The 0-section of the universal abelian variety

I summarize here some results and questions which emerged from discussions in October 2020 with Sam Molcho and Johannes Schmitt about the 0-section of the universal abelian variety over the moduli space of abelian varieties (partially motivated by conversations with Dhruv Ranganathan and Jonathan Wise on double ramification cycles).

I. Background

Let $\mathcal{A}_g$ be the usual moduli space of PPAVs of dimension g, and let

$$\pi : \mathcal{X}_g \to \mathcal{A}_g$$

be the universal abelian variety π equipped with a universal 0-section

$$s : \mathcal{A}_g \to \mathcal{X}_g \,.$$

The 0-section determines an algebraic cycle class[1]

$$Z_g \in \mathsf{CH}^g(\mathcal{X}_g)\,.$$

The study of Z_g is related to the double ramification cycle (especially over curves of compact type), see the articles by Hain [7] and Grushevsky-Zakharov [5]. A central idea there is to use the beautiful formula

$$Z_g = \frac{\Theta^g}{g!} \ \in \mathsf{CH}^g(\mathcal{X}_g)\,, \tag{1}$$

where $\Theta \in \mathsf{CH}^1(\mathsf{X}_g)$ is the universal symmetric theta divisor trivialized along the 0-section. The proof of (1) in Chow uses the Fourier-Mukai transformation and work of Denninger-Murre [4], see [2, 9]. The article [5] provides a more detailed discussion of the history of (1).

II. Compactification $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$

There are various compactifications of $\mathcal{A}_g$, but I am interested in the second Voronoi which has been given a modular interpretation by Alekseev:

$$\mathcal{A}_g \subset \overline{\mathcal{A}}_g^{\mathsf{Alekseev}}\,.$$

[1] All Chow classes are taken here with $\mathbb{Q}$-coefficients.

As explained in [8], Olsson provides a modular interpretation for the normalization

$$\overline{\mathcal{A}}^{\mathsf{Olsson}} \to \overline{\mathcal{A}}_g^{\mathsf{Alekseev}} .$$

Our arguments will be valid for both $\overline{\mathcal{A}}^{\mathsf{Olsson}}$ and $\overline{\mathcal{A}}_g^{\mathsf{Alekseev}}$. We will denote the compactification by

$$\mathcal{A}_g \subset \overline{\mathcal{A}}_g \,,$$

where $\overline{\mathcal{A}}_g$ either the space of Alekseev or the space of Olsson.

The four important properties of the compactification $\overline{\mathcal{A}}_g$ as far as the discussion here is concerned are:

- The points of $\overline{\mathcal{A}}_g$ parameterize (before normalization) stable semiabelic pairs which are quadruples (G, P, L, θ) where G is a semiabelian variety, P is projective variety equipped with a G-action, L is an ample line bundle on P, and $\theta \in H^0(P, L)$. The data (G, P, L, θ) satisfy several further conditions, see Section 4.2.16 of [8].
- There is a universal semiabelian variety

  $$\overline{\pi} : \overline{\mathcal{X}}_g \to \overline{\mathcal{A}}_g$$

  with a 0-section

  $$\overline{s} : \overline{\mathcal{A}}_g \to \overline{\mathcal{X}}_g$$

  corresponding to the semiabelian variety which is the first piece of data of a stable semiabelic pair (the rest of the pair data will not play a role in our study).
- The usual Torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$ extends canonically

  $$\overline{\tau} : \overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g \,,$$

  see [1].
- The $\overline{\tau}$-pullback to $\overline{\mathcal{M}}_g$ of $\overline{\mathcal{X}}_g$ is the universal family

  $$\mathsf{Pic}^0_\epsilon \to \overline{\mathcal{M}}_g$$

  parameterizing line bundles on the fibers of the universal curve

  $$\epsilon : \mathcal{C}_g \to \overline{\mathcal{M}}_g$$

  which have degree 0 *on every component* of the fiber [1].

I have followed the notation of [8]. M. Olsson assures me that the last two Torelli results also hold after his normalization.

The question I am interested in here is to what extent is an equation of the form of (1) is possible over $\overline{\mathcal{A}}_g$. A result by Grushevsky and Zakharov along these line appears in [6]. Let

$$\overline{Z}_g \in \mathsf{CH}^g(\overline{\mathcal{X}}_g)$$

be the class of the 0-section $\overline{s}$. Grushevsky and Zakharov calculate the restriction $\overline{Z}_g|_{\mathcal{U}_g}$ of $\overline{Z}_g$ over a particular open set

$$\mathcal{A}_g \subset \mathcal{U}_g \subset \overline{\mathcal{A}}_g$$

in terms of Θ, a boundary divisor $D \in \mathsf{CH}^1(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$, and a class

$$\Delta \in \mathsf{CH}^2(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})\,.$$

The formula of [6] is a useful extension of (1).

The result of Grushevsky-Zarkhov shows that while the naive extension of (1) does *not* hold over $\mathcal{U}_g$, the class $\overline{Z}_g|_{\mathcal{U}_g}$ lies in the subring of $\mathsf{CH}^*(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$ generated by classes of degree 1 and 2.[2]

III. Bounding the complexity of $\overline{Z}_g$ from below.

Let $\mathsf{CH}^*(\overline{\mathcal{X}}_g)$ be the operational Chow ring. Since the image of the 0-section $\overline{s}$ is a local complete intersection,

$$\overline{Z}_g \in \mathsf{CH}^g(\overline{\mathcal{X}}_g)\,.$$

Let $\mathsf{CH}^*_{\mathsf{Div}}(\overline{\mathcal{X}}_g) \subset \mathsf{CH}^*(\overline{\mathcal{X}}_g)$ be the subring generated by $\mathsf{CH}^1(\overline{\mathcal{X}}_g)$. The first result (proven with S. Molcho and J. Schmitt) is the following:

Theorem 1. For all $g \geq 3$, we have $\overline{Z}_g \notin \mathsf{CH}^*_{\mathsf{Div}}(\overline{\mathcal{X}}_g)$.

As a consequence, no divisor formula extending (1) is possible for $\overline{\mathcal{A}}_g$. As remarked in Footnote 2, Theorem 1 can also be obtained from the analysis of [6].

Proof. The idea is geometrically very simple. Let

$$\mathsf{Pic}^0_\epsilon \to \overline{\mathcal{M}}_g$$

parameterize line bundles on the fibers of the universal curve of multidegree 0 (as discussed in Section II). Let

$$t : \overline{\mathcal{M}}_g \to \mathsf{Pic}^0_\epsilon$$

be the 0-section defined by the trivial line bundle. By the properties of

$$\overline{\pi} : \overline{\mathcal{X}}_g \to \overline{\mathcal{A}}_g$$

listed in Section II,

$$\overline{\tau}^*\overline{s}^*(\overline{Z}_g) = t^*(t[\overline{\mathcal{M}}_g])\,.$$

[2]Though not stated, the analysis of [6] over $\mathcal{U}_g$ can be used to show $\overline{Z}_g|_{\mathcal{U}_g}$ is *not* in the subring of $\mathsf{CH}^*(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$ generated by classes of degree 1. I thank Sam Grushevsky for correspondence about [6].

By standard analysis of the vertical tangent bundle of Pic^0_ϵ,

$$t^*(t[\overline{\mathcal{M}}_g]) = (-1)^g \lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_g)\,.$$

The Theorem is then an immediate consequence of the following Lemma (proven by J. Schmitt). $\Diamond$

Lemma 1. For all $g \geq 3$, we have $\lambda_g \notin \mathsf{CH}^*_{\mathsf{Div}}(\overline{\mathcal{M}}_g)$.

Proof. For $g = 3$, we have complete control of the tautological ring $\mathsf{R}^*(\overline{\mathcal{M}}_3)$. In degree 3,

$$\mathsf{CH}^3_{\mathsf{Div}}(\overline{\mathcal{M}}_3) \subset \mathsf{R}^3(\overline{\mathcal{M}}_3)$$

is a 9-dimensional subspace of a 10-dimensional space. Explicit calculations with the Sage program *admcycles* [3] shows $\lambda_3 \notin \mathsf{CH}^3_{\mathsf{Div}}(\overline{\mathcal{M}}_3)$.

We also understand $\mathsf{R}^*(\overline{\mathcal{M}}_{3,1})$ completely:

$$\mathsf{CH}^3_{\mathsf{Div}}(\overline{\mathcal{M}}_{3,1}) \subset \mathsf{R}^3(\overline{\mathcal{M}}_{3,1})$$

is a 28-dimensional subspace of a 29-dimensional space. But remarkably, a calculation by *admcycles* shows

$$\lambda_3 \in \mathsf{CH}^3_{\mathsf{Div}}(\overline{\mathcal{M}}_{3,1}) \;!$$

The containment appears miraculous. Is there a geometric explanation?

The tautological ring $\mathsf{R}^*(\overline{\mathcal{M}}_{4,1})$ is also completely under control:

$$\mathsf{CH}^4_{\mathsf{Div}}(\overline{\mathcal{M}}_{4,1}) \subset \mathsf{R}^4(\overline{\mathcal{M}}_{4,1})$$

is a 103-dimensional subspace of a 191-dimensional space. An *admcycles* calculation shows

$$\lambda_4 \notin \mathsf{CH}^4_{\mathsf{Div}}(\overline{\mathcal{M}}_{4,1})\,. \tag{2}$$

The result (2) implies

$$\lambda_4 \notin \mathsf{CH}^4_{\mathsf{Div}}(\overline{\mathcal{M}}_4)\,.$$

For $g \geq 5$, a boundary restriction argument is pursued. Suppose, for contradiction,

$$\lambda_g \in \mathsf{CH}^g_{\mathsf{Div}}(\overline{\mathcal{M}}_g)\,. \tag{3}$$

Then, by pull-back, we have

$$\lambda_g \in \mathsf{CH}^g_{\mathsf{Div}}(\overline{\mathcal{M}}_{g,1})\,. \tag{4}$$

Consider the standard boundary inclusion

$$\delta : \overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{g,1}\,.$$

As usual, we have

$$\delta^*(\lambda_g) = \lambda_{g-1} \otimes \lambda_1\,.$$

Then (4) implies

$$\lambda_{g-1} \otimes \lambda_1 \in \mathsf{CH}^g_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2})\,. \tag{5}$$

Since $\overline{\mathcal{M}}_{1,2}$ has a coarse moduli space with an affine stratification,

$$\mathsf{CH}^*_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2}) = \mathsf{CH}^*_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{CH}^*_{\mathsf{Div}}(\overline{\mathcal{M}}_{1,2})\,.$$

We therefore can write $\mathsf{CH}^g_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,1})$ as

$$\begin{aligned} & \mathsf{CH}^g_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{CH}^0_{\mathsf{Div}}(\overline{\mathcal{M}}_{1,1}) \\ \oplus\ & \mathsf{CH}^{g-1}_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{CH}^1_{\mathsf{Div}}(\overline{\mathcal{M}}_{1,1}) \\ \oplus\ & \mathsf{CH}^{g-2}_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1}) \otimes \mathsf{CH}^2_{\mathsf{Div}}(\overline{\mathcal{M}}_{1,1})\,. \end{aligned} \tag{6}$$

After multiplying with ψ_1 (corresponding to the original marking of $\overline{\mathcal{M}}_{g,1}$) and pushing both (5) and (6) to the factor $\overline{\mathcal{M}}_{g-1,1}$, we conclude

$$\lambda_{g-1} \in \mathsf{CH}^{g-1}_{\mathsf{Div}}(\overline{\mathcal{M}}_{g-1,1})\,.$$

By descending induction, we contradict (2). Therefore (4) and hence also (3) must be false. $\diamondsuit$

The proof of Lemma 1 above shows

$$\lambda_g \notin \mathsf{CH}^g_{\mathsf{Div}}(\overline{\mathcal{M}}_{g,1}) \tag{7}$$

for $g \geq 4$. By using (7) as a starting point, we can study

$$\lambda_g \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$$

for $g \geq 4$ and $n \geq 2$ using the boundary restrictions

$$\widehat{\delta} : \overline{\mathcal{M}}_{g,n-1} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n}\,.$$

The argument used in the proof then easily yields a statement with markings:

Lemma 2. For all $g \geq 4$ and $n \geq 0$, we have

$$\lambda_g \notin \mathsf{CH}^g_{\mathsf{Div}}(\overline{\mathcal{M}}_{g,n})\,.$$

IV. Higher degree generators and the Chern classes of the Hodge bundle

Define the subring of tautological classes

$$\mathsf{R}^*_{\mathsf{D}\leq k}(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{R}^*(\overline{\mathcal{M}}_{g,n})$$

generated by classes of (complex) degrees less than or equal to k. Since all divsors are tautological,

$$\mathsf{R}^*_{\mathsf{D}\leq 1}(\overline{\mathcal{M}}_{g,n}) = \mathsf{CH}^*_{\mathsf{Div}}(\overline{\mathcal{M}}_{g,n})\,.$$

Other such relationships are discussed below. The arguments in Section III naturally generalize to address the following question: *when is*

$$\lambda_{g-r} \in \mathsf{R}^{g-r}_{\mathsf{D}\leq k}(\overline{\mathcal{M}}_{g,n})\,?$$

A crucial case of the question (from the point of view of boundary restriction arguments) is for $n = 1$. Let $\mathsf{Q}_g(r,k)$ be the statement

$$\lambda_{g-r} \notin \mathsf{R}^{g-r}_{\mathsf{D}\leq k}(\overline{\mathcal{M}}_{g,1})$$

which may be true or false.

For example, $\mathsf{Q}_g(r, g-r)$ is false essentially by definition. In fact,

$$\mathsf{Q}_g(s, g-r) \quad \text{for all} \quad s \geq r$$

is false for the same reason. By Mumford's formula for the Chern character of the Hodge bundle (and the vanishing of even Chern characters), $\mathsf{Q}_g(r-1, g-r)$ is false whenever $g-r$ is odd.

The boundary arguments used in the proofs of Lemmas 1 and 2 yield the following two results.

Theorem 2. If $\mathsf{Q}_g(r,k)$ is true, then $\mathsf{Q}_{g+1}(r,k)$ and $\mathsf{Q}_{g+1}(r+1,k)$ are true.

Theorem 3. If $\mathsf{Q}_g(r,k)$ is true, then

$$\lambda_{g-r} \notin \mathsf{R}^{g-r}_{\mathsf{D}\leq k}(\overline{\mathcal{M}}_{g,n})$$

for all $n \geq 0$.

Since the $\mathsf{D} \leq 1$ case has already been studied in Section III, we consider $\mathsf{D} \leq 2$. The first relevant *admcycles* calculation is

$$\lambda_3 \notin \mathsf{R}^3_{\mathsf{D}\leq 2}(\overline{\mathcal{M}}_{4,1})\,,$$

so $\mathsf{Q}_4(1,2)$ is true. The corresponding subspace here is of dimension 91 inside a 93 dimensional space. As a consequence of Theorems 2 and 3, we find

$$\lambda_{g-1} \notin \mathsf{R}^{g-1}_{\mathsf{D}\leq 2}(\overline{\mathcal{M}}_{g,n})$$

for all $g \geq 4$ and $n \geq 0$.

A much more complicated *admcycles* calculation (modulo Pixton's conjecture) shows

$$\lambda_5 \notin \mathsf{R}^5_{\mathsf{D}\leq 2}(\overline{\mathcal{M}}_{5,1})\,,$$

so $\mathsf{Q}_5(0,2)$ is true. The corresponding subspace here is of dimension 1314 inside a 1371 dimensional space – a longer *admcycles* calculation should be possible here using Poincaré duality to prove Pixton's conjecture for $\overline{\mathcal{M}}_{5,1}$. As a consequence of Theorems 2 and 3, we find

$$\lambda_g \notin \mathsf{R}^g_{\mathsf{D}\leq 2}(\overline{\mathcal{M}}_{g,n})$$

for all $g \geq 5$ and $n \geq 0$. For large g, the following equality is known[3]

$$\mathsf{R}^2(\overline{\mathcal{M}}_g) = H^4(\overline{\mathcal{M}}_g)\,.$$

We then obtain a new obstruction for generalizing (1) by running the above discussion in cohomology.

Theorem 4. Not only is no divisor formula extending (1) possible for $\overline{\mathcal{A}}_g$, no formula including codimension 2 classes as by Grushevsky-Zakharov [6] can be extended over $\overline{\mathcal{A}}_g$.

Conjecture. No extension of (1) over $\overline{\mathcal{A}}_g$ for all g can be written with classes of uniformly bounded degree.

V. Parallel study in $\mathsf{CH}^*_{\mathsf{log}}(\overline{\mathcal{M}}_{g,n})$

We can consider the parallel questions also in $\mathsf{CH}^*_{\mathsf{log}}(\overline{\mathcal{M}}_{g,n})$ where $\mathsf{CH}^*_{\mathsf{log}}$ is defined via the limit of all iterated blow-ups of $\overline{\mathcal{M}}_{g,n}$ along boundary strata. The most basic question is:

Question. Does λ_g lie in the subring of $\mathsf{CH}^*_{\mathsf{log}}(\overline{\mathcal{M}}_g)$ generated by divisors?

I would guess the answer is no except for perhaps finitely many g. However, David Holmes thinks the answer is yes based on a view of the log Chow groups of LogPic.

Rahul, 19 October 2020

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