$ for curves $C' \rightarrow \mathbb{P}^1$ with the same branch points and ramification profiles as $C \rightarrow \mathbb{P}^1$ gives a tautological class. Hence, we bound $T(g, 0)$ in terms of a suitable Hurwitz number. A similar strategy works for any n by including the markings $p_1, \dots, p_n \in C$ among the ramification data of $C \rightarrow \mathbb{P}^1$.

However, these approaches will likely not yield optimal bounds. In all the cases listed in Figure 1, the space $\overline{\mathcal{M}}_{g,n}$ is rationally connected, so

$$T(g, n) = 1,$$

which is far below the bounds.

A different perspective on the question is to study the behavior of $T(g, n)$ for fixed g as $n \rightarrow \infty$. The following result shows that the asymptotic growth in n is *at most* linear.

Proposition 20 *Let (g, n) satisfy $2g - 2 + n > 0$. Then,*

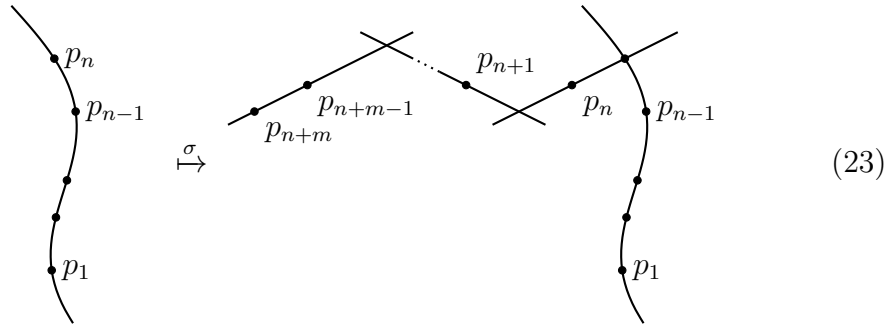
$$T(g, n + m) \leq (gm + 1) \cdot T(g, n) \quad (22)$$

for all $m \geq 0$.

Proof. The natural forgetful map

$$\nu : \overline{\mathcal{M}}_{g, n+m} \rightarrow \overline{\mathcal{M}}_{g, n}$$

has a section σ defined by the following construction: $\sigma((C, p_1, \dots, p_n))$ is the curve obtained by gluing a chain of rational curves containing the markings p_n, \dots, p_{n+m} at the previous position of $p_n \in C$.



The section σ is a composition of suitable boundary gluing maps, so the push-forward of a tautological cycle via σ is tautological.

Let $Q \in \mathcal{M}_{g, n}$ be moduli point with a *nonsingular* domain curve C . We claim: *for every $Q_1 \in \nu^{-1}(Q)$, there exist $Q_2, \dots, Q_{gm+1} \in \nu^{-1}(Q)$ satisfying*

$$[Q_1] + \dots + [Q_{gm+1}] = (gm + 1)[\sigma(Q)] \in A_0(\nu^{-1}(Q)).$$

Assuming the above claim, we can easily finish the proof.

Let $Q_1 \in \mathcal{M}_{g, n+m}$ with $Q = \nu(Q)$ and Q_2, \dots, Q_{gm+1} as in the above claim. By the definition of $T(g, n)$, we can find

$$P_1 = Q, P_2, \dots, P_{T(g, n)} \in \overline{\mathcal{M}}_{g, n}$$

for which

$$[P_1] + \dots + [P_{T(g, n)}] \in R_0(\overline{\mathcal{M}}_{g, n}).$$

We then obtain

$$\begin{aligned}
& [Q_1] + \dots + [Q_{gm+1}] + \sum_{i=2}^{T(g,n)} (gm+1)[\sigma(P_i)] \\
& = (gm+1)[\sigma(Q)] + \sum_{i=2}^{T(g,n)} (gm+1)[\sigma(P_i)] \\
& = (gm+1)\sigma_*([P_1] + \dots + [P_{T(g,n)}]) \in R_0(\overline{M}_{g,n+m}).
\end{aligned}$$

Hence, $T(g, n+m) \leq (gm+1) \cdot T(g, n)$.

We now prove the required claim. For

$$Q = (C, p_1, \dots, p_n) \in \mathcal{M}_{g,n},$$

the fibre $\nu^{-1}(Q)$ is isomorphic to a blow-up of the product C^m . Since the natural map

$$\nu^{-1}(Q) \rightarrow C^m$$

is a birational morphism between nonsingular varieties, we have an induced isomorphism

$$A_0(\nu^{-1}(Q)) \rightarrow A_0(C^m)$$

by [17, Example 16.1.11]. We can therefore verify the claim on C^m instead of $\nu^{-1}(Q)$. The image of $\sigma(Q)$ in C^m is exactly the point

$$(p_n, \dots, p_n) \in C^m.$$

By Riemann-Roch, every line bundle on C of degree at least g is effective. In other words, any divisor of degree at least g can be written as a sum of points on C . Assume we are given

$$Q_1 = (x_1, \dots, x_m) \in C^m.$$

Then, there exist points $x_{1,1}, \dots, x_{1,g} \in C$ satisfying

$$[x_{1,1}] + \dots + [x_{1,g}] = (g+1)[p_n] - [x_1] \in A_0(C).$$

Let $Q_{i+1} = (x_{1,i}, x_2, \dots, x_m) \in C^m$ for $i = 1, \dots, g$. We have

$$[Q_1] + \dots + [Q_{g+1}] = (g+1)[(p_n, x_2, \dots, x_m)] \in A_0(C^m).$$

For the next step, there exist points $x_{2,1}, \dots, x_{2,g} \in C$ satisfying

$$[x_{2,1}] + \dots + [x_{2,g}] = (2g + 1)[p_n] - (g + 1)[x_2] \in A_0(C).$$

Let $Q_{g+1+i} = (p_n, x_{2,i}, x_3, \dots, x_{n'})$ for $i = 1, \dots, g$. We have

$$[Q_1] + \dots + [Q_{2g+1}] = (2g + 1)[(p_n, p_n, x_3, \dots, x_{n'})] \in A_0(C^m).$$

After iterating the above procedure, we find points Q_1, \dots, Q_{gm+1} satisfying

$$[Q_1] + \dots + [Q_{gm+1}] = (gm + 1)[(p_n, \dots, p_n)] \in A_0(C^m)$$

as desired. ◇

Question 21 *Does $T(g, n)$ really grow linearly as $n \rightarrow \infty$?*

By results¹⁴ of Voisin (see Theorem 1.4 of [39]), the analogous T number of an abelian variety A is at least $\dim(A) + 1$. The linear growth there perhaps also suggests a linear lower bound for $T(g, n)$ as $n \rightarrow \infty$.

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¹⁴We thank Qizheng Yin for pointing out the connection.

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