Moduli in Mathematics

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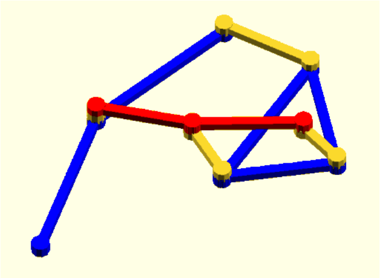
The **moduli space** of a mathematical structure parameterizes all deformations which **respect the defining properties** of the structure.

§I. Mechanical Linkages

As a first example, consider a **mechanical linkage**:



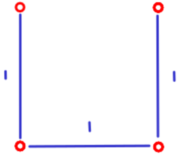
An **abstract linkage** Γ is a connected graph

$$\Gamma = (V, E, \ell)$$

where V and E are the **vertex** and **edge** sets and

$$\ell : E \to \mathbb{R}_{>0}$$

is an **edge length** function.



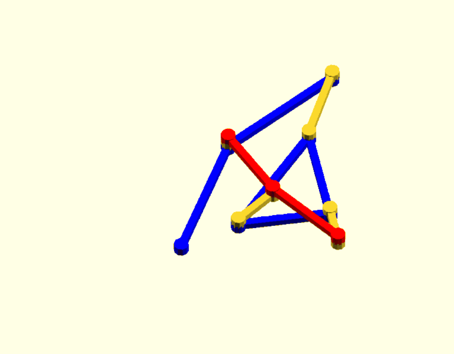
A **planar linkage** ϕ **of type** Γ is a function

$$\phi : V \to \mathbb{R}^2$$

which, for every edge $e = (v, v') \in E$, satisfies the condition

$$\ell(e) = |\phi(v) - \phi(v')| \, .$$

For a fixed **abstract linkage** Γ, there could be many **planar linkages**:



What is the space of all **planar linkages of type** Γ?

Let **Mod**(Γ) be the **moduli space** of **planar linkages of type** Γ,

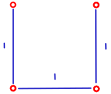
$$\mathbf{Mod}(\Gamma) \subset (\mathbb{R}^2)^{|V|},$$

defined by **real algebraic equations** corresponding to the **edges**.

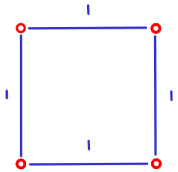
Given a **planar linkage**, we can apply translations and rotations. Let

$$\mathbf{mod}(\Gamma) = \frac{\mathbf{Mod}(\Gamma)}{\mathbb{R}^2 \rtimes \mathrm{SO}(2)}$$

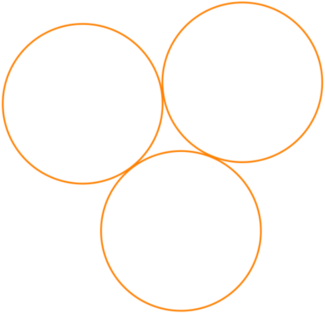
denote the quotient by these simple motions.

A first example: $\mathbf{mod}\Big($  $\Big) = \mathrm{S}^1 \times \mathrm{S}^1$.

A basic exercise is to compute **mod**(Γ) for the square:

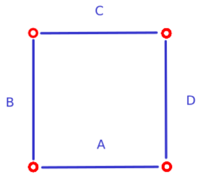
$\Gamma =$ 

The answer is:

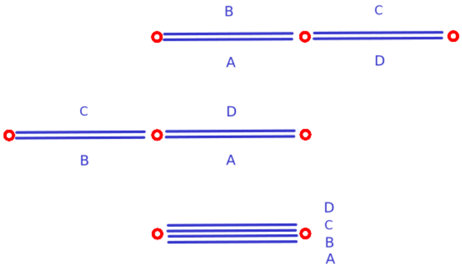
mod(Γ) $=$ 

What are the three **singular points**?

After labelling the **edges**:



the **singular points** can be drawn as:



Theorem (Kapovich-Millson 1999): For every **compact smooth manifold M**, there exists an **abstract linkage** Γ with

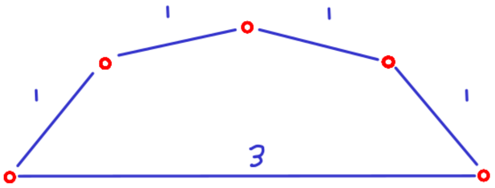
$$\mathbf{mod}(\Gamma) \overset{diffeo}{=} \mathbf{M} \sqcup \cdots \sqcup \mathbf{M},$$

a finite disjoint union.

The result was first imagined by Thurston in the 1980s. The first step of the proof uses the Nash-Tognoli Theorem to realize **M** as a **real algebraic set** in $\mathbb{R}^n$. Once the latter is found, the proof of Kapovich-Millson is constructive.

Can we find an **abstract linkage** Γ with **mod**$(\Gamma) = S^2$?

An answer for Γ is:



The example is taken from the work of Dirk Schütz.

§II. Instantons

Let **M** be a compact, oriented, simply connected, **smooth 4-manifold**. The only interesting cohomology of **M** is $H^2(\mathbf{M}, \mathbb{Z})$ which carries a unimodular symmetric bilinear **intersection form**:

$$H^2(\mathbf{M}, \mathbb{Z}) \times H^2(\mathbf{M}, \mathbb{Z}) \xrightarrow{\cup} H^4(\mathbf{M}, \mathbb{Z}) \cong \mathbb{Z}\,.$$

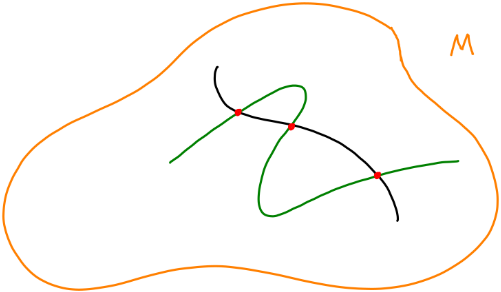
Theorem (Freedman 1982): **M** is classified up to **homeomorphism** by the **intersection form** on $H^2(\mathbf{M}, \mathbb{Z})$.

The algebraic invariants include the **rank** $\mathbb{Z}^{\mathbf{r}} \cong H^2(\mathbf{M}, \mathbb{Z})$ and the **signature** σ of the **intersection form**. The form is **definite** if $\sigma = \pm\mathbf{r}$.

The **intersection form**

$$H^2(\mathbf{M},\mathbb{Z}) \times H^2(\mathbf{M},\mathbb{Z}) \xrightarrow{\cup} H^4(\mathbf{M},\mathbb{Z}) \cong \mathbb{Z}$$

can either be defined via cup product or geometrically via intersection counts of Poincaré dual cycles:



For compact, oriented, simply connected, **topological 4-manifolds**, all unimodular symmetric bilinear forms can arise as **intersection forms**. Is this also true for **smooth 4-manifolds**?

Theorem (Donaldson 1983): In the **smooth** case, if the intersection form of **M** is **definite**, then the intersection form is **diagonalizable** over $\mathbb{Z}$.

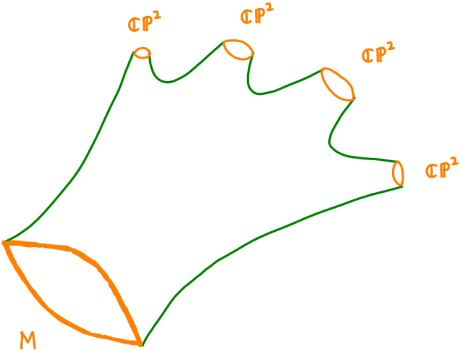
There are many **non-diagonalizable definite** forms, but Donaldson rules them all out for **smooth 4-manifolds**. The remarkable proof uses in a novel way the geometry of the moduli space of **SU(2) instantons** on **M**.

What **possible path** could an argument take?

Suppose there exists an oriented **5-manifold** which bounds **M** together with a disjoint union of **projective planes**

$$\mathbb{CP}^2 \sqcup \cdots \sqcup \mathbb{CP}^2.$$

The **supposed** picture looks like:



Then, we could use **properties** of the **oriented cobordism** between **M** and the disjoint union

$$\mathbb{CP}^2 \sqcup \cdots \sqcup \mathbb{CP}^2 .$$

A fundamental **property** is **signature invariance**,

$$\sigma(\mathbf{M}) = \sigma(\mathbb{CP}^2 \sqcup \cdots \sqcup \mathbb{CP}^2) .$$

So such a **cobordism** yields information about the **intersection form** of **M**.

Hirzebruch's famous Signature Theorem expresses the signature of a 4n-dimensional manifold in terms of explicit oriented cobordism invariants, the Pontryagin classes.

Donaldson's proof in the **postive definite** case:

Equip **M** with a Riemannian **metric** g and a principal **SU(2)-bundle** **P** $\to$ **M** with

$$c_2(\mathbf{P}) \cdot [\mathbf{M}] = -1\,.$$

Consider a moduli space **Mod** of connections **A** on **P**:

• The curvature $\mathbf{F}(\mathbf{A}) \in \Omega^2(\mathbf{Ad})$ is a 2-form on **M** with values in the vector bundle on **M** associated to **P** via the adjoint representation of **SU(2)**.

• The **metric** g together with an **invariant metric** on **Ad** yields a **metric** on $\Omega^2(\mathbf{Ad})$.

• The Yang-Mills functional is defined by

$$\int_{\mathbf{M}} |\mathbf{F}(\mathbf{A})|^2 \, \mathrm{dvol}_g\,.$$

- An **instanton A** is a critical point for the Yang-Mills functional.
- We are interested in **instantons A** which are also **self dual**:

$$\mathbf{F}(\mathbf{A}) = \star \mathbf{F}(\mathbf{A}).$$

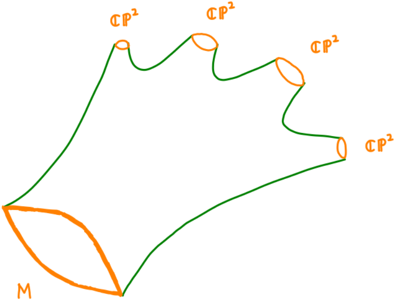
Mod is the moduli space of **self dual instantons** taken up to **gauge transformation**.

Mod is **5-dimensional**, but is **singular** and **not** compact.

By deep results of Taubes, Uhlenbeck, and Donaldson, there is an associated compact oriented moduli space $\widetilde{\mathbf{Mod}}$ of the following form:

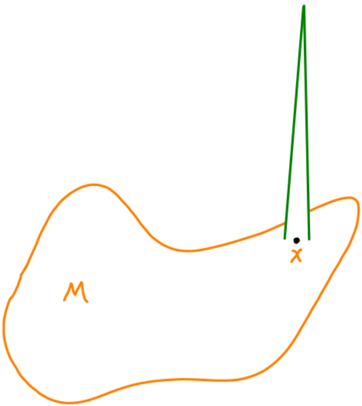
Mod =

$\widetilde{\textbf{Mod}}$ =



- The locus $\mathbf{M} \subset \widetilde{\mathbf{Mod}}$ can be viewed in the following manner:

The point $x \in \mathbf{M} \subset \widetilde{\mathbf{Mod}}$ is the limit of self-dual connections $\mathbf{A}$ where the amplitude $|\mathbf{F}(\mathbf{A})|^2$ of the **curvature** becomes a δ-**function** on $\mathbf{M}$ at the point x.



• The number of singularities of **Mod** is the number **n** of pairs

$$\pm\gamma \in H^2(\mathbf{M}, \mathbb{Z}) \quad \text{satisfying} \quad \int_{\mathbf{M}} \gamma \cup \gamma = 1\,.$$

By simple algebra, $\mathbf{n} \leq \sigma(\mathbf{M})$ for **positive definite** forms with equality only in the **diagonalizable** case.

• **M** is cobordant to a disjoint union of **n** projective planes

$$\mathbb{CP}^2 \sqcup \cdots \sqcup \mathbb{CP}^2.$$

The disjoint union has **signature** $= \mathbf{n}$.

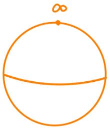
• Since the **signature** is an invariant of oriented cobordism, $\mathbf{n} = \sigma(\mathbf{M})$. □

§III. Riemann surfaces

A **Riemann surface C** is a compact connected **1-dimensional complex manifold**.

The **genus** g is the number of holes as a **topological surface**.

- **genus** 0: there is a **unique** complex structure (up to biholomorphism), the **Riemann sphere**.



- **genus** > 0: the complex structure can be **varied** while keeping the **topology fixed**.

C may also be viewed as an **algebraic curve** defined by the **zero locus** in $\mathbb{C}^2$ of a single **polynomial** equation

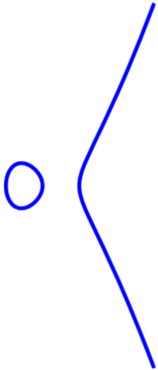
$$F(x, y) = 0$$

in the **complex variables** x, y (up to a few points at infinity).

For example, the **cubic equation**

$$F(x, y) = y^2 - x(x-1)(x-2)$$

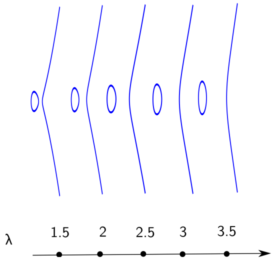
defines a Riemann surface of **genus** 1 with points in $\mathbb{R}^2$ given by:



The **complex structure** can be **varied** by changing the coefficients of the defining polynomial:

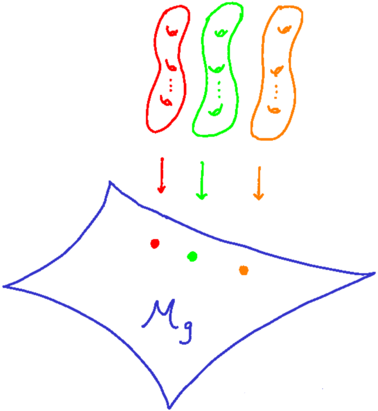
$$F_{\lambda}(x, y) = y^2 - x(x-1)(x-\lambda)$$

provides a **1-parameter family** of Riemann surfaces of genus 1.



We will also be interested in Riemann surfaces with **marked points** $(\mathbf{C}, \mathbf{p_1}, \dots, \mathbf{p_n})$:

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of **genus g**:



Riemann knew $\mathcal{M}_g$ was (essentially) a non-compact **complex manifold** of dimension $3g-3$.

Theorie der *Abel*'schen Functionen.

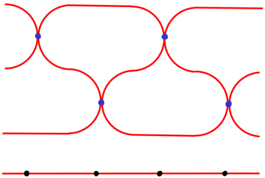
(Von Herrn *B. Riemann*.)

Riemann constructs the **variations** of complex structure, states the **dimension**, and coins the term **moduli** in a single sentence in **Crelle's Journal** in 1857.

Die $3p-3$ übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter μwerthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $\overline{2p+1}$fach zusammenhangender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Gröfsen ab, welche die Moduln dieser Klasse genannt werden sollen.

The remaining $3p-3$ branch values of those systems of μ-valued equally branched functions can therefore take arbitrary values; and thus a class of systems of $(2p+1)$-connected functions and a corresponding class of algebraic equations depend upon $3p-3$ continuously varying quantities, which should be called the moduli of these classes.

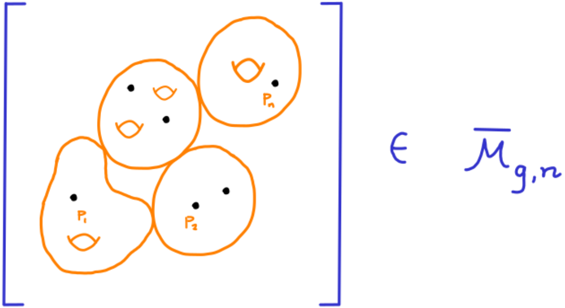
Consider **degree** μ coverings of the Riemann sphere with $2p + 2\mu - 2$ simple branch points:



By the Riemann-Hurwitz formula, the genus of the cover is p. The **variation** of complex structures of the cover is constructed by fixing $-p + 2\mu + 1$ branch points in the Riemann sphere and letting the remaining $3p - 3$ branch points **vary freely**.

Hurwitz later studied these covers systematically around 1900 at ETH Zürich.

Deligne and Mumford in 1969 compactified the moduli space of Riemann surfaces with marked points by the moduli space $\overline{\mathcal{M}}_{g,n}$ of **stable pointed curves**:



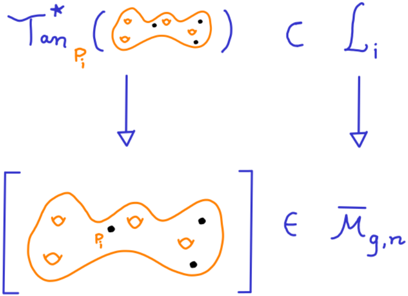
Again, $\overline{\mathcal{M}}_{g,n}$ is (essentially) a **complex manifold** of dimension $3g-3+n$, but is **compact**.

$\overline{\mathcal{M}}_{g,n}$ has been studied from several perspectives (**algebraic**, **hyperbolic**, **symplectic**, **topological**) for more than 50 years.

To each marked point $\mathbf{p_i}$, there is an associated **cotangent line**

$$\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$$

defined by:



Since $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$ is a **complex line bundle**, we can define

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

The Chern class is Poincaré dual to the cycle defined by the **zeros** and **poles** of a **meromorphic section** of $\mathcal{L}_i$.

A fundamental question concerns the **integration** of these **cotangent line classes**:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \psi_2^{k_2} \cdots \psi_n^{k_n} = ?$$

For the dimensions to match: $3g - 3 + n = \sum_{i=1}^{n} k_i$.

A beautiful answer is provided by Witten's conjecture in 1990.

We place the integrals in a **generating series**.

- Let $\left\langle \tau_{k_1}\tau_{k_2}\cdots\tau_{k_n}\right\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n}$.

- Introduce formal variables $t_0, t_1, t_2, \ldots$.

- Define the **generating series** of cotangent line integrals over **moduli spaces** of curves of genus g,

$$\mathsf{F}_g(t_0, t_1, t_2, \ldots) \;=\; \sum_{\{m_i\}} \prod_{i=0}^{\infty} \frac{t_i^{m_i}}{m_i!} \left\langle \tau_0^{m_0}\tau_1^{m_1}\tau_2^{m_2}\cdots\right\rangle_g .$$

- Put them all together:

$$\mathsf{F}(\lambda, t_0, t_1, t_2, \ldots) \;=\; \sum_{g=0}^{\infty} \lambda^{2g-2}\, \mathsf{F}_g .$$

Witten's Conjecture (1990) / Kontsevich's Theorem (1992):

Let $U(\lambda, t_0, t_1, t_2, \ldots) = \frac{\partial^2 F}{\partial t_0^2}$.

The series U satisfies the Korteweg-DeVries equation,

$$\lambda^{-2} \frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}.$$

The KdV equation was written in the 19th century to study **shallow water waves**. The connection to integration over $\overline{\mathcal{M}}_{g,n}$ was proposed by Witten via a **matrix model** approach to **quantum gravity**.

Furthermore, U satifies the KdV hierarchy which (together with the **string equation**) uniquely determines F.

§ Moduli in Mathematics

I. Moduli study transforms the **particular** to the **universal** in mathematics (a **planar linkage** is a particular object in Euclidean geometry, the moduli spaces include the study of all **smooth manifolds**).

II. The study of the moduli space of objects on **M** can reveal hidden structure of **M** (Donaldson's Theorem).

III. Moduli spaces themselves can have an very rich **intrinsic geometry** (Witten's Conjecture / Kontsevich's Theorem).

The goal of the last example will be to show:

IV. The surprising **connections** between seemingly **unrelated** moduli spaces.

§IV. Sheaves

Let **S** be a nonsingular projective **algebraic surface**.

As a topological space, **S** is a **4-manifold**.

An **algebraic analogue** of the instanton moduli space is the **moduli space** $\mathcal{U}_{\mathbf{S}}(\mathbf{c_1}, \mathbf{c_2})$ of rank 2 **stable sheaves** on **S**.

The moduli space $\mathcal{U}_{\mathbf{S}}(\mathbf{c_1}, \mathbf{c_2})$ parameterizes stable sheaves

$$\mathcal{E} \to \mathbf{S}$$

of rank 2 with fixed Chern classes

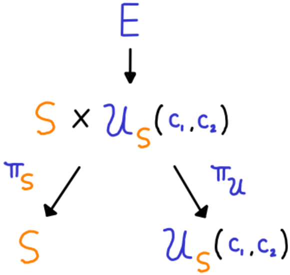
$$c_1(\mathcal{E}) = \mathbf{c_1}\,, \quad c_2(\mathcal{E}) = \mathbf{c_2}\,.$$

Stablity is with respect to a fixed ample line bundle on **S**.

We have **universal structures** which we use to define cohomology classes

$$\tau_k(\gamma) = \pi_{\mathcal{U}*}(\pi_{\mathbf{S}}^*(\gamma) \cup \mathrm{ch}_k(\mathrm{E}))$$

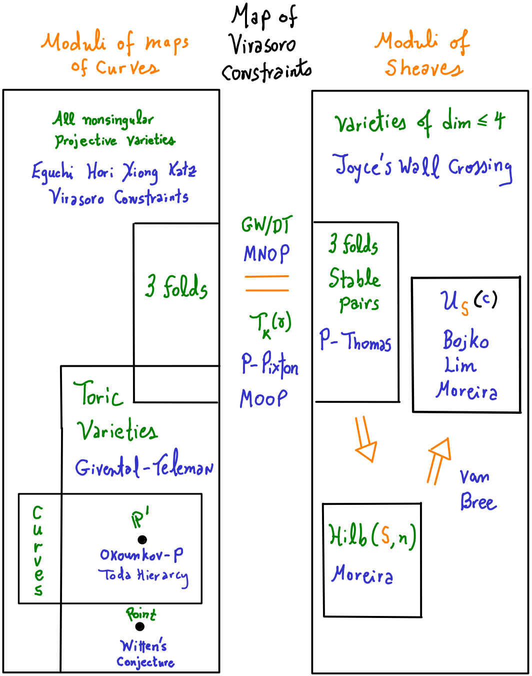
for integers $k \geq 0$ and $\gamma \in H^*(\mathbf{S}, \mathbb{Q})$.



We can then ask the question

$$\int_{\mathcal{U}_{\mathbf{S}}(\mathbf{c_1},\mathbf{c_2})} \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2)\cdots\tau_{k_n}(\gamma_n) = ?$$

Is there any **relationship** to the integrals in Witten's Conjecture?


Map of Virasoro Constraints
Moduli of maps of Curves
Moduli of Sheaves
All nonsingular Projective varieties
Eguchi Hori Xiong Katz Virasoro Constraints
3 folds
Toric Varieties
Givental-Teleman
Curves
P¹
Okounkov-P Toda Hierarcy
Point
Witten's Conjecture
GW/DT
MNOP
T_K(γ)
P-Pixton
MOOP
Varieties of dim ≤ 4
Joyce's Wall Crossing
3 folds
Stable pairs
P-Thomas
U_S(c)
Bojko
Lim
Moreira
Van Bree
Hilb(S,n)
Moreira

For $\mathbf{S} = \mathbb{CP}^2$ and $\mathsf{H} \in H^2(\mathbb{CP}^2)$ the **hyperplane class**, define the following generating series of integrals over $\mathcal{U}_{\mathbf{S}}(\mathbf{c_1}, \mathbf{c_2})$:

$$\mathsf{F} = \sum_{\ell=0}^{\infty} \sum_{\substack{j_1,\dots,j_\ell \\ k_1,\dots,k_\ell}} \prod_{i=1}^{\ell} k_i!\, t_{k_i}^{j_i} \int_{\mathcal{U}_{\mathbf{S}}(\mathbf{c_1},\mathbf{c_2})} \prod_{i=1}^{\ell} \tau_{k_i+2-j_i}(\mathsf{H}^{j_i})\,.$$

Theorem (Bojko-Lim-Moreira 2022): For all $\mathbf{n} \geq -1$,

$$\mathsf{L}_{\mathbf{n}}\, \mathsf{F} = 0$$

for the **differential operators**

$$\mathsf{L}_{\mathbf{n}} = \sum_{j=0}^{2} \sum_{k=0}^{\infty} \left(k t_k^j \frac{\partial}{\partial t_{k+\mathbf{n}}^j} - \frac{k}{2} \frac{\partial}{\partial t_{\mathbf{n}+1}^2} t_k^j \frac{\partial}{\partial t_{k-1}^j} \right) + \sum_{a+b=\mathbf{n}} \left(\frac{\partial}{\partial t_a^0} \frac{\partial}{\partial t_b^2} - \frac{\partial}{\partial t_a^1} \frac{\partial}{\partial t_b^1} + \frac{\partial}{\partial t_a^2} \frac{\partial}{\partial t_b^0} + \frac{\partial}{\partial t_a^2} \frac{\partial}{\partial t_b^2} \right).$$

The End

Acknowledgements

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